# Introduction to Association Schemes

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June 15–16, 2014
Algebraic Combinatorics Summer School, Sendai

## 1 Assumed results

(i) Vandermonde determinant:

$$\begin{vmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_m \\ & \vdots & \\ a_1^{m-1} & \cdots & a_m^{m-1} \end{vmatrix} = \prod_{1\le i<j\le m} (a_j - a_i) \neq 0 \quad \text{if } a_1, \dots, a_m \text{ are distinct.}$$

(ii) If $A_1, \dots, A_n \in \mathrm{Mat}_n(\mathbb{R})$ are pairwise commutative symmetric matrices, then there exists an orthogonal matrix $T \in \mathrm{Mat}_n(\mathbb{R})$ such that $TA_iT^{-1}$ is diagonal for all $i$.

(iii) If $E \in \mathrm{Mat}_n(\mathbb{R})$ is positive semidefinite symmetric matrix, then there exists an $n \times \mathrm{rank}(E)$ matrix $F$ such that $E = FF^\top$.

$J$ stands for a matrix all of whose entries are 1. For a positive integer $m$, denote by $[m]$ the set $\{1, 2, \dots, m\}$. Unless otherwise noted, $X$ will denote a finite set with $|X| = n$.

## 2 Graphs and their adjacency matrices

Let $\Gamma = (X, E)$ be a undirected simple graph. The adjacency matrix $A$ of $\Gamma$ is defined as

$$(A)_{xy} = \begin{cases} 1 & \text{if } \{x, y\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} (A^2)_{xy} &= \sum_{z\in X} (A)_{xz}(A)_{zy} \\ &= \#\text{ path of length 2 from } x \text{ to } y \\ &= \begin{cases} \deg x & \text{if } x = y, \\ \#\text{ common neighbors} & \text{otherwise} \end{cases} \end{aligned}$$

$\partial(x,y)$ = distance between $x$ and $y$. The $i$-th distance matrix is defined as

$$(A_i)_{xy} = \delta_{i,\partial(x,y)}.$$

For the 3-cube,

$$\begin{aligned} A^2 = 3I + 0\cdot A + 2A_2 + 0\cdot A_3 &\implies A^2 \in \langle I, A, A_2\rangle, \\ &\quad\;\; A_2 \in \langle I, A, A^2\rangle,\ A_2 \notin \langle I, A\rangle, \\ AA_2 = 0\cdot I + 2A + 0\cdot A_2 + 3A_3 &\implies A^3 \in \langle I, A, A_2, A_3\rangle, \\ &\quad\;\; A_3 \in \langle I, A, A^2, A^3\rangle,\ A_3 \notin \langle I, A, A^2\rangle, \\ AA_3 = A_2 &\implies A^4 \in \langle I, A, A^2, A^3\rangle \subset \langle I, A, A_2, A_3\rangle. \end{aligned}$$

By induction $A^n \in \langle I, A, A_2, A_3\rangle$ for all $n \in \mathbb{N}$. Thus

$$\mathbb{R}[A] = \langle I, A, A_2, A_3\rangle. \quad (1)$$ 2

where $\mathbb{R}[A]$ is the subalgebra of the full matrix algebra $\mathrm{Mat}_X(\mathbb{R})$ generated by $A$. Observe that $\mathbb{R}[A]$ is an algebra with respect to matrix multiplication, while $\langle I, A, A_2, A_3\rangle$ is in general just a vector subspace. However, by considering the entrywise product, $\langle I, A, A_2, A_3\rangle$ is also an algebra. This implies that (1) is closed under both multiplications.

## 3 Coherent algebras

Let $\mathbb{F}$ be a field. An algebra $\mathcal{A}$ over $\mathbb{F}$ is a commutative ring with 1 which is also an $\mathbb{F}$-vector space, satisfying come compatibility axioms. An example is a subring of $\mathrm{Mat}_n(\mathbb{F})$ which is also closed under scalar multiplications. A simpler example is the vector space $\mathbb{F}^m$ with entrywise multiplication. Its identity as a ring is $(1, \dots, 1) \in \mathbb{F}^m$.

