Representations of association schemes and coherent configurations

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June, 2014, Summer School

Introduction

- We will consider representations of association schemes and coherent configurations.
- Usually, "representations" mean representations of adjacency algebras. But it is not enough. For example, it is known that there are many strongly regular graphs with same parameters. They have isomorphic adjacency algebras.
- The standard module can contain more combinatorial informations. But standard modules over the complex number field are always isomorphic.
- We will consider modular representations, representations over a positive characteristic field.
- As an application, we will consider p-ranks of designs.

Coherent configurations and association schemes

- Let X be a finite set.
- Let S be a partition of $X \times X$. Namely, $X \times X = \bigcup_{s \in S} s$ is disjoint and $s \neq \emptyset$ for $s \in S$.
- For $s \subset X \times X$, put $s^* = \{(y, x) \mid (x, y) \in s\}$.
- A pair (X, S) is called a coherent configuration if
  - there is a subset $\{1_1, \cdots, 1_r\}$ of S such that $\bigcup_{i=1}^r 1_i = \{(x, x) \mid x \in X\}$,
  - if $s \in S$, then $s^* \in S$, and
  - there are nonnegative integers p_{st}^u $(s, t, u \in S)$ such that

$$p_{st}^u = \sharp\{z \in X \mid (x, z) \in s, \ (z, y) \in t\}$$

    when $(x, y) \in u$.
- Put $X_i = \{x \in X \mid (x, x) \in 1_i\}$ $(i = 1, \cdots, r)$ and call X_i a fiber.
- (X, S) is said to be homogeneous if $r = 1$. A homogeneous coherent configuration is also called an association scheme (noncommutaive).

- Let (X, S) be a coherent configuration with fibers $X_1, \cdots X_r$.
- For $s \in S$, we denote by σ_s the adjacency matrix of s.
- By definition, $\mathbb{Z}S = \bigoplus_{s \in S} \mathbb{Z}\sigma_s$ is a subring of $\mathrm{Mat}_X(\mathbb{Z})$.

$$\sigma_s \sigma_t = \sum_{u \in S} p_{st}^u \sigma_u$$

(p_{st}^u : intersection number, structure constant)

- For a commutative ring R with unity, define $RS = R \otimes_{\mathbb{Z}} \mathbb{Z}S$ and call this R-algebra the adjacency algebra of (X, S) over R. We often use the notation σ_s for the corresponding element in RS.

- For $s \in S$, there is a unique pair (i,j) such that $\sigma_{1_i}\sigma_s\sigma_{1_j} = \sigma_s$.
- Put
$$S^{ij} = \{s \in S \mid \sigma_{1_i}\sigma_s\sigma_{1_j} = \sigma_s\}.$$
- Then $S = \bigcup_i \bigcup_j S^{ij}$ is a partition of S.
- (X_i, S^{ii}) is a homogeneous coherent configuration and
$$RS^{ii} = \bigoplus_{s \in S^{ii}} R\sigma_s$$
is a subalgebra of RS (with non-common identity).

Example

- Let G be a permutation group on a finite set X.
- Let S be the set of G-orbits on $X \times X$ by diagonal action of G.
- Then (X, S) is a coherent configuration and the G-orbits of X are fibers.
- Put $G = \langle (12), (34) \rangle$ and $H = \langle (12)(34) \rangle$.
- Then the coherent configurations are

$$\begin{pmatrix} 0 & 1 & 4 & 4 \\ 1 & 0 & 4 & 4 \\ 5 & 5 & 2 & 3 \\ 5 & 5 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 4 & 5 \\ 1 & 0 & 5 & 4 \\ 6 & 7 & 2 & 3 \\ 7 & 6 & 3 & 2 \end{pmatrix}$$

Ordinary representations

- Let (X, S) be a coherent configuration.
- We will consider ordinary representations, representations over the complex number field $\mathbb{C}$, equivalently representations of $\mathbb{C}S$.

Lemma 3.1

Let A be a subalgebra of $\mathrm{Mat}_n(\mathbb{C})$ closed under transposed conjugate. Namely $x \in A$ implies ${}^t\overline{x} \in A$. Then A is semisimple.

Proposition 3.2

The adjacency algebra $\mathbb{C}S$ of a coherent configuration (X, S) is semisimple.

- We denote by $\mathrm{Irr}(S)$ the set of all irreducible characters of (X, S), where a character is the trace function of a representation.

