Introduction to Association Schemes
Part I

Akihiro Munemasa
Tohoku University

June 15–16, 2014
Algebraic Combinatorics Summer School
Sendai

The adjacency matrix of an undirected simple graph $\Gamma = (X, E)$:

$$(A)_{xy} = \begin{cases} 1 & \text{if } \{x, y\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Then

The adjacency matrix of an undirected simple graph $\Gamma = (X, E)$:

$$(A)_{xy} = \begin{cases} 1 & \text{if } \{x, y\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$(A^2)_{xy} = \sum_{z \in X} (A)_{xz} (A)_{zy}$$

The adjacency matrix of an undirected simple graph $\Gamma = (X, E)$:

$$(A)_{xy} = \begin{cases} 1 & \text{if } \{x, y\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Then

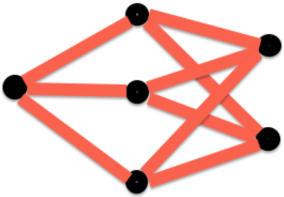
$$\begin{aligned} (A^2)_{xy} &= \sum_{z \in X} (A)_{xz} (A)_{zy} \\ &= \text{\# path of length } 2 \text{ from } x \text{ to } y \end{aligned}$$

The adjacency matrix of an undirected simple graph $\Gamma = (X, E)$:

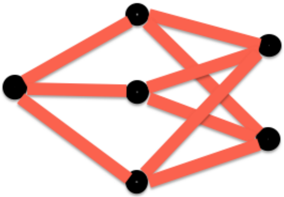
$$(A)_{xy} = \begin{cases} 1 & \text{if } \{x, y\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} (A^2)_{xy} &= \sum_{z \in X} (A)_{xz} (A)_{zy} \\ &= \text{\# path of length } 2 \text{ from } x \text{ to } y \\ &= \begin{cases} \deg x & \text{if } x = y, \\ \text{\# common neighbors} & \text{otherwise.} \end{cases} \end{aligned}$$



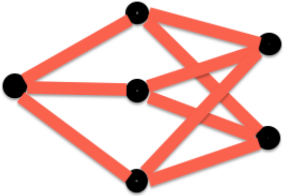
$$(A^2)_{xy} = \begin{cases} \deg x & \text{if } x = y, \\ \text{\# common neighbors} & \text{otherwise} \end{cases}$$



$\nexists$ triangle

$$(A^2)_{xy} = \begin{cases} \deg x & \text{if } x = y, \\ \text{\# common neighbors} & \text{otherwise} \end{cases}$$

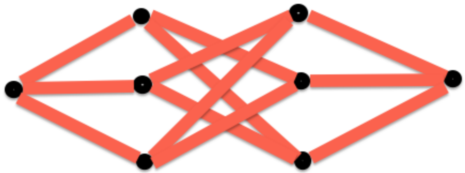
$$A^2 = 3 \cdot I + 0 \cdot A + 3 \cdot (J - I - A).$$

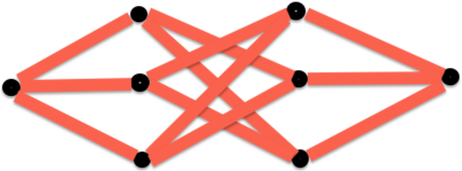


$\forall x, y$ non-adj., $\exists\ 3$ common neighbors

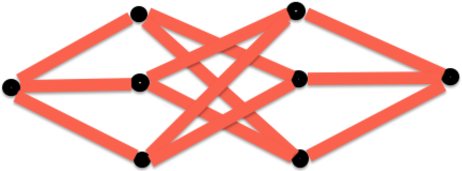
$$(A^2)_{xy} = \begin{cases} \deg x & \text{if } x = y, \\ \text{\# common neighbors} & \text{otherwise} \end{cases}$$

$$A^2 = 3 \cdot I + 0 \cdot A + 3 \cdot (J - I - A).$$





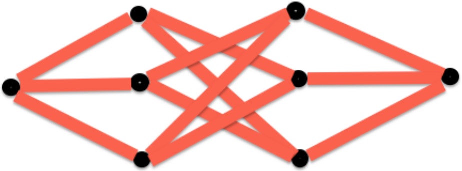
$\nexists$ triangle



$\nexists$ triangle

$$(A^2)_{xy} = \begin{cases} \deg x & \text{if } x = y, \\ \text{\# common neighbors} & \text{otherwise} \end{cases}$$

