Distance-regular graph

距離正則グラフ入門

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• Graph

V : a set **vertex set**

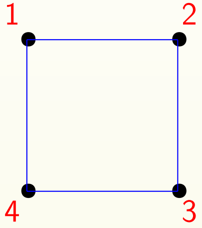
E : a collection of 2-elements subsets of V **edge set**

$\Gamma = (V, E)$ **(simple) graph**

Ex. 1

$$V = \{1, 2, 3, 4\}$$

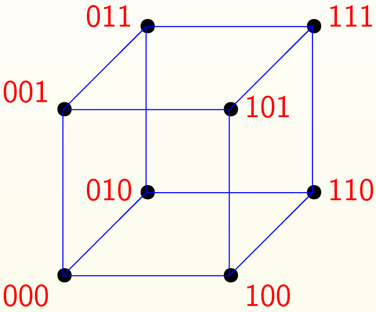
$$E = \Big\{ \{1,2\}, \{2,3\}, \{3,4\}, \{1,4\} \Big\}$$



• Graph

Ex. 2 $\quad V = (\mathbb{Z}_2)^3$

$$E = \Big\{ \{a, b\} \mid \#\{i \mid a_i \neq b_i\} = 1 \Big\}$$

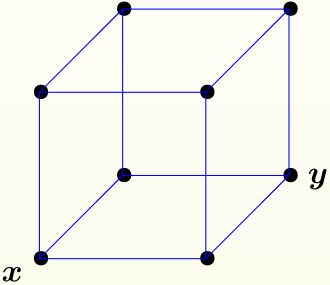


• Graph

$\Gamma = (V, E)$: graph $\qquad x, y \in V$

path of length t from x to y : $(x = x_0, x_1, \ldots, x_t = y)$

$\{x_{i-1}, x_i\} \in E \;\; (i = 1, \ldots, t)$

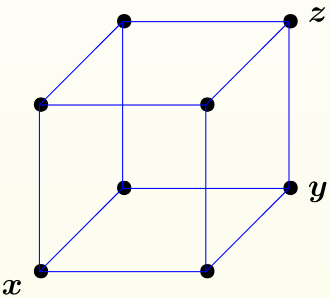


• Graph

$\partial_\Gamma(x,y)$: distance : length of shortest path from x to y

$d := \max\{\partial_\Gamma(x,y) \mid x,y \in V\}$: diameter of Γ

Ex.



$$\partial_\Gamma(x,y) = 2$$

$$\partial_\Gamma(y,z) = 1$$

$$\partial_\Gamma(x,z) = 3$$

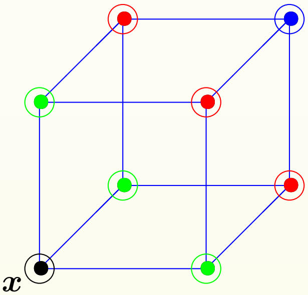
$$d = 3$$

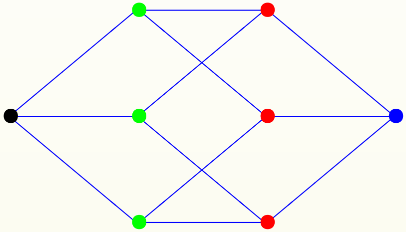
• Graph

$$\Gamma_j(x) \ = \ \{\, v \in V \mid \partial_\Gamma(x,v) = j \,\}$$

Ex.

$\Gamma_0(x)$ $\Gamma_1(x)$ $\Gamma_2(x)$ $\Gamma_3(x)$





• D.R.G

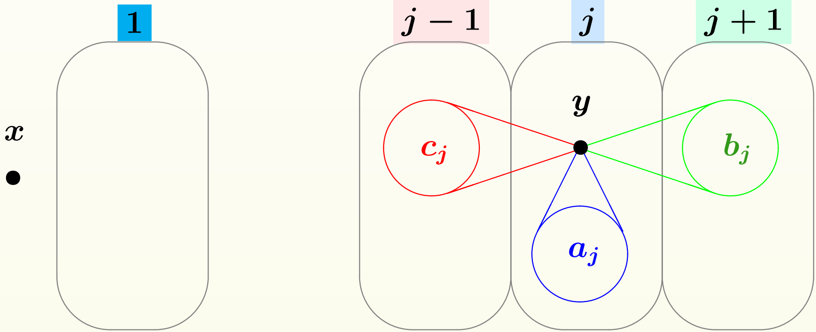
Γ : distance-regular $\Leftrightarrow$

$$c_j := |\Gamma_{j-1}(x) \cap \Gamma_1(y)|$$

$$a_j := |\Gamma_j(x) \cap \Gamma_1(y)|$$

$$b_j := |\Gamma_{j+1}(x) \cap \Gamma_1(y)|$$

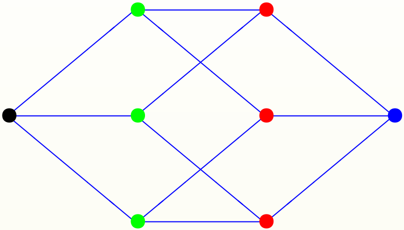
are constants whenever $\partial_\Gamma(x, y) = j$



• D.R.G

Ex. Cube

$\Gamma_0(x)$ $\Gamma_1(x)$ $\Gamma_2(x)$ $\Gamma_3(x)$

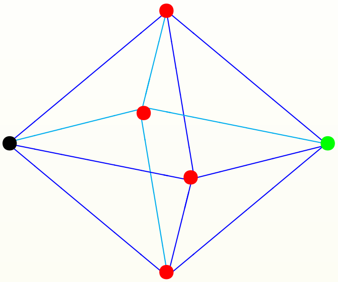


$$\begin{bmatrix} c_j \\ a_j \\ b_j \end{bmatrix} = \begin{bmatrix} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{bmatrix}$$

• D.R.G

Ex. Octahedron

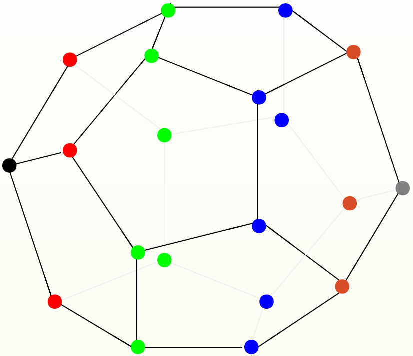
$\Gamma_0(x)$ $\Gamma_1(x)$ $\Gamma_2(x)$

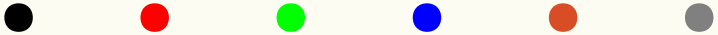


$$\begin{bmatrix} c_j \\ a_j \\ b_j \end{bmatrix} = \begin{bmatrix} * & 1 & 4 \\ 0 & 2 & 0 \\ 4 & 1 & * \end{bmatrix}$$

• D.R.G

Ex. Dodecahedron





$$\begin{bmatrix} c_j \\ a_j \\ b_j \end{bmatrix} = \begin{bmatrix} * & 1 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 1 & * \end{bmatrix}$$