- For $s \in S^{ii}$, define the valency $n_s = p_{ss^*}^{1_i}$.
- Define a map by $\sigma_s \mapsto n_s$ if $s \in S^{ii}$ and $\sigma_s \mapsto 0$ if $s \in S^{ij}$ $(i \neq j)$. Then the map is the character of an irreducible representation. We call this the trivial representation. The trivial representation has degree r.
- The map $\Gamma_S : \mathbb{C}S \to \mathrm{Mat}_X(\mathbb{C})$, $\Gamma_S(\sigma_s) = \sigma_s$ is also a representation. We call this the standard representation. By γ_S we denote the character of Γ_S. Then $\gamma_S(\sigma_s) = |X_i|$ if $s = 1_i$, and 0 otherwise.
- Consider the irreducible decomposition of γ_S :

$$\gamma_S = \sum_{\chi \in \mathrm{Irr}(S)} m_\chi \chi.$$

We call m_χ the multiplicity of χ.

Proposition 3.3

Let (X, S) be a homogeneous coherent configuration. For $\chi \in \mathrm{Irr}(S)$, we denote by e_χ the primitive central idempotent corresponding to χ. Then

$$e_\chi = \frac{m_\chi}{|X|} \sum_{s \in S} \frac{1}{n_s} \chi(\sigma_{s^*}) \sigma_s.$$

Proposition 3.4 (Orthogonality relation)

Let (X, S) be a homogeneous coherent configuration. For $\chi, \varphi \in \mathrm{Irr}(S)$,

$$\frac{m_\chi}{|X|\chi(1)} \sum_{s \in S} \frac{1}{n_s} \chi(\sigma_{s^*}) \varphi(\sigma_s) = \delta_{\chi\varphi}.$$

- There are similar formulas also for non-homogeneous coherent configurations.
- Note that the multiplicity m_χ is determined by character values. So, if two coherent configurations have the same intersection numbers, then the standard modules are isomorphic.

Problem 3.5

Find a good algorithm to compute character tables of (homogeneous) coherent configurations.

- For finite groups, Dixon–Schneider algorithm is good.
- When (X, S) is commutative, the intersections of eigenspaces determine the table. But it is not so easy to compute.

Question 3.6 (Bannai-Ito)

Let (X, S) be a (commutative) coherent configuration. Is it true that all eigenvalues of σ_s $(s \in S)$ are cyclotomic numbers ?

- Suppose that (X, S) be a homogeneous coherent configuration with $|X|$ prime.
- If an eigenvalue of σ_s $(s \in S)$ is non-cyclotomic, then the field generated by the value have some strong properties.
- Toru Komatsu (2006) constructed such field and possible intersection numbers (character table).
- By the same method, Sho Teranishi (2012) constructed many examples.

Example : $p = 2875$

$$\begin{aligned}
f_1(X) &= X^4 + X^3 - 1071X^2 - 7321X - 8850,\\
f_2(X) &= X^4 + X^3 - 1071X^2 - 4464X + 102573,\\
f_3(X) &= X^4 + X^3 - 1071X^2 + 1250X - 279,\\
f_4(X) &= X^4 + X^3 - 1071X^2 + 1250X + 85431
\end{aligned}$$

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 714 & 185 & 162 & 188 & 178 \\ 0 & 162 & 186 & 183 & 183 \\ 0 & 188 & 183 & 166 & 177 \\ 0 & 178 & 183 & 177 & 176 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 162 & 186 & 183 & 183 \\ 714 & 186 & 182 & 180 & 165 \\ 0 & 183 & 180 & 177 & 174 \\ 0 & 183 & 165 & 174 & 192 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 188 & 183 & 166 & 177 \\ 0 & 183 & 180 & 177 & 174 \\ 714 & 166 & 177 & 176 & 194 \\ 0 & 177 & 174 & 194 & 169 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 178 & 183 & 177 & 176 \\ 0 & 183 & 165 & 174 & 192 \\ 0 & 177 & 174 & 194 & 169 \\ 714 & 176 & 192 & 169 & 176 \end{pmatrix}$$

Question 3.7

Is there an association scheme with the above intersection numbers ?

Modular representations

- "Modular represntation" means representation of an adjacency algebra over a positive characteristic field F with non-semisimple FS.