Suppose that $\mathcal{A} \subset \mathbb{F}^m$ is an algebra over $\mathbb{F}$. Using the Vandermonde determinant, we find

$$\mathbb{F}^m \supset \mathcal{A} \ni \exists \boldsymbol{a},\ \forall i \neq \forall j,\ a_i \neq a_j \implies \mathcal{A} = \mathbb{F}^m.$$

The same proof shows

$$\mathbb{F}^m \supset \mathcal{A} \ni a,\ \forall i \in [m],\ \sum_{\substack{j\in[m]\\ a_j = a_i}} \boldsymbol{e}_j \in \mathcal{A}. \quad (2)$$ 1

lem:1 **Lemma 1.** *Let $\mathcal{A} \subset \mathbb{F}^m$ be a subalgebra. Define $i \sim j \iff \forall \boldsymbol{a} \in \mathcal{A},\ a_i = a_j$, and denote by $I_1, \dots I_d$ its equivalence classes. Then*

$$\mathcal{A} = \langle \sum_{i\in I_1} \boldsymbol{e}_i, \dots, \sum_{i\in I_d} \boldsymbol{e}_i\rangle.$$

*Proof.* We may assume $1 \in I_1$, and it suffices to show

$$\sum_{i\in I_1} \boldsymbol{e}_i \in \mathcal{A}.$$

For $\boldsymbol{b} \in \mathcal{A}$, set $I(\boldsymbol{b}) = \{i \in [m] \mid b_i = b_1\}$. Then by (2),

$$\sum_{i\in I(\boldsymbol{b})} \boldsymbol{e}_i \in \mathcal{A}.$$

For any $k \in [m] \setminus I_1$, there exists $\boldsymbol{b}^{(k)} \in \mathcal{A}$ such that $k \notin I(\boldsymbol{b}^{(k)})$. Then

$$\bigcap_{k \in [m] \setminus I_1} I(\boldsymbol{b}^{(k)}) = I_1, \text{ so } \sum_{i \in I_1} \boldsymbol{e}_i = \prod_{k \in [m] \setminus I_1} \sum_{i \in I(\boldsymbol{b}^{(k)})} \boldsymbol{e}_i \in \mathcal{A}.$$

□

**Definition 2.** A *coherent algebra* $\mathcal{A}$ is a subalgebra of $\mathrm{Mat}_n(\mathbb{C})$ containing $J$, closed under the entrywise product $\circ$ and transposition $\top$.

**Definition 3.** A *coherent configuration* is a pair $(X, \{R_i\}_{i=0}^d)$ where $\{R_i\}_{i=0}^d$ is a partition of $X \times X$ such that

(i) $\{(x,x) \mid x \in X\}$ is a union of some $R_i$'s,

(ii) For each $i \in \{0, 1, \ldots, d\}$, ${}^tR_i = R_{i'}$ for some $i' \in \{0, 1, \ldots, d\}$, where ${}^tR_i = \{(x,y) \in X \times X \mid (y,x) \in R_i\}$.

(iii) For $h, i, j \in \{0, 1, \ldots, d\}$, the number of $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ is constant whenever $(x,y) \in R_h$. This constant is denoted by $p_{ij}^h$. These constants are called *intersection numbers*.

The adjacency matrices of the relations of a coherent configuration form a basis of a coherent algebra, and every coherent algebra arises in this way.

$$\begin{aligned} \text{partition of } X \times X &\iff J \in \mathcal{A} \\ \text{(i)} &\iff I \in \mathcal{A} \\ \text{(ii)} &\iff \text{closed under transposition} \\ \text{(iii)} &\iff \text{closed under multiplication,} \quad A_i A_j = \sum_{h=0}^d p_{ij}^h A_h. \end{aligned}$$

**Definition 4.** If a coherent configuration $(X, \{R_i\}_{i=0}^d)$ satisfies the additional property

(iv) $R_0 = \{(x,x) \mid x \in X\}$

then it is called an *association scheme*. If an association scheme satisfies the additional property

(v) $p_{ij}^h = p_{ji}^h$ for all $h, i, j \in \{0, 1, \ldots, d\}$,

then it is called *commutative*. If it satisfies the additional property

(vi) ${}^tR_i = R_i$ for all $i \in \{0, 1, \ldots, d\}$

then it is called *symmetric*. A symmetric association scheme is commutative.