• Association Scheme

X : a set

$R_0 \cup R_1 \cup \cdots \cup R_d$: a partition of $X \times X$

$R_0 = \{(x,x) \mid x \in X\}$

$(x,y) \in R_i \Leftrightarrow (y,x) \in R_i$

$p^h_{i,j} := |\{z \in X \mid (x,z) \in R_i, (z,y) \in R_j\}|$

are constants whenever $(x,y) \in R_h$ $(\forall i,j,h)$

$\left(X, \{R_i\}_{0 \le i \le d}\right)$: (Symmetric) Association Scheme

• Association Scheme

$X = V$: vertex set of D.R.G. $\Gamma = (V, E)$ of diameter d

$R_i = \{(x, y) \mid \partial_\Gamma(x, y) = i\}$

$R_0 = \{(x, x) \mid x \in X\}$

$(x, y) \in R_i \Leftrightarrow (y, x) \in R_i$

$p^h_{i,j} := |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|$

are constants whenever $(x, y) \in R_h$ $\qquad (\forall i, j, h)$

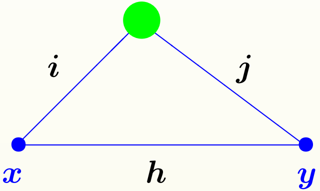
$(X, \{R_i\}_{0 \le i \le d})$: (Symmetric) Association Scheme

• Association Scheme

$X = V$: vertex set of D.R.G. $\Gamma = (V, E)$ of diameter d

$R_i = \{(x, y) \mid \partial_\Gamma(x, y) = i\}$

$p^h_{i,j}$



$\left(X, \{R_i\}_{0 \le i \le d}\right)$: (Symmetric) Association Scheme

• Association Scheme

$\Gamma = (X, E)$: D.R.G. of diameter d

$\Downarrow$

$(X, \{R_i\}_{0 \leq i \leq d})$: (Symmetric) Association Scheme

• Association Scheme

$\Gamma = (X, E)$: D.R.G. of diameter d

$\Downarrow$ $\Uparrow$

$\left(X, \{R_i\}_{0 \leq i \leq d}\right)$: "P-polynomial" Association Scheme

• Distance-Transitive

$\Gamma = (V, E)$: graph

σ : automorphism of Γ $\Leftrightarrow$ σ : bijection on V such that

$(\sigma(x), \sigma(y)) \in E$ if $(x, y) \in E$

$Aut(\Gamma)$: automorphism group of Γ

Γ : distance-transitive $\Leftrightarrow$ $\forall x, y, z, w \in V$ with

$\partial_\Gamma(x, y) = \partial_\Gamma(z, w)$

$\exists \sigma \in Aut(\Gamma)$ such that

$\sigma(x) = z, \quad \sigma(y) = w$

• D.T $\Rightarrow$ D.R

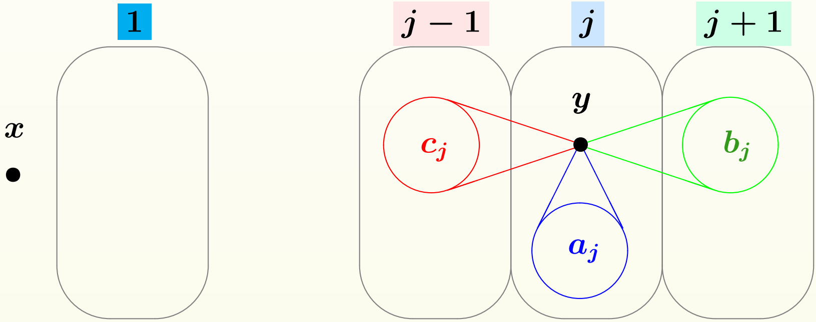
Γ : distance-regular $\Leftrightarrow$

$$c_j := |\Gamma_{j-1}(x) \cap \Gamma_1(y)|$$

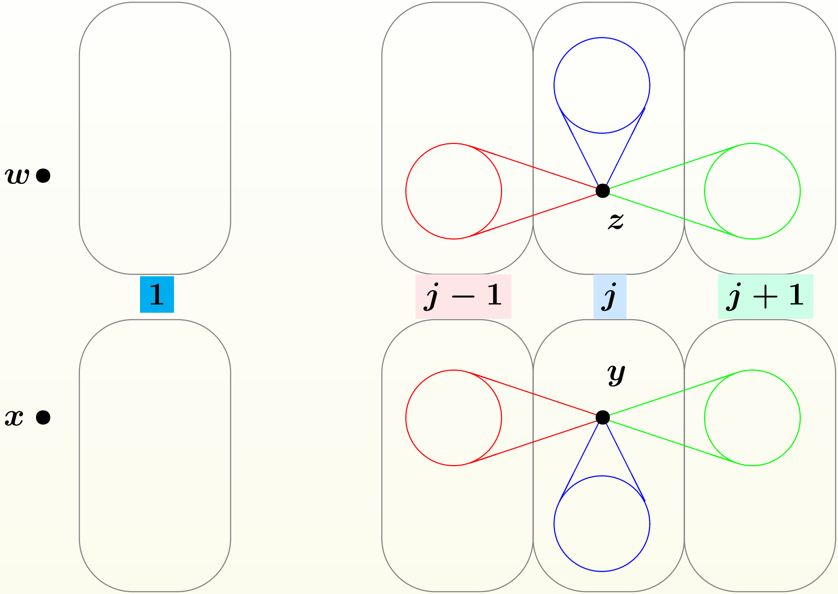
$$a_j := |\Gamma_j(x) \cap \Gamma_1(y)|$$

$$b_j := |\Gamma_{j+1}(x) \cap \Gamma_1(y)|$$

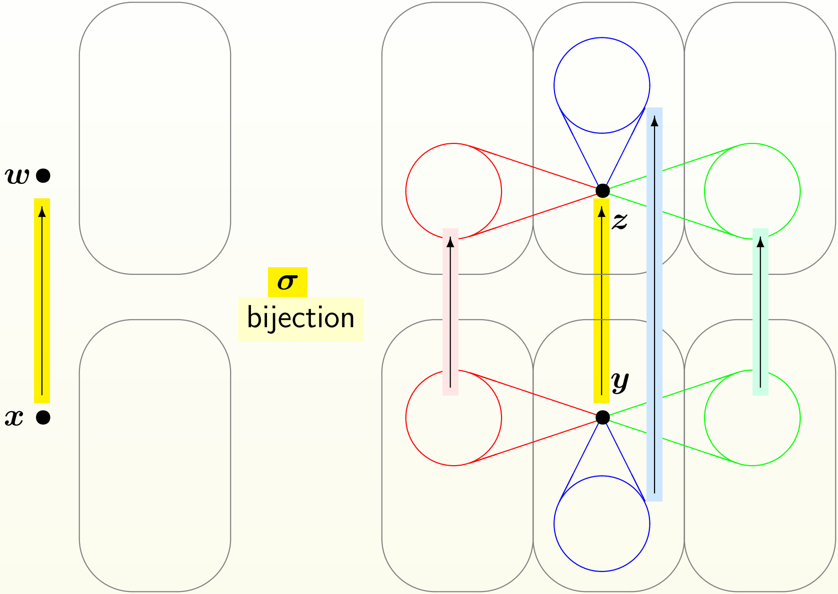
are constants whenever $\partial_\Gamma(x,y) = j$



• D.T ⇒ D.R



• D.T ⇒ D.R



• Hamming graph

$$V = (\mathbb{Z}_n)^d$$

Hamming graph $H(d, n)$

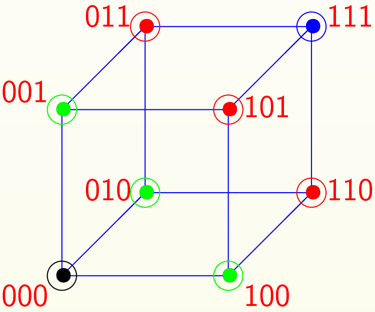
$$E = \Big\{ \{a, b\} \mid \#\{i \mid a_i \neq b_i\} = 1 \Big\}$$