Theorem 4.1

For a homogeneous coherent configuration (X, S) and a field of characteristic p, FS is semisimple if and only if p does not divide $\mathcal{F}(S)$. Here $\mathcal{F}(S)$ is the Frame number :

$$\mathcal{F}(S) = |X|^{|S|} \frac{\prod_{s \in S} n_s}{\prod_{\chi \in \mathrm{Irr}(S)} {m_\chi}^{(\chi(1)^2)}}.$$

- We also consider relations between representations over $\mathbb{C}$ and F.

p-Modular systems

- Let p be a prime number.
- Let R be a complete discrete valuation ring with valuation ideal πR.
- Let K be the quotient field of R, and let F be the residue field $R/\pi R$.
- Suppose that K is of characteristic 0 and F is of characteristic p.
- Then we say that (K, R, F) is a p-modular system.
- For a coherent configuration (X, S), we say that (K, R, F) is a splitting p-modular system of (X, S) if all adjacency algebras of subconfigurations and quotient configurations over K and F are splitting algebras. (In general, a finite dimensional k-algebra A is called a splitting algebra if $A/J(A)$ is isomorphic to a direct sum of full matrix k-algebras.)
- It is enough to consider the case that K and F are big enough.
- (If you do not understand, consider $(\mathbb{Q}, \mathbb{Z}, GF(p))$, though this is not a p-modular system.)

- Let (X, S) be a coherent configuration, and let (K, R, F) is a splitting p-modular system of (X, S).
- For a KS-module M, there is an RS-lattice $\widetilde{M}$ such that $K \otimes_R \widetilde{M} \cong M$. We call $\widetilde{M}$ an R-form of M. (An R-form is not uniue.)
- This means that, for every representation Φ of KS, there exists a similar representation Φ' such that $\Phi'(\sigma_s) \in \mathrm{Mat}_d(R)$ $(s \in S)$.
- Through an R-form, we can get an FS-module $M^* = \widetilde{M}/\pi\widetilde{M}$.
- M^* is not uniquely determined, but its composition factors are unique.
- So the modular character, the trace of the representation, is determined.
- For FS-modules V and W, we write $V \leftrightarrow W$ if they have the same composition factors.

Question 4.2

Suppose that we know the (ordinary) character table of (X, S). Can we determine modular irreducible characters (modules) ?

- The answer is NO, in general.
- It is difficult even for finite groups. (Problem to determine decomposition numbers.)
- If a group G is solvable, then every simple FG-module has a form M^* for some simple KG-module (Fong-Swan's Theorem).

Problem 4.3

Consider a good definition of "solveble coherent configurations". (Definition by French-Zieschang ?)

Decomposition matrices and Cartan matrices

- Let M be a simple KS-module.
- Determine $d_{M,V}$ by $M^* \leftrightarrow \bigoplus_{V\in \mathrm{IRR}(FS)} d_{M,V}V$.
- We call $d_{M,V}$ the decomposition number.
- Put $D = (d_{M,V})_{\mathrm{IRR}(KS)\times\mathrm{IRR}(FS)}$ and call this matrix the decomposition matrix.

- Let V be a simple FS-module.
- There is a primitive idempotent e_V such that $e_V FS/e_V J(FS) \cong V$.
- In this case, $e_V FS$ is the projective cover $P(V)$ of V.
- For V, $W \in \mathrm{IRR}(FS)$, define $c_{V,W} = \dim_F \mathrm{Hom}_{FS}(e_V FS, e_W FS) = \dim_F e_W FS e_V$.
- We call $c_{V,W}$ the Cartan invariant.
- Remark that $c_{V,W}$ is the number of V in $P(W)$ as simple constituents.
- Put $C = (c_{V,W})_{\mathrm{IRR}(FS)\times\mathrm{IRR}(FS)}$ and call this matrix the Cartan matrix.

- It is known that

$$ {}^{t}DD = C. $$

  Especially, the Cartan matrix C is symmetic.

Problem 4.4

Consider properties of decomposition matrices and Cartan matrices.

- For group representations,
  - C is non-singular, and
  - elementary divisors of C are p-power.
- But they are not true for (homogeneous) coherent configurations.

Modular irreducible representations of commutative schemes

- Let (X, S) be a commutative association scheme (homogeneous coherent configuration).
- Every irreducible representation over K has degree 1 and has values in R.
- Taking values modulo πR, we can get modular irreducible characters.
- Every modular irreducible character is obtained in this way.

Example

- Let (X, S) be an association scheme defined by a Fano plane.

$$\left(\begin{array}{ccccccc|ccccccc}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 3 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 2 & 2 & 2 & 3 & 3 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 2 & 2 & 3 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 3 & 3 & 2 & 3 & 3 & 2 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 3 & 2 & 3 & 3 & 2 & 3 & 2 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 3 & 3 & 2 & 3 & 3 & 2 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 3 & 2 & 3 & 2 & 3 & 2 & 3 \\
\hline
2 & 3 & 2 & 2 & 3 & 3 & 3 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 3 & 3 & 3 & 2 & 3 & 2 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 3 & 3 & 3 & 2 & 3 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
3 & 2 & 3 & 2 & 3 & 3 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
3 & 2 & 2 & 3 & 2 & 3 & 3 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
3 & 3 & 2 & 3 & 3 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
3 & 3 & 3 & 2 & 2 & 2 & 3 & 1 & 1 & 1 & 1 & 1 & 1 & 0
\end{array}\right)$$