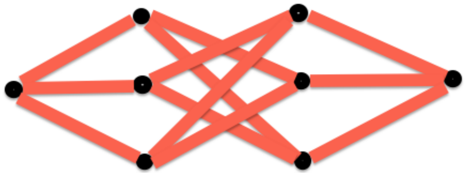
$$A^2 = 3 \cdot I + 0 \cdot A + (2 \text{ or } 0) \cdot (J - A - I)$$



$\nexists$ triangle

$$(A^2)_{xy} = \begin{cases} \deg x & \text{if } x = y, \\ \text{\# common neighbors} & \text{otherwise} \end{cases}$$

$$A^2 = 3 \cdot I + 0 \cdot A + (2 \text{ or } 0) \cdot (J - A - I)$$



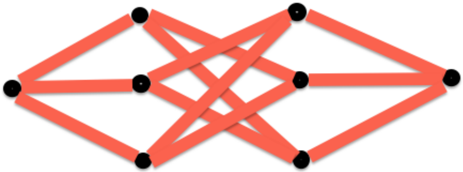
$\nexists$ triangle

$$(A^2)_{xy} = \begin{cases} \deg x & \text{if } x = y, \\ \text{\# common neighbors} & \text{otherwise} \end{cases}$$

$$A^2 = 3 \cdot I + 0 \cdot A + (2 \text{ or } 0) \cdot (J - A - I)$$

Need “distance matrices”. $\partial(x, y) =$ distance between x, y.

$$(A_i)_{xy} = \delta_{i,\partial(x,y)}.$$



$\nexists$ triangle

$$(A^2)_{xy} = \begin{cases} \deg x & \text{if } x = y, \\ \text{\# common neighbors} & \text{otherwise} \end{cases}$$

$$A^2 = 3 \cdot I + 0 \cdot A + (2 \text{ or } 0) \cdot (J - A - I)$$

Need "distance matrices". $\partial(x, y) =$ distance between x, y.

$$(A_i)_{xy} = \delta_{i,\partial(x,y)}.$$

$$A^2 = 3 \cdot I + 0 \cdot A + 2 \cdot A_2 + 0 \cdot A_3.$$

$$A^2 = 3 \cdot I + 0 \cdot A + 2 \cdot A_2 + 0 \cdot A_3 \implies A_2 = \frac{1}{2}(A^2 - 3I).$$

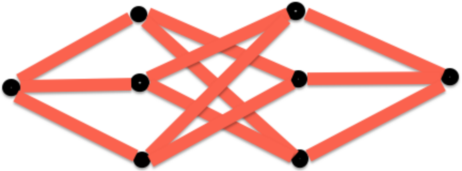
$$A^2 = 3 \cdot I + 0 \cdot A + 2 \cdot A_2 + 0 \cdot A_3 \implies A_2 = \frac{1}{2}(A^2 - 3I).$$

$$(A_2 A)_{xy} = \sum_{z \in X} (A_2)_{xz} (A)_{zy}$$

$$A^2 = 3 \cdot I + 0 \cdot A + 2 \cdot A_2 + 0 \cdot A_3 \implies A_2 = \frac{1}{2}(A^2 - 3I).$$

$$(A_2 A)_{xy} = \sum_{z \in X} (A_2)_{xz} (A)_{zy}$$

$$A_2 A = 0 \cdot I + 2 \cdot A + 0 \cdot A_2 + 3 \cdot A_3.$$



$$A^2 = 3 \cdot I + 0 \cdot A + 2 \cdot A_2 + 0 \cdot A_3 \implies A_2 = \frac{1}{2}(A^2 - 3I).$$