• Hamming graph

$$V = (\mathbb{Z}_2)^3$$

Hamming graph $H(3,2)$

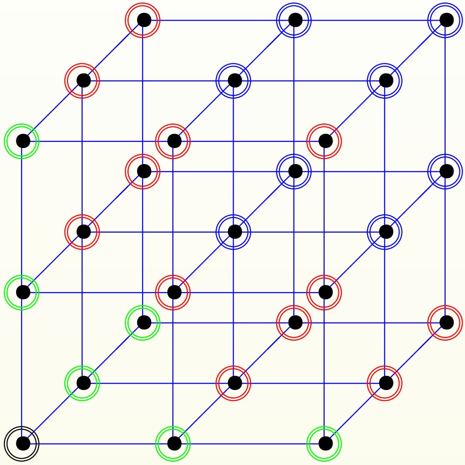
$$E = \Big\{ \{a,b\} \mid \#\{i \mid a_i \neq b_i\} = 1 \Big\}$$



$$\begin{bmatrix} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{bmatrix}$$

• Hamming graph

Hamming graph $\boldsymbol{H(3,3)}$



$$\begin{bmatrix} * & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 6 & 4 & 2 & * \end{bmatrix}$$

• Hamming graph

$$V = (\mathbb{Z}_n)^d$$

Hamming graph $H(d, n)$

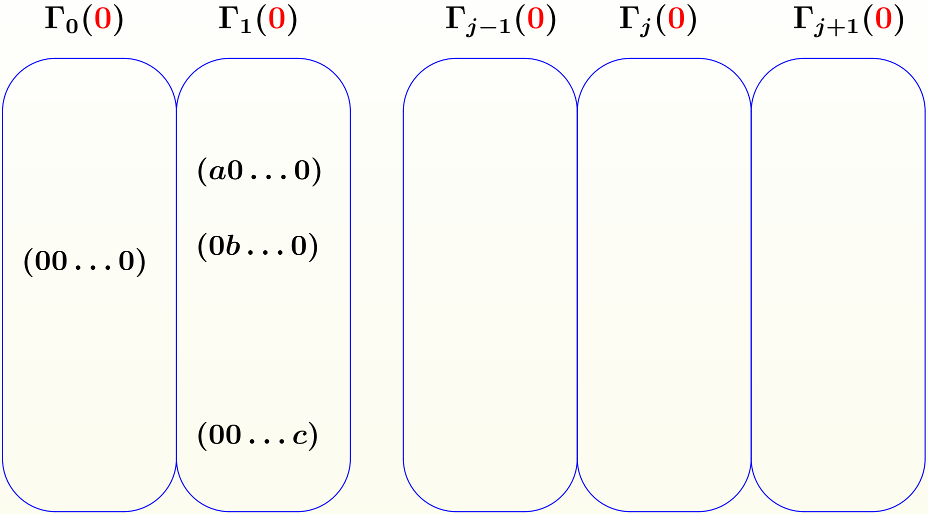
$$E = \Big\{ \{a, b\} \mid \#\{i \mid a_i \neq b_i\} = 1 \Big\}$$