- The character table is

	g_0	g_1	g_2	g_3	m_i
χ_1	1	6	3	4	1
χ_2	1	6	-3	-4	1
χ_3	1	-1	$\sqrt{2}$	$-\sqrt{2}$	6
χ_4	1	-1	$-\sqrt{2}$	$\sqrt{2}$	6

- Let $p = 7$.
- Then
  - $\chi_1^* = \chi_4^*$ and $\chi_2^* = \chi_3^*$ if we choose a prime ideal containing $3 + \sqrt{2}$, and
  - $\chi_1^* = \chi_3^*$ and $\chi_2^* = \chi_4^*$ if we choose a prime ideal containing $3 - \sqrt{2}$.
- It depends on the choice of the prime ideal.

Irreducible representations of coherent configurations and their fibers

- Irreducible representations of (non-homogeneous) a coherent configuration can be determined by them of its fibers.
- We denote by $\mathrm{IRR}(A)$ the set of representatives of isomorphism classes of simple right-A modules.

Theorem 4.5

Let K be an algebraically closed field. Let (X, S) be a coherent configuration with fibers X_i $(i = 1, 2, \cdots, r)$. Put $\mathcal{A} = \bigoplus_{i=1}^{r} KS^{ii}$. Then there are injections $\Phi_i : \mathrm{IRR}(KS^{ii}) \to \mathrm{IRR}(S)$ $(i = 1, 2, \cdots, r)$ satisfying the following properties.

1. *Define $\Phi : \bigcup_{i=1}^{r} \mathrm{IRR}(KS^{ii}) \to \mathrm{IRR}(KS)$ by $\Phi(W) = \Phi_i(W)$ if $W \in \mathrm{IRR}(KS^{ii})$. Then Φ is surjective.*
2. *For $V \in \mathrm{IRR}(KS)$, $V\downarrow_{\mathcal{A}} \cong \bigoplus_{W \in \Phi^{-1}(V)} W$.*
3. *The map Φ preserves the multiplicities.*

- What is "multiplicity".
- It is the number of the simple module in the standard module KX as simple constituents.
- Namely,

$$m_V = \dim_K \mathrm{Hom}_{KS}(P(V), KX) = \dim_K KSe = \mathrm{rank}(e),$$

where $P(V)$ is the projective cover of V and e is a primitive idempotent corresponding to V.
- If $K = \mathbb{C}$, then the multiplicity is just the usual multiplicity of a character.

- How can we get a partition of $\bigcup_{i=1}^{r} \mathrm{IRR}(KS^{ii})$?

Proposition 4.6

For V, $W \in \mathrm{IRR}(\mathcal{A})$, let e, f be primitive idempotents corresponding to V, W, respectively. Namely $e\mathcal{A}/eJ(\mathcal{A}) \cong V$ and $f\mathcal{A}/fJ(\mathcal{A}) \cong V$. Then $\Phi(V) = \Phi(W)$ if and only if $eKSf \not\subset J(KS)$.

Examples – Coherent configurations defined by quasi-symmetric designs

- Higman defined the type of coherent configurations.
- Let (X, S) be a coherent configuration with fibers X_i $(i = 1, \cdots, r)$.
- The type of (X, S) is an $r \times r$ matrix $T = (t_{ij})$ with $t_{ij} = |S^{ij}|$.
- It is clear that T is symmetric. We omit entries below the main diagonal.
- For example, a symmetric design defines a coherent configuration of type $\begin{pmatrix} 2 & 2 \\ & 2 \end{pmatrix}$.
- We also write the type as $(2, 2; 2)$.

- A design is said to be quasi-symmetric if any two blocks intersect in either x of y points $(x \neq y)$.
- For example, a 2-$(v, k, 1)$-design is quasi-symmetric with $x = 0$ and $y = 1$.
- The block graph of a quasi-symmetric design is defined as follows :
  - Points of the graph are blocks of the design.
  - Two blocks are adjacent in the graph if they intersect in y points.
- The block graph of a quasi-symmetric design is strongly-regular.
- The incidence matrix of a quasi-symmetric design defines a coherent configuration of type $(2, 2, ; 3)$.
- Conversely, a coherent configuration of type $(2, 2; 3)$ gives a quasi-symmetric design.
- We will consider some 2-$(v, k, 1)$-designs, especially 80 isomorphism classes of Steiner triple systems 2-$(15, 3, 1)$.

