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$\boxed{1}$ Let n be a nonnegative integer.

(1) Find the maximum of the function

$$f(x) = x^n e^{-x}$$

on $x \geq 0$.

(2) Calculate the improper integral

$$I_n = \int_0^\infty x^n e^{-x} dx.$$

(3) Let C_n be the smallest among the constants $C > 0$ such that the inequality

$$a^n \int_a^\infty e^{-bx} dx \leq C \int_0^\infty x^n e^{-bx} dx$$

holds for all $a > 0$ and $b > 0$. Find C_n.

$\boxed{2}$ Let $D = (0,1) \times (0,\pi)$ be a region in the plane $\mathbb{R}^2$. Define the functions $R(x,t)$, $\Theta(x,t)$ on D by

$$R(x,t) = \frac{x \sin t}{\sqrt{1 - x^2 \cos^2 t}},$$
$$\Theta(x,t) = \arccos(x \cos t).$$

(1) Find the partial derivatives $\dfrac{\partial R}{\partial x}(x,t)$, $\dfrac{\partial R}{\partial t}(x,t)$, $\dfrac{\partial \Theta}{\partial x}(x,t)$, and $\dfrac{\partial \Theta}{\partial t}(x,t)$.

(2) Show that the Jacobian of the map $F(x,t) = (R(x,t), \Theta(x,t))$ is

$$\frac{\partial(R(x,t), \Theta(x,t))}{\partial(x,t)} = \frac{x}{\sin^2(\Theta(x,t))}.$$

(3) Show that $F(x,t) = (R(x,t), \Theta(x,t))$ is a bijection from D onto itself.

(4) For an integer $k \geq 2$, prove

$$\iint_{\bar{D}} \frac{r^k \sin^{k+2}\theta}{r^2 \sin^2\theta + \cos^2\theta} dr d\theta = \frac{1}{k} \int_0^\pi \sin^k t \, dt,$$

where $\bar{D}$ is the closure of D.

$\boxed{3}$ Let p, q be real numbers, and put $A = \begin{pmatrix} 1 & p & 4 \\ 3 & 3 & 2 \\ -1 & 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & p & 4 \\ 0 & 3 & 3 & 2 \\ 0 & -1 & 3 & 2 \\ q & q & q & q \end{pmatrix}$.

(1) Find the determinants of A and B.

(2) Find the condition on p under which the matrix A is nonsingular.

(3) Under the condition of (2), find the inverse of A.

(4) Let f be the linear map from $\mathbb{R}^4$ to $\mathbb{R}^4$ defined by $f(\boldsymbol{x}) = B\boldsymbol{x}$. Find the dimension and a basis of the image of f.

$\boxed{4}$ Consider the following mathematical model for the epidemic dynamics of a transmissible disease. $S(t)$ and $I(t)$ denote respectively the density of susceptible (non-infected) individuals and the density of infective (infected) individuals. The total population density is given by $N(t) = S(t) + I(t)$.

Let λ, μ, β, and γ be positive constants. Suppose that there is a specific time t_0 such that $S(t) = \lambda/\mu$ and $I(t) = 0$ for any $t < t_0$, while $S(t_0) = \lambda/\mu - I_0$ and $I(t_0) = I_0$ with a sufficiently small positive value of I_0 ($\ll \lambda/\mu$). The epidemic dynamics for $t \geq t_0$ is assumed to be governed by the following system of ordinary differential equations:

$$\frac{dS(t)}{dt} = \lambda - \mu S(t) - \beta S(t)I(t) + \gamma I(t),$$

$$\frac{dI(t)}{dt} = \beta S(t)I(t) - \mu I(t) - \gamma I(t).$$

Answer the following questions about the epidemic dynamics of a transmissible disease for $t \geq t_0$, given by the above mathematical model.

(1) Describe the meaning of the parameters λ, μ, and γ. In addition, give an explanation about the nature of the disease when the value of the parameter γ is large.

(2) What is the meaning of the time t_0 for the epidemic dynamics?

(3) Explain that the total population density $N(t)$ is a constant independent of time.