$$\partial_\Gamma(x, y) = \#\{i \mid a_i \neq b_i\}$$

Hamming distance

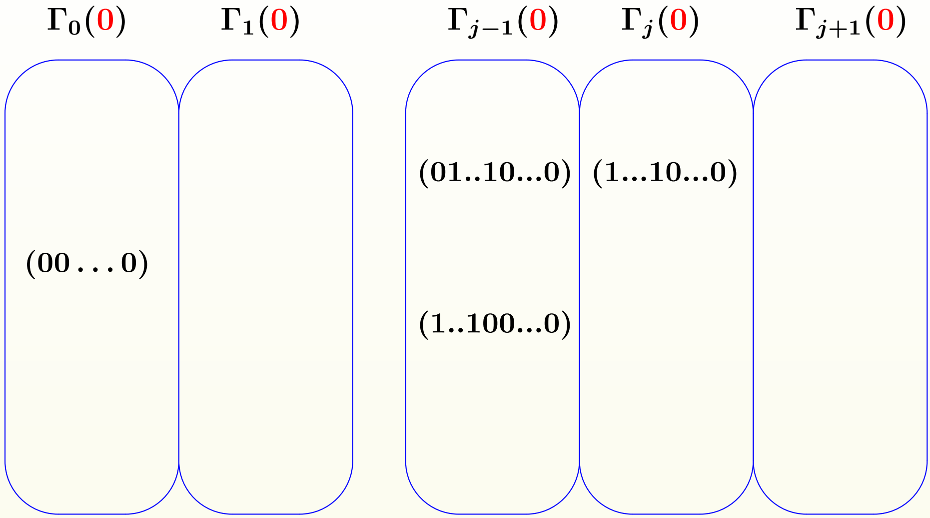
d : diameter

• Hamming graph



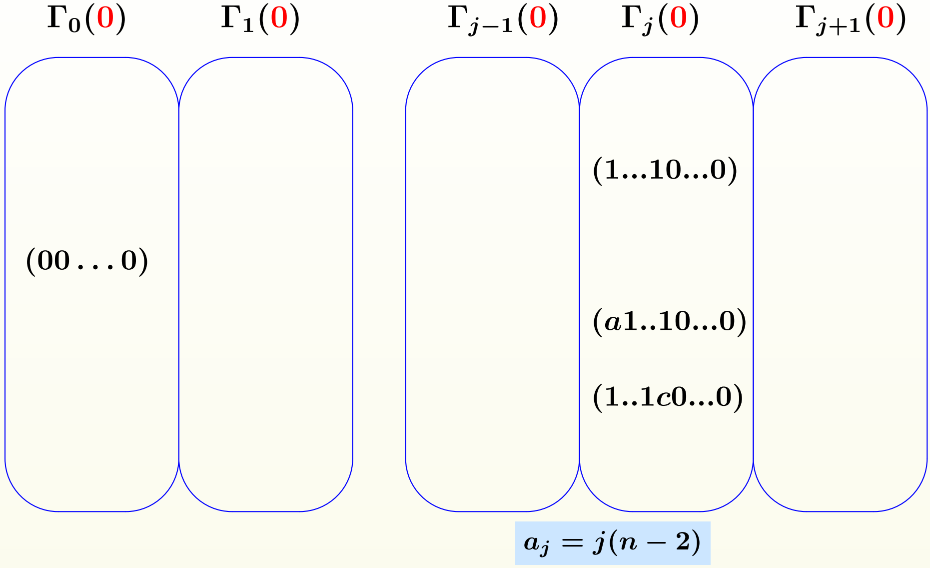
$b_0 = d(n-1)$

• Hamming graph

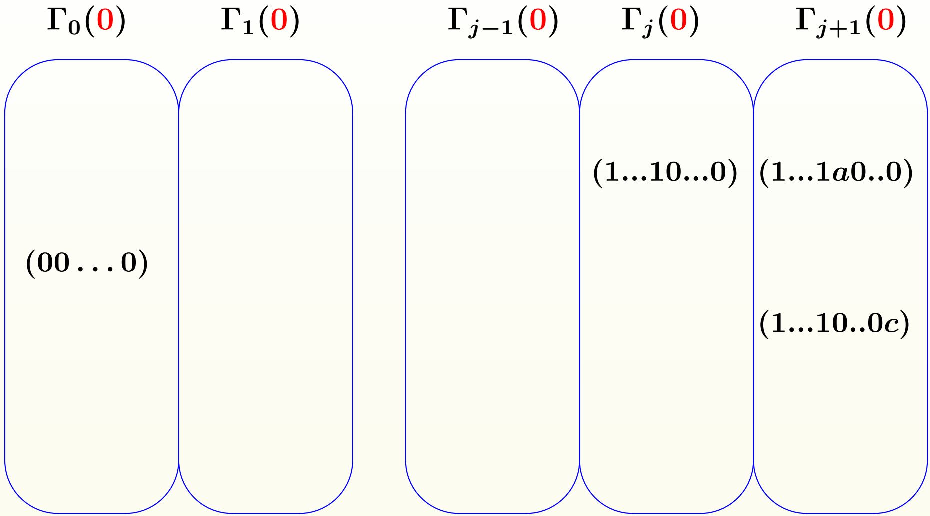


$c_j = j$

• Hamming graph

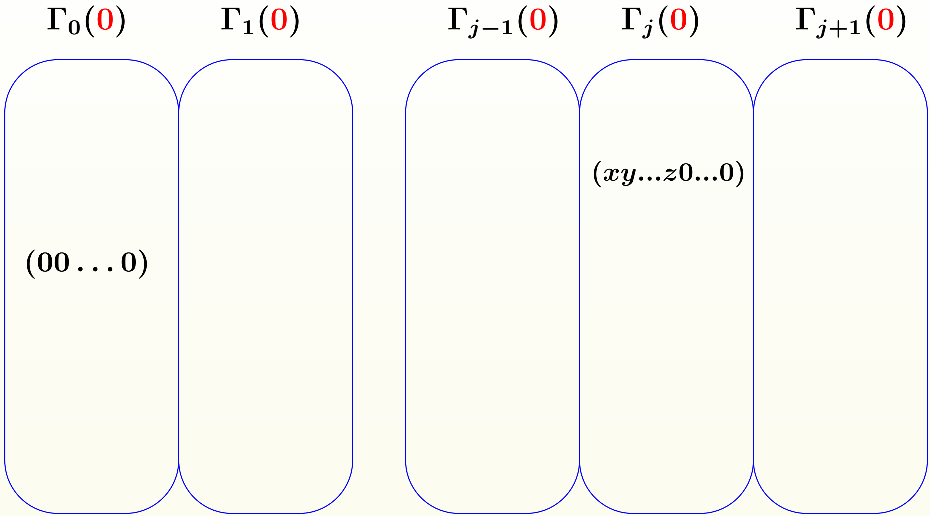


• Hamming graph



$b_j = (d-j)(n-1)$

• Hamming graph



$c_j = j$ $a_j = j(n-2)$ $b_j = (d-j)(n-1)$

• Hamming graph

$$V = (\mathbb{Z}_n)^d$$

Hamming graph $H(d, n)$

$$E = \Big\{ \{a, b\} \mid \#\{i \mid a_i \neq b_i\} = 1 \Big\}$$

$$\partial_\Gamma(x, y) = \#\{i \mid a_i \neq b_i\}$$

Hamming distance

d : diameter

• Hamming graph

$$V = (\mathbb{Z}_n)^d$$

Hamming graph $H(d, n)$

$$\mathtt{Wt}(x) = \#\{i \mid x_i \neq 0\}$$

Hamming weight

$$\partial_\Gamma(x, y) = \#\{i \mid x_i \neq y_i\}$$

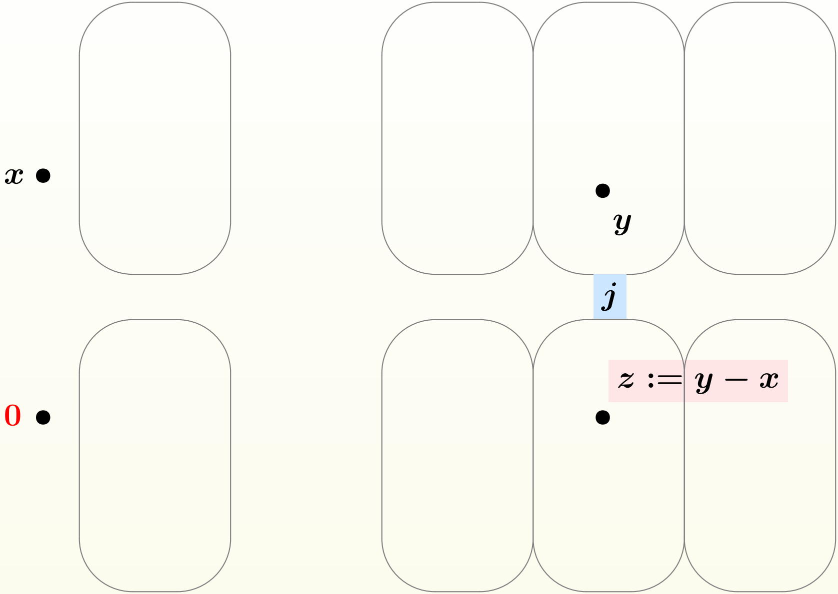
Hamming distance

$$= \#\{i \mid x_i - y_i \neq 0\} = \mathtt{Wt}(x - y)$$

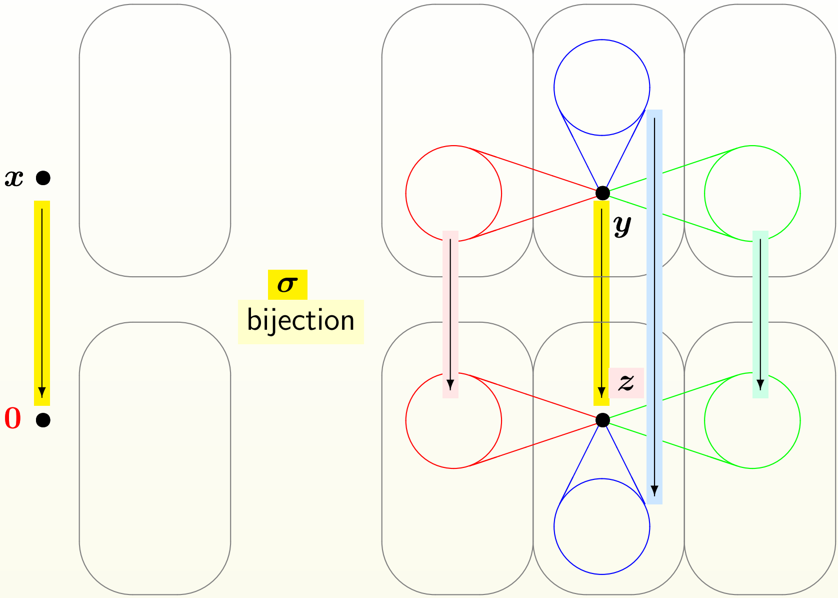
$$\partial_\Gamma(x - z, y - z) = \mathtt{Wt}\Big((x - z) - (y - z)\Big)$$

$$= \mathtt{Wt}\Big(x - y\Big) = \partial_\Gamma(x, y)$$

• D.T $\Rightarrow$ D.R



• D.T ⇒ D.R



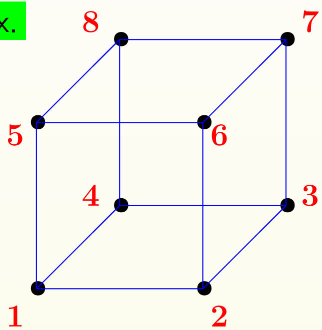
• Eigenvalues

$\Gamma = (V, E)$: graph

A : $|V| \times |V|$-matrix with $A_{xy} = \begin{cases} 1 & \text{if } (x,y) \in E \\ 0 & \text{if } (x,y) \notin E \end{cases}$

adjacency matrix

Ex.