$$\text{type}\begin{pmatrix} 2 & 2 \\ & 2 \end{pmatrix} \longleftrightarrow \text{symmetric design}$$

$$\text{type}\begin{pmatrix} 2 & 2 \\ & 3 \end{pmatrix} \longleftrightarrow \text{quasi-symmetric design}$$

$$\text{type}\begin{pmatrix} 3 & 2 \\ & 3 \end{pmatrix} \longleftrightarrow \text{strongly regular design}$$

- strongly regular design : Higman, Klin–Reichard

- Let C be an incidence matrix of a combinatorial design.
- The p-ranks, the ranks of matrices in characteristic $p > 0$, of designs with same parameters are not constant, in general.
- For 80 nonisomorphic 2-$(15, 3, 1)$-designs, the 2-ranks of incidence matrices are 11, 12, 13, 14, and 15.
- The 3-ranks are 14, and p-ranks are 15 for $p \neq 2, 3$.
- We will focus on the 2-$(15, 3, 1)$-designs and $p = 2$.

- The parameters of a 2-$(v,\ell,1)$-design and strongly regular graph defined by the design are :

$$\begin{aligned}
r &= \frac{v-1}{\ell-1}, \\
b &= \frac{v(v-1)}{\ell(\ell-1)}, \\
n &= b, \\
k &= \ell\left(\frac{v-1}{\ell-1}-1\right), \\
a &= \frac{v-1}{\ell-1}-2+(\ell-1)^2, \\
c &= \ell^2.
\end{aligned}$$

- Let (X_1, X_2, F) be a 2-$(v, \ell, 1)$-design, where X_1 is the set of points, X_2 is the set of blocks, and F is the set of flags.
- Put $X = X_1 \cup X_2$.
- Define binary relations s_i $(i = 1, \cdots, 9)$ on X by

$$
\begin{aligned}
s_1 &= \{(x,x) \mid x \in X_1\}, \quad s_2 = \{(x,x) \mid x \in X_2\}, \\
s_3 &= X_1^2 - s_1, \\
s_4 &= \{(x,y) \in X_2^2 \mid \sharp(x \cap y) = 1\}, \\
s_5 &= \{(x,y) \in X_2^2 \mid \sharp(x \cap y) = 0\}, \\
s_6 &= F, \qquad s_7 = X_1 \times X_2 - F, \\
s_8 &= {}^t s_6 = \{(y,x) \mid x, y \in s_6\}, \\
s_9 &= {}^t s_7 = \{(y,x) \mid x, y \in s_7\}.
\end{aligned}
$$

- Put $S = \{s_1, \cdots, s_9\}$.
- Then (X, S) is a coherent configuration.
- Note that the subconfiguration $(X_2, \{s_2, s_4, s_5\})$ defines a strongly regular graph.

- We can compute the table of multiplications :

	σ_1	σ_3	σ_6	σ_7
σ_1	σ_1	σ_3	σ_6	σ_7
σ_3	σ_3	$(v-1)\sigma_1 + (v-2)\sigma_3$	$(\ell-1)\sigma_6 + \ell\sigma_7$	$(v-\ell)\sigma_6 + (v-\ell-1)\sigma_7$
σ_8	σ_8	$(\ell-1)\sigma_8 + \ell\sigma_9$	$\ell\sigma_2 + \sigma_4$	$(\ell-1)\sigma_4 + \ell\sigma_5$
σ_9	σ_9	$(v-\ell)\sigma_8 + (v-\ell-1)\sigma_9$	$(\ell-1)\sigma_4 + \ell\sigma_5$	$(v-\ell)\sigma_2 + (v-2\ell+1)\sigma_4 + (v-2\ell)\sigma_5$

	σ_2	σ_4	σ_5	σ_8	σ_9
σ_2	σ_2	σ_4	σ_5	σ_8	σ_9
σ_4	σ_4	$k\sigma_2 + a\sigma_4 + \ell^2\sigma_5$	$(k-a-1)\sigma_4 + (k-\ell^2)\sigma_5$	$(r-1)\sigma_8 + \ell\sigma_9$	$(k-r+1)\sigma_8 + (k-\ell)\sigma_9$
σ_5	σ_5	$(k-a-1)\sigma_4 + (k-\ell^2)\sigma_5$	$(b-k-1)\sigma_2 + (b+a-2k)\sigma_4 + (b-2k-2+\ell^2)\sigma_5$	$(r-\ell)\sigma_9$	$(b-k-1)\sigma_8 + (b-r-k+\ell-1)\sigma_9$
σ_6	σ_6	$(r-1)\sigma_6 + \ell\sigma_7$	$(r-\ell)\sigma_7$	$r\sigma_1 + \sigma_3$	$(r-1)\sigma_3$
σ_7	σ_7	$(k-r+1)\sigma_6 + (k-\ell)\sigma_7$	$(b-k-1)\sigma_6 + (b-r-k+\ell-1)\sigma_7$	$(r-1)\sigma_3$	$(b-r)\sigma_1 + (b-2r+1)\sigma_3$

- We remark that the coefficients are polynomial of v, ℓ, k, a, r, and b.