$$(A_2 A)_{xy} = \sum_{z \in X} (A_2)_{xz} (A)_{zy}$$

$$A_2 A = 0 \cdot I + 2 \cdot A + 0 \cdot A_2 + 3 \cdot A_3.$$

$\implies A_3 =$ polynomial of degree 3 in A

$$A^2 = 3 \cdot I + 0 \cdot A + 2 \cdot A_2 + 0 \cdot A_3 \implies A_2 = \frac{1}{2}(A^2 - 3I).$$

$$(A_2 A)_{xy} = \sum_{z \in X} (A_2)_{xz} (A)_{zy}$$

$$A_2 A = 0 \cdot I + 2 \cdot A + 0 \cdot A_2 + 3 \cdot A_3.$$

$\implies A_3 =$ polynomial of degree 3 in A

Similarly, $A_3 A = A_2$, and this ($\deg 4 = \deg 2$) implies A satisfies a polynomial of degree 4.

$$A^2 = 3 \cdot I + 0 \cdot A + 2 \cdot A_2 + 0 \cdot A_3 \implies A_2 = \frac{1}{2}(A^2 - 3I).$$

$$(A_2 A)_{xy} = \sum_{z \in X} (A_2)_{xz} (A)_{zy}$$

$$A_2 A = 0 \cdot I + 2 \cdot A + 0 \cdot A_2 + 3 \cdot A_3.$$

$\implies A_3 =$ polynomial of degree 3 in A

Similarly, $A_3 A = A_2$, and this ($\deg 4 = \deg 2$) implies A satisfies a polynomial of degree 4.

$$\implies A^4 \in \langle I, A, A^2, A^3 \rangle \subset \langle I, A, A_2, A_3 \rangle$$

$$A^2 = 3 \cdot I + 0 \cdot A + 2 \cdot A_2 + 0 \cdot A_3 \implies A_2 = \frac{1}{2}(A^2 - 3I).$$

$$(A_2 A)_{xy} = \sum_{z \in X} (A_2)_{xz} (A)_{zy}$$

$$A_2 A = 0 \cdot I + 2 \cdot A + 0 \cdot A_2 + 3 \cdot A_3.$$

$\implies A_3 =$ polynomial of degree 3 in A

Similarly, $A_3 A = A_2$, and this ($\deg 4 = \deg 2$) implies A satisfies a polynomial of degree 4.

$$\implies A^4 \in \langle I, A, A^2, A^3 \rangle \subset \langle I, A, A_2, A_3 \rangle$$

By induction $A^n \in \langle I, A, A_2, A_3 \rangle$ for all $n \in \mathbb{N}$.

$$\mathbb{R}[A] = \langle I, A, A_2, A_3 \rangle.$$

where $\mathbb{R}[A]$ is the subalgebra of the full matrix algebra $\mathrm{Mat}_X(\mathbb{R})$ generated by A.

$$A^2 = 3 \cdot I + 0 \cdot A + 2 \cdot A_2 + 0 \cdot A_3 \implies A_2 = \frac{1}{2}(A^2 - 3I).$$

$$(A_2A)_{xy} = \sum_{z \in X} (A_2)_{xz}(A)_{zy}$$

$$A_2A = 0 \cdot I + 2 \cdot A + 0 \cdot A_2 + 3 \cdot A_3.$$

$\implies A_3 =$ polynomial of degree 3 in A

Similarly, $A_3A = A_2$, and this ($\deg 4 = \deg 2$) implies A satisfies a polynomial of degree 4.

$$\implies A^4 \in \langle I, A, A^2, A^3 \rangle \subset \langle I, A, A_2, A_3 \rangle$$

By induction $A^n \in \langle I, A, A_2, A_3 \rangle$ for all $n \in \mathbb{N}$.

$$\mathbb{R}[A] = \langle I, A, A_2, A_3 \rangle.$$

where $\mathbb{R}[A]$ is the subalgebra of the full matrix algebra $\mathrm{Mat}_X(\mathbb{R})$ generated by A.

algebra = vector space + ring

algebra = vector space + ring
$\mathbb{F}$: field $\implies$ $\mathbb{F}^m$: vector space $\implies$ $\mathbb{F}^m$: algebra with entrywise multiplication, identity = $(1, \ldots, 1)$.

algebra = vector space + ring

$\mathbb{F}$: field $\implies$ $\mathbb{F}^m$: vector space $\implies$ $\mathbb{F}^m$: algebra with entrywise multiplication, identity = $(1, \ldots, 1)$.