$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

• Eigenvalues

$\Gamma = (V, E)$: graph

A : $|V| \times |V|$-matrix with $A_{xy} = \begin{cases} 1 & \text{if} \quad (x,y) \in E \\ 0 & \text{if} \quad (x,y) \notin E \end{cases}$

adjacency matrix

Eigenvalues of Γ is eigenvalues of its adjacency matrix A

• Eigenvalues

Ex. Γ : $H(d,n)$ Hamming graph

$$V = (\mathbb{Z}_n)^d \qquad |V| = n^d$$

$$|H(3,2)| = 2^3 = 8$$

$$|H(3,3)| = 3^3 = 27$$

$$|H(5,5)| = 5^5 = 3125$$

Eigenvalues of Γ is eigenvalues of its adjacency matrix A

• Eigenvalues

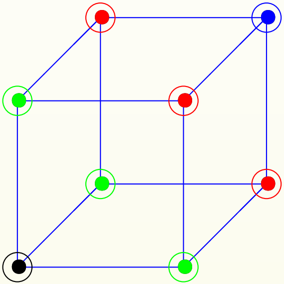
Γ : distance-regular graph of diameter d with c_i, a_i, b_i

$$B = \begin{bmatrix} a_0 & b_0 & & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ \mathbf{0} & & & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & & c_d & a_d \end{bmatrix}$$

intersection matrix

• Eigenvalues

Hamming graph $H(3, 2)$

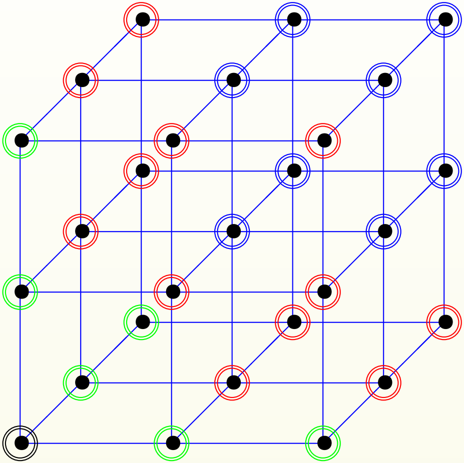


$$B = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{bmatrix}$$

• Eigenvalues

Hamming graph $\boldsymbol{H}(3,3)$



$$B = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 6 & 4 & 2 & * \end{bmatrix}$$

• Eigenvalues

Eigenvalues

$$B = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$3,\ 1,\ -1,\ -3$$

$$B = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$$6,\ 3,\ 0,\ -3$$

• Subgraph

$\Gamma = (V, E)$ graph

$\emptyset \neq V' \subset V$

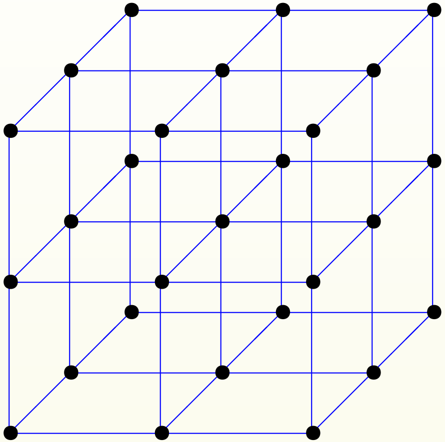
$E' := \{(x, y) \in E \mid x, y \in V'\}$

$\Gamma' = (V', E')$ induced subgraph on V'

• Subgraph

$\Gamma = (V, E)$

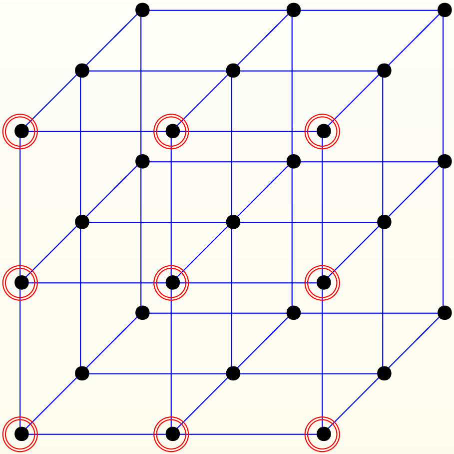
Hamming graph $H(3,3)$



• Subgraph

$\Gamma = (V, E)$

Hamming graph $H(3,3)$



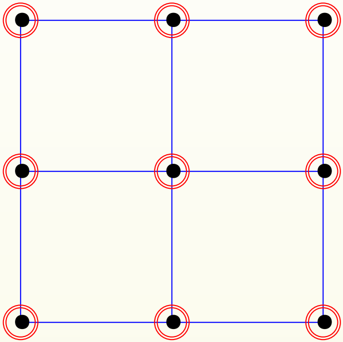
• Subgraph

$\Gamma = (V, E)$

Hamming graph $H(3,3)$

$\Gamma' = (V', E')$

Hamming graph $H(2,3)$

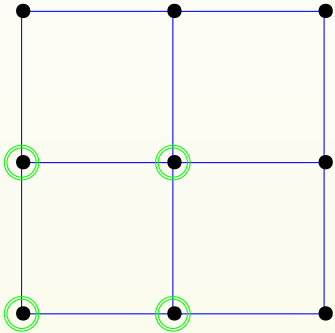


• Subgraph

$\Gamma = (V, E)$ Hamming graph $H(3,3)$

$\Gamma' = (V', E')$ Hamming graph $H(2,3)$

$\Gamma'' = (V'', E'')$ Hamming graph $H(2,2)$



• Subgraph

$$V = (\mathbb{Z}_n)^d$$

Hamming graph $H(d,n)$

$$E = \Big\{ \{a,b\} \mid \#\{i \mid a_i \neq b_i\} = 1 \Big\}$$

$\forall t \leq d$

$\forall m \leq n$

$$M = \{0, 1, \ldots, m-1\} \subset \mathbb{Z}_n$$

$$V' = \{x \in (\mathbb{Z}_n)^d \mid x_1, \ldots, x_t \in M, x_{t+1} = \cdots = x_d = 0\}$$

$$x = (x_1, \ldots, x_t, 0, \ldots, 0)$$
$$y = (y_1, \ldots, y_t, 0, \ldots, 0)$$

$$V^* = (M)^t$$

Hamming graph $H(t,m)$

$$E^* = \Big\{ \{x,y\} \mid \#\{i \mid x_i \neq y_i\} = 1 \Big\}$$

• Subgraph

$\Gamma = (V, E)$ **graph** $\emptyset \neq W \subset V$

$$\partial_\Gamma(W, x) = \partial_\Gamma(x, W) := \min\left\{\partial_\Gamma(x, w) \mid w \in W\right\}$$

$\rho(W) = \max\left\{\partial_\Gamma(x, W) \mid x \in V\right\}$ **covering radius of** W

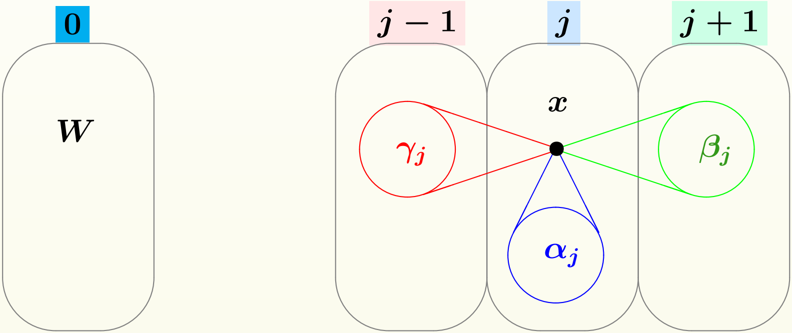
$$\Gamma_j(W) = \left\{x \in V \mid \partial_\Gamma(W, x) = j\right\}$$