Lemma 5.1

If ℓ and $r = (v-1)/(\ell-1)$ are odd, then v, a, and b are odd and k is even.

Theorem 5.2

Let F be a field of characteristic 2. Let $\mathcal{A}$ be the adjacency algebra of a coherent configuration defined by a 2-$(15, 3, 1)$-design over F. Suppose that ℓ and $r = (v-1)/(\ell-1)$ are odd. Then the adjacency algebra of a coherent configuration defined by a 2-$(v, \ell, 1)$-design over F is isomorphic to $\mathcal{A}$.

- We determine the structure of the algebra $\mathcal{A}$.
- What should we do ?
- Give informations enough to apply representation theory of finite dimensional algebras.

- Morita equivalence – changing the dimensions of simple modules such that the module categories are equivalent.
- Example : K and $\mathrm{Mat}_n(K)$ are Morita equivalent.
- A finite dimensional algebra is said to be basic if every simple modules are one-dimensional.
- Every finite dimensional algebra is Morita equivalent to a basic algebra :

  Put $\mathrm{IRR}(A) = \{V_i \mid i = 1, \cdots, \ell\}$*. Let* e_i *be a primitive idempotent corresponding to* V_i*. Put* $e = \sum_{i=1}^{r} e_i$*. Then* eAe *is basic and Morita equivelent to* A*.*

  For $\mathrm{Mat}_n(K)$*,* $e_{11}\mathrm{Mat}_n(K)e_{11} \cong K$*, where* e_{11} *is the matrix unit.*

- Let $Q = (V, P, s, t)$ be a quiver where V is a set of vertices, P is a set of arrows, $s : P \to V$, and $t : P \to V$.
- A quiver is a directed graph with loops and multiple edges.
- The quiver algebra KQ is a K-algebra whose basis is the set of all paths and multiplication is composition of paths.
- The quiver algebra is not necessary finite dimensional.
- Let J be the ideal of KQ generated by all arrows.
- An ideal I of KQ is said to be admissible if $J^2 \supset I \supset J^n$ for some n.

- Let A be a basic algebra.
- For $V \in \mathrm{IRR}(A)$, we denote by e_V a primitive idempotent corresponding to V.
- We define the Gabriel quiver $Q(A)$ of A.
- The point set is $\mathrm{IRR}(A)$.
- The number of arrows from V to W is $\dim_K e_V(J(A)/J^2(A))e_W$.

Theorem 5.3 (Gabriel)

Let A be a basic algebra. Then there is an admissible ideal I of $KQ(A)$ such that $A \cong KQ(A)/I$.

- We will compute the Gabriel quiver and the admissible ideal for adjacency algebra of a coherent configuration defined by a 2-$(15, 3, 1)$-design.

- Let (X, S) be a coherent configuration defined by a 2-$(15, 3, 1)$-design.
- The character table of fibers (X_i, S^{ii}) $(i = 1, 2)$ are

	s_1	s_3	multiplicity
φ_1	1	14	1
φ_2	1	-1	14

	s_2	s_4	s_5	multiplicity
ψ_1	1	18	16	1
ψ_2	1	3	-4	14
ψ_3	1	-3	2	20

- The possibilities of irreducible characters of (X, S) are $\{\varphi_1 + \psi_1, \varphi_2, \psi_2, \psi_3\}$ and $\{\varphi_1 + \psi_1, \varphi_2 + \psi_2, \psi_3\}$.
- Degrees are $\{2, 1, 1, 1\}$ and $\{2, 2, 1\}$.
- Since $|S| = 9$ is the sum of squares of degrees, we have the character table.

	s_1	s_3	s_2	s_4	s_5	multiplicity
χ_1	1	14	1	18	16	1
χ_2	1	-1	1	3	-4	14
χ_3	0	0	1	-3	2	20