For simplicity, we consider symmetric association schemes in what follows.

## 4 Primitive idempotents

Let $(X, \{R_i\}_{i=0}^d)$ be a symmetric association scheme. The coherent algebra $\mathcal{A}$ spanned by its adjacency matrices is called the Bose–Mesner algebra. Since the adjacency matrices are pairwise commutative symmetric matrices, there exists an orthogonal matrix $T$ such that

$TAT^{-1}$ is a subalgebra of diagonal matrices, that is, $TAT^{-1} \subset \mathbb{R}^n$. By Lemma 1, $TAT^{-1}$ has a basis of the form

$$\begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \end{bmatrix}, \text{ etc.}$$

Call these matrices $D_i$. Then $D_i D_j = \delta_{ij} D_i$. Define $E_i = T^{-1} D_i T \in \mathcal{A}$. Then $E_i E_j = \delta_{ij} E_i$. Since $\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle$ has dimension $d+1$, we have $\dim T\mathcal{A}T^{-1} = d+1$, so there are $d+1$ $E_i$'s. These $E_i$'s are called the *primitive idempotents*. Since $J \in \mathcal{A}$, writing $J = \sum_{i=0}^{d} c_i E_i$, squaring both sides gives $c_i \in \{0, 1\}$. Comparing rank shows $\frac{1}{n} J = E_i$ for some $i$, so we may assume $\frac{1}{n} J = E_0$ without loss of generality.

We have

$$A_i \circ A_j = \delta_{ij} A_i,$$
$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h,$$
$$E_i E_j = \delta_{ij} E_i.$$

To be complete, we need

$$E_i \circ E_j = \frac{1}{n} \sum_{h=0}^{d} q_{ij}^h E_h.$$

The coefficients $q_{ij}^h$ are called *Krein parameters*. It is known that $q_{ij}^h \geq 0$.

The nonsingular matrix $P$ defined by

$$(A_0, A_1, \ldots, A_d) = (E_0, E_1, \ldots, E_d) P$$

is called the *first eigenmatrix*, and $Q = nP^{-1}$ is called the *second eigenmatrix*.

$$A_j = \sum_{i=0}^{d} P_{ij} E_i,$$
$$E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i.$$

**Example 5.**

$$A_0 = I, \quad A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$E_0 = \frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{pmatrix}, \quad E_1 = \frac{1}{2}\begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 1 & -1 & 0\\ 0 & -1 & 1 & 0\\ -1 & 0 & 0 & 1\end{pmatrix}, \quad E_2 = \frac{1}{4}\begin{pmatrix}1 & -1 & -1 & 1\\ -1 & 1 & 1 & -1\\ -1 & 1 & 1 & -1\\ 1 & -1 & -1 & 1\end{pmatrix}.$$

Moreover,

$$\begin{aligned} A_1^2 &= 2A_0 + 2A_2, & A_1A_2 &= A_1, & A_2^2 &= A_0,\\ E_1 \circ E_1 &= \frac{1}{4}(2E_0 + 2E_2), & E_1 \circ E_2 &= \frac{1}{4}E_1, & E_2 \circ E_2 &= \frac{1}{4}E_0. \end{aligned}$$

$$P = Q = \begin{pmatrix}1 & 2 & 1\\ 1 & 0 & -1\\ 1 & -2 & 1\end{pmatrix}.$$

## 5 Orthogonality relations

Write

$$\begin{aligned} k_j &= P_{0j} \quad \text{that is, } A_jJ = k_jJ,\\ m_j &= \operatorname{rank} E_j = \operatorname{tr} E_j = \frac{1}{n}\operatorname{tr}\sum_{i=0}^{d} Q_{ij}A_i = Q_{0j}. \end{aligned}$$

Then

$$\begin{aligned} Q_{hj}k_h &= Q_{hj}\frac{1}{n}\operatorname{tr} A_h^2\\ &= \frac{1}{n}\operatorname{tr}\sum_{i=0}^{d} Q_{ij}A_hA_i\\ &= \operatorname{tr} A_hE_j\\ &= \operatorname{tr}\sum_{i=0}^{d} P_{ih}E_iE_j\\ &= \operatorname{tr} P_{jh}E_j\\ &= P_{jh}m_j. \end{aligned}$$