• Completely Regular

W : completely regular $\Leftrightarrow$

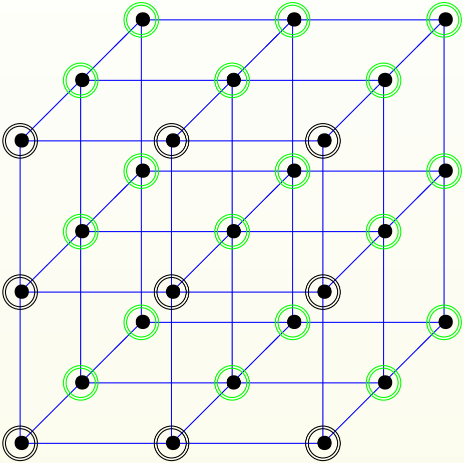
$$\gamma_j := |\Gamma_{j-1}(W) \cap \Gamma_1(x)|$$
$$\alpha_j := |\Gamma_j(W) \cap \Gamma_1(x)|$$
$$\beta_j := |\Gamma_{j+1}(W) \cap \Gamma_1(x)|$$

are constants whenever $\partial_\Gamma(W, x) = j$



• Completely Regular

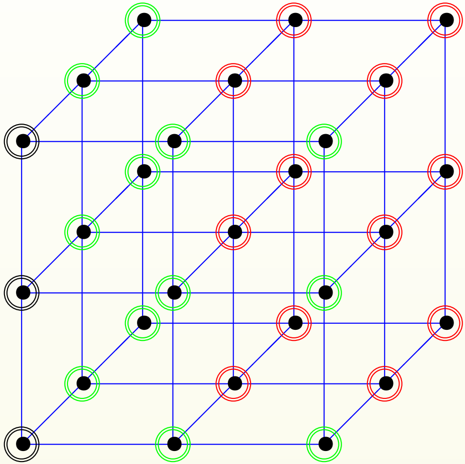
Hamming graph $H(3,3)$



$$\begin{bmatrix} * & 1 \\ 4 & 5 \\ 2 & * \end{bmatrix}$$

• Completely Regular

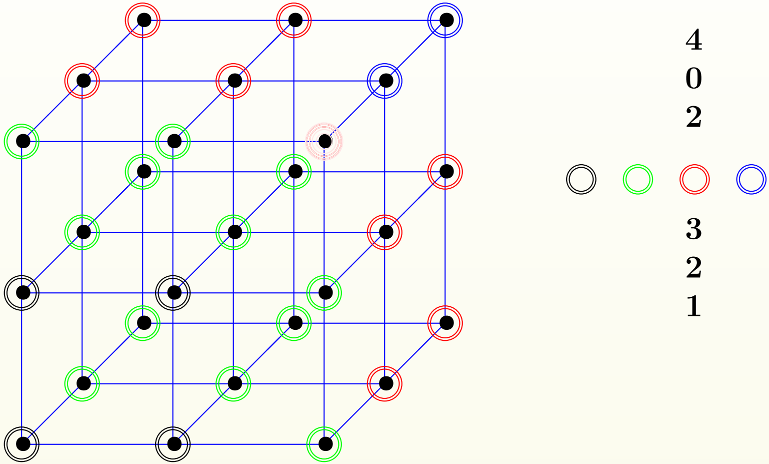
Hamming graph $H(3,3)$



$$\begin{bmatrix} * & 1 & 2 \\ 2 & 3 & 4 \\ 4 & 2 & * \end{bmatrix}$$

• Not Completely Regular

Hamming graph $H(3,3)$



• Completely Regular

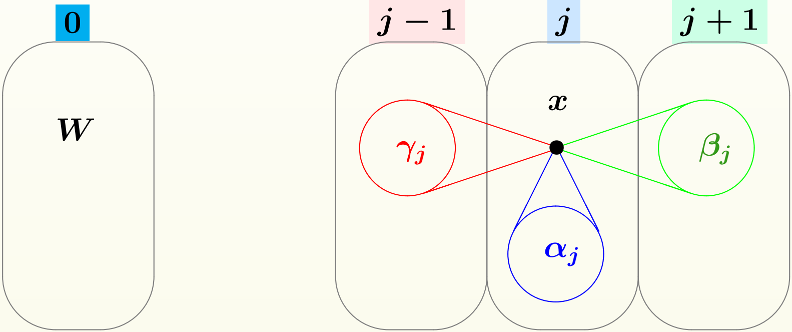
W : completely regular $\Leftrightarrow$

$$\gamma_j := |\Gamma_{j-1}(W) \cap \Gamma_1(x)|$$

$$\alpha_j := |\Gamma_j(W) \cap \Gamma_1(x)|$$

$$\beta_j := |\Gamma_{j+1}(W) \cap \Gamma_1(x)|$$

are constants whenever $\partial_\Gamma(W, x) = j$



• Eigenvalues

W : completely regular of $\rho = \rho(W)$ with $\gamma_j, \alpha_j, \beta_j$ in Γ

$$Q = \begin{bmatrix} \alpha_0 & \beta_0 & & & & \mathbf{0} \\ \gamma_1 & \alpha_1 & \beta_1 & & & \\ & \gamma_2 & \alpha_2 & \beta_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \gamma_{\rho-1} & \alpha_{\rho-1} & \beta_{\rho-1} \\ \mathbf{0} & & & & \gamma_\rho & \alpha_\rho \end{bmatrix}$$

quotient matrix with respect to W

• Theorem 1

Theorem 1

W : completely regular subset in a D.R.G. Γ ($\rho = \rho(W)$)

Q : quotient matrix with respect to W

Then each eigenvalue of Q is an eigenvalue of Γ

Proof. S : $|V| \times (\rho + 1)$-matrix characteristic matrix of W

with $S_{x,j} = \begin{cases} 1 & \text{if } x \in \Gamma_j(W) \\ 0 & \text{if } x \notin \Gamma_j(W) \end{cases}$

Then $AS = SQ$ θ : eigenvalue of Q

$\Rightarrow$ $0 \neq {}^{\exists}u$ s.t. $Qu = \theta u$

$\Rightarrow$ $0 \neq {}^{\exists}Su$ s.t. $A(Su) = (SQ)u = S(\theta u) = \theta(Su)$

• Theorem 1

$$AS = SQ$$

$$(AS)_{x,j} = \sum_{y \in V} A_{x,y} S_{y,j}$$

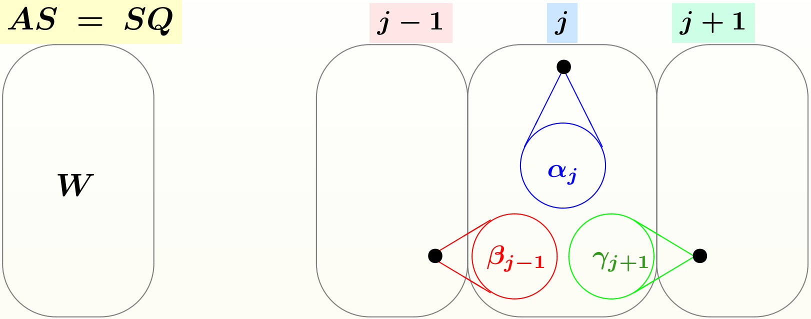
$$A_{x,y} = 1 \Leftrightarrow (x, y) \in E$$

$$S_{y,j} = 1 \Leftrightarrow y \in \Gamma_j(W)$$

$$= |\{y \in V \mid (x, y) \in E,\ y \in \Gamma_j(W)\}|$$

• Theorem 1

$AS = SQ$



$$= |\{y \in V \mid (x,y) \in E,\ y \in \Gamma_j(W)\}|$$

$$(AS)_{x,j} = \begin{cases} \beta_{j-1} & \text{if } x \in \Gamma_{j-1}(W) \\ \alpha_j & \text{if } x \in \Gamma_j(W) \\ \gamma_{j+1} & \text{if } x \in \Gamma_{j+1}(W) \\ 0 & \text{if otherwise} \end{cases}$$