$\mathcal{A} \subset \mathbb{R}^3$: subalgebra

If $(1, 2, 3) \in \mathcal{A} \implies$

algebra = vector space + ring

$\mathbb{F}$: field $\implies$ $\mathbb{F}^m$: vector space $\implies$ $\mathbb{F}^m$: algebra with entrywise multiplication, identity = $(1, \ldots, 1)$.

$\mathcal{A} \subset \mathbb{R}^3$: subalgebra

$$\text{If } (1,2,3) \in \mathcal{A} \implies (1,4,9) \in \mathcal{A}, \text{ also } (1,1,1) \in \mathcal{A}$$
$$\implies$$

algebra = vector space + ring

$\mathbb{F}$: field $\implies$ $\mathbb{F}^m$: vector space $\implies$ $\mathbb{F}^m$: algebra with entrywise multiplication, identity = $(1, \ldots, 1)$.

$\mathcal{A} \subset \mathbb{R}^3$: subalgebra

$$\text{If } (1,2,3) \in \mathcal{A} \implies (1,4,9) \in \mathcal{A}, \text{ also } (1,1,1) \in \mathcal{A} \implies \dim \mathcal{A} = 3 \implies \mathcal{A} = \mathbb{R}^3.$$

algebra = vector space + ring

$\mathbb{F}$: field $\implies$ $\mathbb{F}^m$: vector space $\implies$ $\mathbb{F}^m$: algebra with entrywise multiplication, identity = $(1, \ldots, 1)$.

$\mathcal{A} \subset \mathbb{R}^3$: subalgebra

$$\text{If } (1,2,3) \in \mathcal{A} \implies (1,4,9) \in \mathcal{A}, \text{ also } (1,1,1) \in \mathcal{A} \implies \dim \mathcal{A} = 3 \implies \mathcal{A} = \mathbb{R}^3.$$

Using Vandermonde's determinant,

$$\mathbb{F}^m \supset \mathcal{A} \ni \exists \boldsymbol{a},\ \forall i \neq \forall j,\ a_i \neq a_j \implies \mathcal{A} = \mathbb{F}^m.$$

$$\begin{vmatrix} 1 & \cdots & 1 \\ a_1 & \cdots & a_m \\ & \vdots & \\ a_1^{m-1} & \cdots & a_m^{m-1} \end{vmatrix} = \prod_{1 \leq i < j \leq m} (a_j - a_i) \neq 0.$$

algebra = vector space + ring

$\mathbb{F}$: field $\implies$ $\mathbb{F}^m$: vector space $\implies$ $\mathbb{F}^m$: algebra with entrywise multiplication, identity = $(1, \ldots, 1)$.

$\mathcal{A} \subset \mathbb{R}^3$: subalgebra

$$\text{If } (1,2,3) \in \mathcal{A} \implies (1,4,9) \in \mathcal{A}, \text{ also } (1,1,1) \in \mathcal{A}$$
$$\implies \dim \mathcal{A} = 3 \implies \mathcal{A} = \mathbb{R}^3.$$

Using Vandermonde's determinant,

$$\mathbb{F}^m \supset \mathcal{A} \ni \exists \boldsymbol{a}, \ \forall i \neq \forall j, \ a_i \neq a_j \implies \mathcal{A} = \mathbb{F}^m.$$

More generally

$$\mathbb{F}^m \supset \mathcal{A} \ni \boldsymbol{a}, \ \forall i \in [m], \ \sum_{\substack{j \in [m] \\ a_j = a_i}} \boldsymbol{e}_j \in \mathcal{A}.$$

$$\boldsymbol{a} = (a, \ldots, a, a', \ldots, a', \ldots) \in \mathcal{A}$$
$$\implies (1, \ldots, 1, \ 0, \ldots, \ 0, \ldots) \in \mathcal{A}$$

