March, 2015

$\boxed{1}$

(1) Prove

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi},$$

where e stands for the base of the natural logarithm.

(2) Calculate the double integral

$$\iint_D x\, e^{-x^2-2xy-2y^2}\, dxdy,$$

where $D = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0\}$.

$\boxed{2}$

Let C be the curve in the xy-plane determined by $x^4 + y^4 - 4xy = 0$.

(1) Show that the curve C lies in the set

$$\{(x, y) \mid x \geq 0,\ y \geq 0\} \cup \{(x, y) \mid x \leq 0,\ y \leq 0\}.$$

(2) Show that the curve C lies in the set

$$\{(x, y) \mid x^2 + y^2 \leq 4\}.$$

(3) Find the smallest constant R such that $|y| \leq R$ holds for every point (x, y) on the curve C.

$\boxed{3}$

Consider the following matrix A and polynomial $f(x)$:

$$A = \begin{pmatrix} -3 & 0 & -2 \\ 8 & -1 & 8 \\ 4 & 0 & 3 \end{pmatrix}, \quad f(x) = x^5 + x^4 - 2x^3 - x^2 + 2x + c,$$

where c is a real number.

(1) Express the eigenvalues of the matrix $f(A)$ by using c.

(2) Find a necessary and sufficient condition for c in order that the matrix $f(A)$ is invertible.

(3) Assume that $c = 0$. For a positive integer n, find $f(A)^n$.

$\boxed{4}$ The following system of ordinary differential equations describes one of simplest mathematical models for a dynamics of disease transmission in a population:

$$(*)\qquad \begin{aligned} \frac{dS}{dt} &= -aSI \\ \frac{dI}{dt} &= aSI - rI \\ \frac{dR}{dt} &= rI, \end{aligned}$$

where a and r are positive constants, and $S(0) > 0$, $I(0) > 0$ and $R(0) = 0$.

(1) In model $(*)$, S means the density of population susceptible for the transmissible disease, and I does the density of infected population which has capacity to infect the other. What is the biological meaning of variable R? Describe one of possible meanings.

(2) Show that the quantity $V(S, I)$ defined by

$$V(S, I) = S + I - \frac{r}{a}\log S$$

is a constant independent of time t. Then draw the outline of a trajectory of the solution for $(*)$ in (S, I)–plane.

(3) If the temporal variation of infected population density $I(t)$ has a peak at a time, we say it the outbreak of transmissible disease. Describe the condition that the outbreak occurs for $(*)$.

(4) Consider a modification of model $(*)$ with the substitution of a function of I for parameter a: $a = a(I)$ which satisfies that $a(I) > 0$ and $a'(I) < 0$. Give the meaning of this modification as any addition of assumptions.

$\boxed{5}$ Let A be a subset of the direct product $X \times Y$ of two finite sets X and Y. Assume that

$$\bigl|\{y \in Y \mid (x, y) \in A\}\bigr|$$

is a constant r independent of $x \in X$, and that

$$\bigl|\{x \in X \mid (x, y) \in A\}\bigr|$$

is a constant s independent of $y \in Y$. Assume further that, for any distinct elements $x, x' \in X$, there exists a unique element $y \in Y$ such that $(x, y) \in A$ and $(x', y) \in A$. Here, the cardinality of a finite set S is denoted by $\bigl|S\bigr|$.

(1) Show $r|X| = s|Y|$.

(2) Show $\big|X\big| - 1 = r(s-1)$.

(3) Assume that there exist a subset X_1 of X, a partition $Y = Y_1 \cup Y_2$ of Y, and distinct integers s_1 and s_2 such that

$$y \in Y_i \implies \big|\{x \in X_1 \mid (x, y) \in A\}\big| = s_i \quad (i = 1, 2).$$

Show that $\big|\{y \in Y_1 \mid (x_1, y) \in A\}\big|$ is a constant independent of $x_1 \in X_1$.

$\boxed{6}$ Let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables obeying the uniform distribution on the interval $[0, 1]$, and set

$$L = \max\{X_1, X_2, \ldots, X_n\}, \qquad S = \min\{X_1, X_2, \ldots, X_n\}.$$

(1) Find the distribution function of L, $F_L(x) = P(L \leq x)$, and its density function $f_L(x)$.

(2) Calculate the mean value and variance of L.

(3) Find the joint density function $f_{SL}(x, y)$ of S and L.

(4) Calculate the correlation coefficient between S and L.

$\boxed{7}$ We define a meromorphic function f on the complex plane by

$$f(z) = \frac{z^2}{5 + 2(z + z^{-1})}.$$

(1) Find all the poles and their residues of f on the complex plane.

(2) Evaluate the following definite integral:

$$\int_0^{2\pi} \frac{\cos 3\theta}{5 + 4\cos\theta}\, d\theta.$$

$\boxed{8}$ Let $y = y(x)$ be a real-valued function of class C^2 on $\mathbb{R}$ solving the initial value problem

$$(2 + x^2)y'' - y = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

(1) Show that $y(x)$ is an odd function.

(2) When $y(x)$ is given by the power series $y(x) = \sum_{n=0}^{\infty} a_n x^n$ in some neighborhood of $x = 0$, find the relations satisfied by the coefficients $\{a_n\}$.

(3) Find the radius of convergence of the power series given by (2).

$\boxed{9}$ Let (S_1, d_1) and (S_2, d_2) be metric spaces, where d_1 and d_2 are distance functions on S_1 and S_2, respectively. Denote by $\mathcal{O}_1$ and $\mathcal{O}_2$ the topologies (the families of open sets) of S_1 and S_2, respectively. Define a distance function d on the product set $S_1 \times S_2$ by

$$d\big((x_1, x_2), (y_1, y_2)\big) = d_1(x_1, y_1) + d_2(x_2, y_2),$$

and introduce a topology $\mathcal{O}$ in $S_1 \times S_2$ by d. Show that the topology $\mathcal{O}$ coincides with the product topology $\mathcal{O}_{12}$ of $\mathcal{O}_1$ and $\mathcal{O}_2$. Here, the product topology of $\mathcal{O}_1$ and $\mathcal{O}_2$ is the topology induced by the family of open sets with base $\{U_1 \times U_2 \mid U_1 \in \mathcal{O}_1,\ U_2 \in \mathcal{O}_2\}$.

$\boxed{10}$ Let G be a group. For $a, b \in G$, set $[a, b] = a^{-1}b^{-1}ab$. Let $[G, G]$ be the subgroup of G generated by the subset $\{[a, b] \mid a, b \in G\}$ of G.

(1) For $g, x, y \in G$, find a pair of $s, t \in G$ such that $g^{-1}[x, y]g = [s, t]$.

(2) Prove that $[G, G]$ is a normal subgroup of G.

(3) Prove that the quotient group $G/[G, G]$ is abelian.

(4) Let N be a normal subgroup of G. Prove that the quotient group G/N is abelian if and only if $[G, G] \subset N$.