• Theorem 1

$$AS = SQ$$

$$S_{x,t} = 1 \Leftrightarrow x \in \Gamma_t(W)$$

$$(SQ)_{x,j} = \sum_{i=0}^{d} S_{x,i} Q_{i,j}$$

$$= Q_{t,j}$$

$$Q = \begin{bmatrix} \alpha_0 & \beta_0 & & & & \\ \gamma_1 & \alpha_1 & \beta_1 & & & \\ & \gamma_2 & \alpha_2 & \beta_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \gamma_{\rho-1} & \alpha_{\rho-1} & \beta_{\rho-1} \\ & & & & \gamma_\rho & \alpha_\rho \end{bmatrix}$$

$$(AS)_{x,j} = \begin{cases} \beta_{j-1} & \text{if } x \in \Gamma_{j-1}(W) \\ \alpha_j & \text{if } x \in \Gamma_j(W) \\ \gamma_{j+1} & \text{if } x \in \Gamma_{j+1}(W) \\ 0 & \text{if otherwise} \end{cases}$$

• Eigenvalues

W : completely regular of $\rho = \rho(W)$ with $\gamma_j, \alpha_j, \beta_j$ in Γ

$$Q = \begin{bmatrix} \alpha_0 & \beta_0 & & & & & \\ \gamma_1 & \alpha_1 & \beta_1 & & & \mathbf{0} & \\ & \gamma_2 & \alpha_2 & \beta_2 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \gamma_{\rho-1} & \alpha_{\rho-1} & \beta_{\rho-1} \\ & \mathbf{0} & & & \gamma_\rho & \alpha_\rho \end{bmatrix}$$

quotient matrix with respect to W

• Eigenvalues

Γ : distance-regular graph of diameter d with c_i, a_i, b_i

$$B = \begin{bmatrix} a_0 & b_0 & & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ \mathbf{0} & & & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & & c_d & a_d \end{bmatrix}$$

intersection matrix of Γ

• Eigenvalues

$W = \{x\}$: C. R. of $\rho = d$ with c_i, a_i, b_i in Γ

$$B = \begin{bmatrix} a_0 & b_0 & & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & & \\ & c_2 & a_2 & b_2 & & \\ & & \ddots & \ddots & \ddots & \\ \mathbf{0} & & & c_{d-1} & a_{d-1} & b_{d-1} \\ & & & & c_d & a_d \end{bmatrix}$$

intersection matrix of Γ

• Facts

- T : tri-diagonal matrix with $c_i > 0,\ b_i > 0\ \ (\forall i)$

  Then $d+1$ eigenvalues of T are distinct.

$$
T = \begin{bmatrix}
a_0 & b_0 & & & & \mathbf{0} \\
c_1 & a_1 & b_1 & & & \\
 & c_2 & a_2 & b_2 & & \\
 & & \ddots & \ddots & \ddots & \\
 & \mathbf{0} & & c_{d-1} & a_{d-1} & b_{d-1} \\
 & & & & c_d & a_d
\end{bmatrix}
$$

• Facts

- T : tri-diagonal matrix with $c_i > 0,\ b_i > 0\ \ (\forall i)$

  Then $d+1$ eigenvalues of T are distinct.

- Γ : D.R.G. of diameter d

  Then Γ has exactly $d+1$ eigenvalues

• Theorem 1

Theorem 1

W : completely regular subset in a D.R.G. Γ

Q : quotient matrix with respect to W

Then each eigenvalue of Q is an eigenvalue of Γ

Corollary 2

B : intersection matrix of D.R.G. Γ of diameter d

$\theta_0, \theta_1, \ldots, \theta_d$: eigenvalues of B

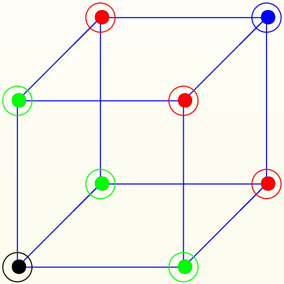
Then $\{\theta_0, \theta_1, \ldots, \theta_d\}$ are the set of all distinct eigenvalues of Γ

In particular, the multiplicity $m(\theta)$ of θ as an eigenvalue of Γ

can be calculated by B

• Eigenvalues of H(3,2)

Hamming graph $H(3,2)$



$$B = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{bmatrix}$$

• Eigenvalues of H(3,2)

$$\begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \theta \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$x_0 \neq 0$

WLOG $x_0 = 1$

$$\begin{cases} 3x_1 & = \theta x_0 \\ 1x_0 + 2x_2 & = \theta x_1 \\ 2x_1 + 1x_3 & = \theta x_2 \\ 3x_2 & = \theta x_3 \end{cases}$$

$x_1 = \frac{1}{3}\theta$

$x_2 = \frac{1}{6}(\theta^2 - 3)$

$x_3 = \frac{1}{6}(\theta^3 - 7\theta)$

$(\theta^2 - 9)(\theta^2 - 1) = 0$

$\theta = 3, 1, -1, -3$

• Eigenvalues

$$\begin{bmatrix} 0 & b_0 & 0 & 0 \\ c_1 & a_1 & b_1 & 0 \\ 0 & c_2 & a_2 & b_2 \\ 0 & 0 & c_3 & a_3 \end{bmatrix} \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix} = \theta \begin{bmatrix} 1 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\begin{cases} b_0 u_1 & = \theta u_0 \\ c_1 u_0 + a_1 u_1 + b_1 u_2 & = \theta u_1 \\ c_2 u_1 + a_2 u_2 + b_2 u_3 & = \theta u_2 \\ c_3 u_2 + a_3 u_3 & = \theta u_3 \end{cases}$$

$$u_{i+1} = \frac{1}{b_i}\Big((\theta - a_i) u_i - c_i u_{i-1} \Big)$$

$$(\theta - a_d) u_d - c_d u_{d-1} = 0$$

• Eigenvalues of H(3,2)

$$(y_0, y_1, y_2, y_3) \begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \theta(y_0, y_1, y_2, y_3)$$

WLOG $y_0 = 1$

$$\begin{cases} y_1 & = \theta y_0 \\ 3y_0 + 2y_2 & = \theta y_1 \\ 2y_1 + 3y_3 & = \theta y_2 \\ y_2 & = \theta y_3 \end{cases}$$

$y_1 = \theta$

$y_2 = \frac{1}{2}(\theta^2 - 3)$

$y_3 = \frac{1}{6}(\theta^3 - 7\theta)$

$(\theta^2 - 9)(\theta^2 - 1) = 0$

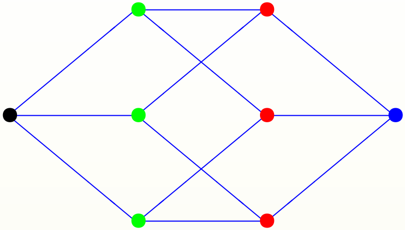
$\theta = 3, 1, -1, -3$

• Eigenvalues of H(3,2)

$$(y_0, y_1, y_2, y_3)\begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix} = \theta(y_0, y_1, y_2, y_3)$$