Lemma

Let $\mathcal{A} \subset \mathbb{F}^m$ be a subalgebra. Define $i \sim j \iff \forall a \in \mathcal{A},\ a_i = a_j$, and denote by $I_1, \ldots I_d$ its equivalence classes. Then

$$\mathcal{A} = \langle \sum_{i \in I_1} \boldsymbol{e}_i, \ldots, \sum_{i \in I_d} \boldsymbol{e}_i \rangle.$$

Lemma

Let $\mathcal{A} \subset \mathbb{F}^m$ be a subalgebra. Define $i \sim j \iff \forall a \in \mathcal{A},\ a_i = a_j$, and denote by $I_1, \dots I_d$ its equivalence classes. Then

$$\mathcal{A} = \langle \sum_{i \in I_1} \boldsymbol{e}_i, \dots, \sum_{i \in I_d} \boldsymbol{e}_i \rangle.$$

Proof. Clearly, "$\subset$" holds, so we only prove "$\supset$". We may assume $1 \in I_1$, and it suffices to show

$$\sum_{i \in I_1} \boldsymbol{e}_i \in \mathcal{A}.$$

Lemma

Let $\mathcal{A} \subset \mathbb{F}^m$ be a subalgebra. Define $i \sim j \iff \forall a \in \mathcal{A},\ a_i = a_j$, and denote by $I_1, \dots I_d$ its equivalence classes. Then

$$\mathcal{A} = \langle \sum_{i \in I_1} \boldsymbol{e}_i, \dots, \sum_{i \in I_d} \boldsymbol{e}_i \rangle.$$

Proof. Clearly, "$\subset$" holds, so we only prove "$\supset$". We may assume $1 \in I_1$, and it suffices to show

$$\sum_{i \in I_1} \boldsymbol{e}_i \in \mathcal{A}.$$

For $\boldsymbol{b} \in \mathcal{A}$, set $I(\boldsymbol{b}) = \{i \in [m] \mid b_i = b_1\}$. Then

$$\sum_{i \in I(\boldsymbol{b})} \boldsymbol{e}_i \in \mathcal{A}.$$

$$
\begin{aligned}
k \in [m] \setminus I_1 &\iff k \not\sim 1 \\
&\iff \exists \boldsymbol{b}^{(k)} \in \mathcal{A},\ k \notin I(\boldsymbol{b}^{(k)})
\end{aligned}
$$

Thus

$$
\begin{aligned}
k \in [m] \setminus I_1 &\iff k \not\sim 1 \\
&\iff \exists \boldsymbol{b}^{(k)} \in \mathcal{A},\ k \notin I(\boldsymbol{b}^{(k)})
\end{aligned}
$$

Thus

$$
\bigcap_{k \in [m] \setminus I_1} I(\boldsymbol{b}^{(k)}) = I_1.
$$

$$
\begin{aligned}
k \in [m] \setminus I_1 &\iff k \not\sim 1 \\
&\iff \exists \boldsymbol{b}^{(k)} \in \mathcal{A},\ k \notin I(\boldsymbol{b}^{(k)})
\end{aligned}
$$

Thus

$$\bigcap_{k \in [m] \setminus I_1} I(\boldsymbol{b}^{(k)}) = I_1.$$

$$\prod_{k \in [m] \setminus I_1} \sum_{i \in I(\boldsymbol{b}^{(k)})} \boldsymbol{e}_i = \sum_{i \in I_1} \boldsymbol{e}_i$$

$$\begin{aligned} k \in [m] \setminus I_1 &\iff k \not\sim 1 \\ &\iff \exists \boldsymbol{b}^{(k)} \in \mathcal{A},\ k \notin I(\boldsymbol{b}^{(k)}) \end{aligned}$$

Thus

$$\bigcap_{k \in [m] \setminus I_1} I(\boldsymbol{b}^{(k)}) = I_1.$$

$$\mathcal{A} \ni \prod_{k \in [m] \setminus I_1} \sum_{i \in I(\boldsymbol{b}^{(k)})} \boldsymbol{e}_i = \sum_{i \in I_1} \boldsymbol{e}_i$$

since

$$\mathcal{A} \ni \sum_{i \in I(\boldsymbol{b}^{(k)})} \boldsymbol{e}_i.$$

Definition

A coherent algebra $\mathcal{A}$ is a subalgebra of $\mathrm{Mat}_n(\mathbb{C})$ containing J, closed under the entrywise product $\circ$ and transposition $\top$.