- We consider the modular character table for $p = 2$.
- The character table of fibers are

	s_1	s_3	multiplicity
φ_1	1	14	1
φ_2	1	-1	14

	s_2	s_4	s_5	multiplicity
ψ_1	1	18	16	1
ψ_2	1	3	-4	14
ψ_3	1	-3	2	20

- Consider them in characteristic 2. We have

s_1	s_3	multiplicity
1	0	1
1	1	14

s_2	s_4	s_5	multiplicity
1	0	0	1
1	1	0	34

- We can see that the modular character table of (X, S) is

	s_1	s_3	s_2	s_4	s_5	multiplicity
ξ_1	1	0	1	0	0	1
ξ_2	1	1	0	0	0	14
ξ_3	0	0	1	1	0	34

- By

	s_1	s_3	s_2	s_4	s_5	multiplicity
χ_1	1	14	1	18	16	1
χ_2	1	-1	1	3	-4	14
χ_3	0	0	1	-3	2	20

	s_1	s_3	s_2	s_4	s_5	multiplicity
ξ_1	1	0	1	0	0	1
ξ_2	1	1	0	0	0	14
ξ_3	0	0	1	1	0	34

we can determine the decomposition matrix and the Cartan matrix.

$$D = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right) \quad C = {}^tDD = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 1 & 2 \end{array}\right)$$

- We write A, B, C for the simple modules corresponding to ξ_1, ξ_2, and ξ_3.
- By the Cartan matrix $\left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 1 & 1 \\ 0 & 1 & 2 \end{array}\right)$, we have the Loewy structures of projective covers $P(A)$ and $P(B)$.

$$P(A) = A, \quad P(B) = \left[\begin{array}{c} B \\ C \end{array}\right]$$

- The projective cover $P(C)$ is either $\left[\begin{array}{c} C \\ B \\ C \end{array}\right]$ or $\left[\begin{array}{cc} \multicolumn{2}{c}{C} \\ B & C \end{array}\right]$.
- Note that the algebra has two blocks $\mathcal{B}_1$ and $\mathcal{B}_2$. The block $\mathcal{B}_1$ (containing A) is isomorphic to $\mathrm{Mat}_2(F)$. So we want to know $\mathcal{B}_2$.
- Remark that the algebra is not basic but $\mathcal{B}_2$ is basic.

- Now we consider the Gabriel quiver.
- We can choose primitive idempotent $e_B = \sigma_3$ and $e_C = \sigma_4 + \sigma_5$.
- Since $e_B F S e_C = F\sigma_7$ and $e_C F S e_B = F\sigma_9$, we put $\alpha = \sigma_7$ and $\beta = \sigma_9$.
- We have

$$\begin{array}{rclcc} e_B & = & \sigma_3, & e_B\alpha = \sigma_7, & (e_2\alpha)\beta = 0 \\ e_C & = & \sigma_4 + \sigma_5, & e_C\beta = \sigma_9, & (e_C\beta)\alpha = \sigma_5. \end{array}$$

- This means that the Gabriel quiver is

$$Q: \; e_B \underset{\beta}{\overset{\alpha}{\rightleftarrows}} e_C$$

and the relation is $\alpha\beta = 0$.

Theorem 5.4

Let F be a field of characteristic 2. Under the above notations,

$$FS \cong \mathrm{Mat}_2(F) \oplus FQ/(\alpha\beta)$$

where Q is the following quiver :

$$Q : \bullet \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \bullet$$

There are three simple module A, B, C and the Loewy structures of the projective covers are

$$P(A) = [A], \quad P(B) = \begin{bmatrix} B \\ C \end{bmatrix}, \quad P(C) = \begin{bmatrix} C \\ B \\ C \end{bmatrix}.$$

- Now we apply representation theory of finite dimensional algebras and consider the standard module FX.
- There are three representation types (very rough definition!):
  - finite : there are finitely many indecomposable modules
  - (infinite) tame : there are infinitely many indecomposable modules but they can be classified
  - (infinite) wild : there are infinitely many indecomposable modules but they can not be classified

Problem 5.5

Consider representation types of adjacency algebras of coherent configurations (association schemes).

- Our case is very easy and we can see that the representation type is finite.

- All indecomposable $\mathcal{B}_2$-modules are

$$M_1 = \left[\ C\ \right], \quad M_2 = \left[\begin{array}{c} C \\ B \end{array}\right], \quad M_3 = \left[\begin{array}{c} C \\ B \\ C \end{array}\right],$$

$$N_1 = \left[\ B\ \right], \quad N_2 = \left[\begin{array}{c} B \\ C \end{array}\right].$$

- So we can write

$$FX \cong [A] \oplus g_1 M_1 \oplus g_2 M_2 \oplus g_3 M_3 \oplus h_1 N_1 \oplus h_2 N_2.$$

for some nonnegative integers g_1, g_2, g_3, h_1, h_2.