$x_0 = 1$		$y_0 = 1$
$x_1 = \frac{1}{3}\theta$	$\times 3$	$y_1 = \theta$
$x_2 = \frac{1}{6}(\theta^2 - 3)$	$\times 3$	$y_2 = \frac{1}{2}(\theta^2 - 3)$
$x_3 = \frac{1}{6}(\theta^3 - 7\theta)$	$\times 1$	$y_3 = \frac{1}{6}(\theta^3 - 7\theta)$
$(\theta^2 - 9)(\theta^2 - 1) = 0$		$(\theta^2 - 9)(\theta^2 - 1) = 0$
$\theta = 3, 1, -1, -3$		$\theta = 3, 1, -1, -3$

• Eigenvalues of H(3,2)



$x_1 = \frac{1}{3}\theta$ $\times 3$ $y_1 = \theta$

$x_2 = \frac{1}{6}(\theta^2 - 3)$ $\times 3$ $y_2 = \frac{1}{2}(\theta^2 - 3)$

$x_3 = \frac{1}{6}(\theta^3 - 7\theta)$ $\times 1$ $y_3 = \frac{1}{6}(\theta^3 - 7\theta)$

$(\theta^2 - 9)(\theta^2 - 1) = 0$

$\theta = 3, 1, -1, -3$

$(\theta^2 - 9)(\theta^2 - 1) = 0$

$\theta = 3, 1, -1, -3$

• Eigenvalues

$$(1, v_1, v_2, v_3) \begin{bmatrix} 0 & b_0 & 0 & 0 \\ c_1 & a_1 & b_1 & 0 \\ 0 & c_2 & a_2 & b_2 \\ 0 & 0 & c_3 & a_3 \end{bmatrix} = \theta(1, v_1, v_2, v_3)$$

$$\begin{cases} c_1 v_1 & = \theta \\ b_0 v_0 + a_1 v_1 + c_2 v_2 & = \theta v_1 \\ b_1 v_1 + a_2 v_2 + c_3 v_3 & = \theta v_2 \\ b_2 v_2 + a_3 v_3 & = \theta v_3 \end{cases}$$

$$v_{i+1} = \frac{1}{c_{i+1}}\Big((\theta - a_i)v_i - b_{i-1}v_{i-1}\Big)$$

$$(\theta - a_d)v_d - b_{d-1}v_{d-1} = 0$$

• Eigenvalues

If $\theta = b_0$, then $v_i(\theta) = |\Gamma_i(x)| =: k_i$

$$v_i(\theta) = k_i u_i(\theta) \qquad (\forall\theta, \ \forall i)$$

$$v_{i+1} = \frac{1}{c_{i+1}}\Big((\theta - a_i)v_i - b_{i-1}v_{i-1}\Big)$$

$$(\theta - a_d)v_d - b_{d-1}v_{d-1} = 0$$

• Eigenvalues

If $\theta = b_0$, then $v_i(\theta) = |\Gamma_i(x)| =: k_i$

$$v_i(\theta) = k_i u_i(\theta) \qquad (\forall\theta, \ \forall i)$$

$$\mathbb{V}_\theta := \Big(1, v_1(\theta), v_2(\theta), \ldots, v_d(\theta)\Big), \ \mathbb{U}_\theta = \begin{bmatrix} 1 \\ u_1(\theta) \\ u_2(\theta) \\ \vdots \\ u_d(\theta) \end{bmatrix}$$

$$\langle \mathbb{V}_\theta, \mathbb{U}_\theta \rangle = \sum_{i=0}^{d} v_i(\theta) u_i(\theta)$$

• Multiplicity formula

Γ : D.R.G.

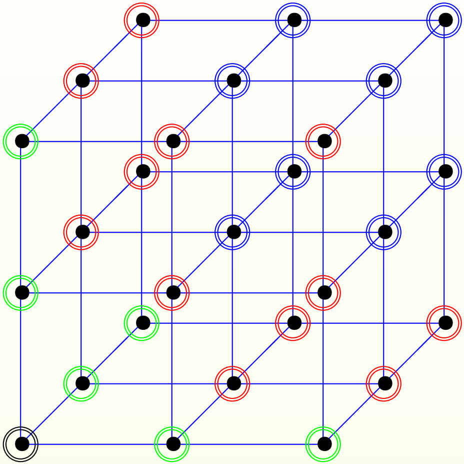
the multiplicity $m(\theta)$ of θ as an eigenvalue of Γ

can be calculated by

$$m(\theta) = \frac{|V|}{\langle \mathbb{V}_{\theta}, \mathbb{U}_{\theta} \rangle}$$

• Eigenvalues of H(3,3)

Hamming graph $\boldsymbol{H}(3,3)$



$$B = \begin{bmatrix} 0 & 6 & 0 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$$\begin{bmatrix} * & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 6 & 4 & 2 & * \end{bmatrix}$$

$$k_i = 1, \quad 6, \quad 12, \quad 8$$

• Eigenvalues of H(3,3)

$k_i = 1, \ 6, \ 12, \ 8$

$$(1, \theta, v_2, v_3) \begin{bmatrix} 0 & 6 & 0 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \end{bmatrix} = \theta(1, \theta, v_2, v_3)$$

$$\begin{cases} 6 + 1\theta + 2v_2 & = \theta^2 \\ 4\theta + 2v_2 + 3v_3 & = \theta v_2 \\ 2v_2 + 3v_3 & = \theta v_3 \end{cases}$$

$$v_2 = \frac{1}{2}(\theta - 3)(\theta + 2)$$

$$(\theta - 6)(\theta^2 - 9)\theta = 0$$

$$v_3 = \frac{1}{3}((\theta - 2)v_2 - 4\theta)$$

$$\theta = 6, \ 3, \ 0, \ -3$$

• Eigenvalues of H(3,3)

$k_i = 1,\ 6,\ 12,\ 8$

$\theta = 6 \quad \Rightarrow \quad (v_i) = (1,\ 6,\ 12,\ 8)$

$\theta = 3 \quad \Rightarrow \quad (v_i) = (1,\ 3,\ 0,\ -4)$

$\theta = 0 \quad \Rightarrow \quad (v_i) = (1,\ 0,\ -3,\ 2)$

$\theta = -3 \quad \Rightarrow \quad (v_i) = (1,\ -3,\ 3,\ -1)$

$$v_2 = \tfrac{1}{2}(\theta - 3)(\theta + 2)$$

$$v_3 = \tfrac{1}{3}((\theta - 2)v_2 - 4\theta)$$

$\theta = 6,\ 3,\ 0,\ -3$

• Eigenvalues of H(3,3)

		$\langle\ ,\ \rangle$	$m(\theta)$
$\theta = 6$	$\Rightarrow\quad (v_i) = (1,\ 6,\ 12,\ 8)$	27	1
$\theta = 3$	$\Rightarrow\quad (v_i) = (1,\ 3,\ 0,\ -4)$	$\frac{9}{2}$	6
$\theta = 0$	$\Rightarrow\quad (v_i) = (1,\ 0,\ -3,\ 2)$	$\frac{9}{4}$	12
$\theta = -3$	$\Rightarrow\quad (v_i) = (1,\ -3,\ 3,\ -1)$	$\frac{27}{8}$	8

$$\begin{bmatrix} 1 & 6 & 12 & 8 \\ 1 & 3 & 0 & -4 \\ 1 & 0 & -3 & 2 \\ 1 & -3 & 3 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & 0 & -\frac{1}{4} & \frac{1}{4} \\ 1 & -\frac{1}{2} & \frac{1}{4} & -\frac{1}{8} \end{bmatrix}$$