Definition

A coherent algebra $\mathcal{A}$ is a subalgebra of $\mathrm{Mat}_n(\mathbb{C})$ containing J, closed under the entrywise product $\circ$ and transposition $\top$.

Definition

A coherent configuration is a pair $(X, \{R_i\}_{i=0}^d)$ where $\{R_i\}_{i=0}^d$ is a partition of $X \times X$ such that
(i) $\{(x,x) \mid x \in X\}$ is a union of some R_i's,
(ii) $\forall i \in \{0,1,\dots,d\}$, $\exists i'$, $R_i^\top = R_{i'}$, where

$$R_i^\top = \{(x,y) \in X \times X \mid (y,x) \in R_i\}.$$

(iii) $\forall h, i, j \in \{0,1,\dots,d\}$,

$$|\{z \in X \mid (x,z) \in R_i,\ (z,y) \in R_j\}| = \text{constant}$$

independent of $(x,y) \in R_h$. This constant is denoted by p_{ij}^h. These constants are called the intersection numbers.

The adjacency matrices of the relations of a coherent configuration form a basis of a coherent algebra, and every coherent algebra arises in this way.

The adjacency matrices of the relations of a coherent configuration form a basis of a coherent algebra, and every coherent algebra arises in this way.

$$X \times X = \bigcup_{i=0}^{d} R_i \iff J = \sum_{i=0}^{d} A_i \in \mathcal{A}$$

$$\{(x,x) \mid x \in X\} = \bigcup_{i \in I_0} R_i \iff I = \sum_{i \in I_0} A_i \in \mathcal{A}$$

$$\forall i,\ \exists i',\ R_i^{\top} = R_{i'} \iff \text{closed under } \top$$

$$p_{ij}^{h} \text{ independent} \iff A_i A_j = \sum_{h=0}^{d} p_{ij}^{h} A_h$$

$$\iff \text{closed under multiplication.}$$

Definition

If a coherent configuration $(X, \{R_i\}_{i=0}^d)$ satisfies the additional property

(iv) $R_0 = \{(x, x) \mid x \in X\}$

$\implies$ association scheme.

If an association scheme satisfies

(v) $p_{ij}^h = p_{ji}^h$ for all $h, i, j \in \{0, 1, \ldots, d\}$,

$\implies$ commutative.

If it satisfies

(vi) $R_i^\top = R_i$ for all $i \in \{0, 1, \ldots, d\}$

$\implies$ symmetric.

Definition

If a coherent configuration $(X, \{R_i\}_{i=0}^d)$ satisfies the additional property

(iv) $R_0 = \{(x,x) \mid x \in X\}$

$\implies$ association scheme.

If an association scheme satisfies

(v) $p_{ij}^h = p_{ji}^h$ for all $h, i, j \in \{0, 1, \ldots, d\}$,

$\implies$ commutative.

If it satisfies

(vi) $R_i^\top = R_i$ for all $i \in \{0, 1, \ldots, d\}$

$\implies$ symmetric.

symmetric $\implies$ commutative.

For a symmetric association scheme $(X, \{R_i\}_{i=0}^{d})$, the coherent algebra $\mathcal{A}$ spanned by its adjacency matrices A_i's is called the Bose–Mesner algebra.

For a symmetric association scheme $(X, \{R_i\}_{i=0}^{d})$, the coherent algebra $\mathcal{A}$ spanned by its adjacency matrices A_i's is called the Bose–Mesner algebra.