(4) For the above model, if $\beta > (\mu + \gamma)\mu/\lambda$, then $I(t)$ converges to a finite positive constant I^* as t tends to infinity, which means that the transmissible disease keeps existing in the population (i.e., becomes endemic), whereas if $\beta \leq (\mu + \gamma)\mu/\lambda$, the density of infective individuals monotonically decreases over time, that is, no outbreak of the disease transmission occurs. Show these features of the model. In addition, express I^* by the parameters.

$\boxed{5}$ For a positive integer k and a real number x, let

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$

(1) Fix k. Show that, as a function of a positive integer x, $\binom{x}{k}$ is monotone non-decreasing.

(2) Let x_1, x_2 be positive integers, and set $x = \dfrac{x_1 + x_2}{2}$. Show that the inequality

$$\binom{x_1}{k} + \binom{x_2}{k} \geq 2\binom{x}{k}$$

holds for every positive integer k such that $k \leq x + 1$.

$\boxed{6}$ Choose a random point from the unit disc $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ equipped with the uniform distribution and let (X, Y) be its coordinate.

(1) Find the probability density function $f_X(x)$ of the random variable X.

(2) Calculate the mean value $\mathbf{E}[X]$ and the variance $\mathbf{V}[X]$ of X.

(3) Calculate the covariance of X and Y, which is defined by

$$\mathbf{Cov}(X, Y) = \mathbf{E}[(X - \mathbf{E}[X])(Y - \mathbf{E}[Y])].$$

(4) Are the random variables X and Y independent? Why?

$\boxed{7}$ Consider the meromorphic function f on the complex plane given by

$$f(z) = \frac{1+z}{1-z}.$$

For a positive number r, let D_r denote the open disc

$$D_r = \{z \in \mathbb{C} \mid |z| < r\}.$$

(1) For $0 < r \leq 1$, determine the image $f(D_r)$ of the disc D_r under the mapping f.

(2) Find the Laurent series expansions of the function $f(z)$ about the origin on the regions $0 < |z| < 1$ and $1 < |z|$, respectively.

(3) For a positive number $r \neq 1$, define a closed curve C_r to be the circle ∂D_r oriented anticlockwise. For an integer $n \geq 2$, find the value of the complex integral

$$\int_{C_r} \frac{f(z)}{z^n} dz.$$

$\boxed{8}$ Let k be a real number.

(1) Find the general solution $y = y(x)$ to the differential equation

$$y'' - (k+1)y' + ky = 0.$$

(2) Find a particular solution $y = y(x)$ to the differential equation

$$y'' - (k+1)y' + ky = 2xe^x.$$

(3) Let $r(x)$ be a smooth function with $r(0) = 0$. Suppose that there exists a commun solution $y = y(x)$ to the differential equations

$$y'' - 2y' + y = 2xe^x$$

and

$$y'' - 3y' + 2y = r(x).$$

Find the function $r(x)$.

$\boxed{9}$ Consider the ellipsoid in $\mathbb{R}^3$ defined by

$$\mathcal{E} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1 \right\}.$$

(1) Define the function $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(x, y, z) = \dfrac{x^2}{4} + \dfrac{y^2}{9} + \dfrac{z^2}{16} - 1$. Show that the gradient ∇f of f does not vanish at each point of the ellipsoid $\mathcal{E}$.

(2) Show that at each point of $\mathcal{E}$ the tangent plane of $\mathcal{E}$ and the gradient ∇f meet orthogonally.

(3) Show that at each point of $\mathcal{E}$ the tangent plane of $\mathcal{E}$ can be identified with the kernel $(df)^{-1}(0)$ of the differential df at the point.

$\boxed{10}$ Let $\mathrm{GL}(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{Z},\ |ad - bc| = 1 \right\}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(1) Prove that $\mathrm{GL}(2, \mathbb{Z})$ is a group under the matrix multiplication.

(2) Prove that for any $n \in \mathbb{Z}$, the matrix $\begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}$ is contained in the subgroup $\langle S, T, D \rangle$ generated by S, T and D.

(3) Prove that $\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{Z},\ |ad| = 1 \right\} \subset \langle S, T, D \rangle$.

(4) Prove that $\mathrm{GL}(2, \mathbb{Z}) = \langle S, T, D \rangle$.