This implies the orthogonality relations:

$$\begin{aligned} \delta_{ij}n &= (PQ)_{ij} = \sum_{h=0}^{d} P_{ih}Q_{hj} = \frac{1}{m_i}\sum_{h=0}^{d} Q_{hi}Q_{hj}k_h,\\ \delta_{ij}n &= (QP)_{ij} = \sum_{h=0}^{d} Q_{ih}P_{hj} = \frac{1}{k_i}\sum_{h=0}^{d} P_{hi}P_{hj}m_h \end{aligned}$$

## 6 Examples

(i) Complete graphs. $\mathcal{A} = \langle I, J - I \rangle$.

(ii) Polygons. $A = C + C^\top$, where

$$C = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}$$

$$\mathcal{A} = \begin{cases} \langle I, C + C^\top, C^2 + (C^\top)^2, \dots, C^m + (C^\top)^m \rangle & (2m+1)\text{-gon}, \\ \langle I, C + C^\top, C^2 + (C^\top)^2, \dots, C^m \rangle & 2m\text{-gon} \end{cases}$$

(iii) Let $G$ be a finite group of order $n$. Define $(A_g)_{xy} = \delta_{x,gy}$. Then $A_g A_h = A_{gh}$. $\mathcal{A} = \langle A_g \mid g \in G \rangle$ is commutative if and only if $G$ is abelian, $\mathcal{A}$ is symmetric if and only if $g^2 = 1$ for all $g \in G$. If $H$ is a subgroup of $\operatorname{Aut} G$, with orbits $S_0 = \{1\}, S_1, \cdots S_d$, then

$$\mathcal{A}' = \langle \sum_{g \in S_i} A_g \mid 0 \leq i \leq d \rangle$$

is also a coherent algebra, defining an association scheme.

(iv) Let $G$ be a transitive permutation group. Let $\{R_i\}_{i=0}^d$ be the $G$-orbits on $X \times X$. Then $(X, \{R_i\}_{i=0}^d)$ is an association scheme.

(v) Let $F$ be a finite set with $q \geq 2$ elements, and set $X = F^d$. Then $X$ is a metric space with respect to the Hamming distance $d_H(x, y) = |\{i \mid i \in [d],\ x_i \neq y_i\}|$. Then $(X, \{R_i\}_{i=0}^d)$ is a symmetric association scheme called the *Hamming scheme* $H(d, q)$, where

$$R_i = \{(x, y) \in X \times X \mid d_H(x, y) = i\}.$$

(vi) Let $\Omega$ be a finite set with $v$ elements, and let $X$ be the collection of all $d$-element subsets of $\Omega$. Define

$$R_i = \{(x, y) \in X \times X \mid |x \cap y| = d - i\}, \quad (i = 0, 1, \dots, d).$$

Then $(X, \{R_i\}_{i=0}^d)$ is a symmetric association scheme called the *Johnson scheme* $J(v, d)$.

(vii) Let $\Gamma$ be a connected regular graph of diameter 2. If there are integers $\lambda, \mu$ such that any adjacent (resp. non-adjacent) pair of vertices have $\lambda$ (resp. $\mu$) common neighbors, then $\Gamma$ is called a *strongly regular graph*. Let $A$ be the adjacency matrix. Then one obtains an association scheme with Bose–Mesner algebra $\mathcal{A} = \langle I, A, J - I - A \rangle$.

(viii) Let $(\mathcal{P}, \mathcal{B})$ be a *quasi-symmetric* 2-*design*, that is, in addition to being a 2-design, we assume that two distinct blocks intersect with $x$ or $y$ points, where $x$ and $y$ are distinct integers. Then $X = \mathcal{B}$ naturally carries a structure of a strongly regular graph.

(ix) A Steiner triple system is a 2-$(v, 3, 1)$ design. any pair of distinct blocks intersect with 0 or 1 points, so it is a quasi-symmetric design, and hence carries a structure of a strongly regular graph.