A_i's are pairwise commutative symmetric matrices
$\implies \exists T :$ orthogonal, $T\mathcal{A}T^{-1} \subset$ {diagonal matrices}

For a symmetric association scheme $(X, \{R_i\}_{i=0}^d)$, the coherent algebra $\mathcal{A}$ spanned by its adjacency matrices A_i's is called the Bose–Mesner algebra.

A_i's are pairwise commutative symmetric matrices
$\implies \exists T:$ orthogonal, $T\mathcal{A}T^{-1} \subset \{$diagonal matrices$\}$

As an algebra

$\{$diagonal matrices$\} \cong \mathbb{R}^n$ with entrywise multiplication.

For a symmetric association scheme $(X, \{R_i\}_{i=0}^d)$, the coherent algebra $\mathcal{A}$ spanned by its adjacency matrices A_i's is called the Bose–Mesner algebra.

A_i's are pairwise commutative symmetric matrices
$\implies \exists T:$ orthogonal, $T\mathcal{A}T^{-1} \subset$ {diagonal matrices}

As an algebra

{diagonal matrices} $\cong \mathbb{R}^n$ with entrywise multiplication.

Lemma

Let $\mathcal{A} \subset \mathbb{F}^m$ be a subalgebra. Then

$$\mathcal{A} = \langle \sum_{i \in I_1} \boldsymbol{e}_i, \ldots, \sum_{i \in I_d} \boldsymbol{e}_i \rangle.$$

For a symmetric association scheme $(X, \{R_i\}_{i=0}^{d})$, the coherent algebra $\mathcal{A}$ spanned by its adjacency matrices A_i's is called the Bose–Mesner algebra.

A_i's are pairwise commutative symmetric matrices
$\implies \exists T:$ orthogonal, $T\mathcal{A}T^{-1} \subset \{$diagonal matrices$\}$

As an algebra

$\{$diagonal matrices$\} \cong \mathbb{R}^n$ with entrywise multiplication.

Lemma

Let $\mathcal{A} \subset \mathbb{R}^n$ be a subalgebra. Then

$$\mathcal{A} = \langle \sum_{i \in I_1} \boldsymbol{e}_i, \ldots, \sum_{i \in I_d} \boldsymbol{e}_i \rangle.$$

By Lemma, $T\mathcal{A}T^{-1}$ has a basis of the form

$$\begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \end{bmatrix},$$

By Lemma, $T\mathcal{A}T^{-1}$ has a basis of the form

$$\begin{bmatrix} 1 & & & & & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & & & & & & & \\ & \ddots & & & & & & \\ & & 0 & & & & & \\ & & & 1 & & & & \\ & & & & \ddots & & & \\ & & & & & 1 & & \\ & & & & & & 0 & \\ & & & & & & & \ddots \end{bmatrix},$$

Call these matrices D_i. Then

$$T\mathcal{A}T^{-1} = \langle D_0, D_1, \ldots \rangle, \quad D_i D_j = \delta_{ij} D_i.$$

$$T\mathcal{A}T^{-1} = \langle D_0, D_1, \ldots \rangle, \qquad D_i D_j = \delta_{ij} D_i.$$

$$T\mathcal{A}T^{-1} = \langle D_0, D_1, \ldots \rangle, \qquad D_i D_j = \delta_{ij} D_i.$$

Define $E_i = T^{-1} D_i T \in \mathcal{A} \implies E_i E_j = \delta_{ij} E_i.$

$$T\mathcal{A}T^{-1} = \langle D_0, D_1, \ldots \rangle, \qquad D_i D_j = \delta_{ij} D_i.$$

Define $E_i = T^{-1} D_i T \in \mathcal{A} \implies E_i E_j = \delta_{ij} E_i.$

Since

$$\begin{aligned} d + 1 &= \dim\langle A_0, A_1, \ldots, A_d \rangle = \dim \mathcal{A} \\ &= \dim T\mathcal{A}T^{-1} = \dim\langle D_0, D_1, \ \ldots \rangle, \end{aligned}$$

$$T\mathcal{A}T^{-1} = \langle D_0, D_1, \ldots \rangle, \qquad D_i D_j = \delta_{ij} D_i.$$