- Now we consider a 2-$(v, \ell, 1)$-design with odd ℓ and r. As we saw, the adjacency algebras are isomorphic to the algebra defined by a 2-$(15, 3, 1)$-design.
- The multiplicities are $m_A = 1$, $m_B = v - 1$, $m_C = b - 1$.
- We have

$$g_2 + g_3 + h_1 + h_2 = b - 1, \quad g_1 + g_2 + gs_3 + h_2 = v - 1.$$

- Since σ_9 is the transposed matrix of σ_7, their ranks are equal. This means $g_2 = h_2$.
- Put $s = \mathrm{rank}(\sigma_7)$ and $t = \mathrm{rank}(\sigma_5)$. Then we have

$$(g_1, g_2, g_3, h_1, h_2) = (b - 2s - 1,\ s - t,\ t,\ v - 2s + t - 1,\ s - t).$$

- Remark that the usual 2-rank of the design is $\mathrm{rank}(\sigma_6)$ and $\mathrm{rank}(\sigma_6) = 1 + \mathrm{rank}(\sigma_7) = 1 + s$.
- The structures of standard modules "can" contain more information than p-ranks.

Example 5.6

For 80 nonisomorphic 2-$(15, 3, 1)$-designs, we have the following parameters (by computation) :

$\sharp$	$s = \text{rank}(\sigma_7)$	$t = \text{rank}(\sigma_5)$	g_1	g_2	g_3	h_1	h_2	$\text{rank}(\sigma_6)$
1	10	6	14	4	6	0	4	11
1	11	8	12	3	8	0	3	12
5	12	10	10	2	10	0	2	13
15	13	12	8	1	12	0	1	14
58	14	14	6	0	14	0	0	15

Question 5.7

Is one parameter enough ?

Example 5.8

In [The CRC Handbook of Combinatorial Designs], we can find a list of designs with odd ℓ and r :

No.	v	b	r	ℓ	$\sharp$
14	15	35	7	3	80
29	19	57	9	3	$\geq 1.1 \times 10^9$
57	45	99	11	5	≥ 16
86	27	117	13	3	$\geq 10^{11}$
114	31	155	15	3	$\geq 6 \times 10^{16}$
120	61	183	15	5	≥ 10
129	91	195	15	7	≥ 2

- I want to compute 2-ranks of them. But I do not have data.

Characteristic 3, 5, and 7 for 2-$(15,3,1)$-designers

Theorem 5.9

Let F be a field of characteristic 3. Under the above notations, the Loewy series of the projective covers of simple FS-modules of coherent configurations obtained by 2-$(15,3,1)$-designs are

$$P(A) = \begin{bmatrix} & A & \\ B & & C \\ & A & \end{bmatrix}, \quad P(B) = \begin{bmatrix} B \\ A \end{bmatrix}, \quad P(C) = \begin{bmatrix} C \\ A \\ C \end{bmatrix}$$

and the structures of standard FS-modules are

$$FX \cong \begin{bmatrix} & A & \\ B & & C \\ & A & \end{bmatrix} \oplus 13 \begin{bmatrix} C \\ A \\ C \end{bmatrix} \oplus 7\,[C]$$

for all 80 designs. (By computation)

Characteristic 3, 5, and 7 for 2-$(15, 3, 1)$-designers

Theorem 5.10

Let F be a field of characteristic 5. There are two simple FS-modules A and B with $\dim_F A = 2$ and $\dim_F B = 1$. The Loewy series of the projective covers of simple FS-modules of coherent configurations obtained by 2-$(15, 3, 1)$-designs are

$$P(A) = \begin{bmatrix} A \\ A \end{bmatrix}, \quad P(B) = [B]$$

and the structure of the standard FS-module is

$$FX \cong \begin{bmatrix} A \\ A \end{bmatrix} \oplus 13\,[A] \oplus 20\,[B]\,.$$

Characteristic 3, 5, and 7 for 2-$(15,3,1)$-designers

Theorem 5.11

Let F be a field of characteristic 7. There are three simple FS-modules A, B, and C with $\dim_F A = \dim_F B = 1$ and $\dim_F C = 2$. The Loewy series of the projective covers of simple FS-modules of coherent configurations obtained by 2-$(15,3,1)$-designs are

$$P(A) = \begin{bmatrix} A \\ B \end{bmatrix}, \quad P(B) = \begin{bmatrix} B \\ A \\ B \end{bmatrix}, \quad P(C) = [C]$$

and the structure of the standard FS-module is

$$FX \cong \begin{bmatrix} B \\ A \\ B \end{bmatrix} \oplus 19\,[B] \oplus 14\,[C]\,.$$

Thank you very much !