Define $E_i = T^{-1} D_i T \in \mathcal{A} \implies E_i E_j = \delta_{ij} E_i.$

Since

$$\begin{aligned} d+1 &= \dim\langle A_0, A_1, \ldots, A_d \rangle = \dim \mathcal{A} \\ &= \dim T\mathcal{A}T^{-1} = \dim\langle D_0, D_1, \ldots \rangle, \end{aligned}$$

$$T\mathcal{A}T^{-1} = \langle D_0, D_1, \ldots \rangle, \qquad D_i D_j = \delta_{ij} D_i.$$

Define $E_i = T^{-1} D_i T \in \mathcal{A} \implies E_i E_j = \delta_{ij} E_i$.

Since

$$\begin{aligned} d+1 &= \dim\langle A_0, A_1, \ldots, A_d \rangle = \dim \mathcal{A} \\ &= \dim T\mathcal{A}T^{-1} = \dim\langle D_0, D_1, \ldots \rangle, \end{aligned}$$

there are $d+1$ D_i's, and $d+1$ E_i's.

$$T\mathcal{A}T^{-1} = \langle D_0, D_1, \ldots \rangle, \qquad D_i D_j = \delta_{ij} D_i.$$

Define $E_i = T^{-1} D_i T \in \mathcal{A} \implies E_i E_j = \delta_{ij} E_i$.

Since

$$\begin{aligned} d + 1 &= \dim\langle A_0, A_1, \ldots, A_d \rangle = \dim \mathcal{A} \\ &= \dim T\mathcal{A}T^{-1} = \dim\langle D_0, D_1, \ \ldots \rangle, \end{aligned}$$

there are $d + 1$ D_i's, and $d + 1$ E_i's.

$E_0, E_1, \ldots, E_d$ are called the primitive idempotents of $\mathcal{A} = \langle E_0, E_1, \ldots, E_d \rangle$.

$J \in \mathcal{A} \implies$

$$J \in \mathcal{A} \implies \frac{1}{n}J = \sum_{i=0}^{d} c_i E_i \quad (\exists c_i)$$

$$
\begin{aligned}
J \in \mathcal{A} &\implies \frac{1}{n}J = \sum_{i=0}^{d} c_i E_i \quad (\exists c_i) \\
&\implies (\frac{1}{n}J)^2 = (\sum_{i=0}^{d} c_i E_i)^2 \\
&\implies \frac{1}{n}J = \sum_{i=0}^{d} c_i^2 E_i
\end{aligned}
$$

$$
\begin{aligned}
J \in \mathcal{A} \implies & \frac{1}{n}J = \sum_{i=0}^{d} c_i E_i \quad (\exists c_i) \\
\implies & \left(\frac{1}{n}J\right)^2 = \left(\sum_{i=0}^{d} c_i E_i\right)^2 \\
\implies & \frac{1}{n}J = \sum_{i=0}^{d} c_i^2 E_i \\
\implies & c_i^2 = c_i \implies c_i \in \{0,1\} \implies \frac{1}{n}J = \sum_{c_i=1} E_i
\end{aligned}
$$

$$
\begin{aligned}
J \in \mathcal{A} &\implies \frac{1}{n} J = \sum_{i=0}^{d} c_i E_i \quad (\exists c_i) \\
&\implies (\frac{1}{n} J)^2 = (\sum_{i=0}^{d} c_i E_i)^2 \\
&\implies \frac{1}{n} J = \sum_{i=0}^{d} c_i^2 E_i \\
&\implies c_i^2 = c_i \implies c_i \in \{0, 1\} \implies \frac{1}{n} J = \sum_{c_i=1} E_i
\end{aligned}
$$

Since

$$
1 = \operatorname{rank} J = \operatorname{rank} \sum_{c_i=1} E_i = \operatorname{rank} \sum_{c_i=1} D_i = \sum_{c_i=1} \operatorname{rank} D_i
$$

$\exists! i, \ \frac{1}{n} J = E_i$.

$$
\begin{aligned}
J \in \mathcal{A} &\implies \frac{1}{n}J = \sum_{i=0}^{d} c_i E_i \quad (\exists c_i) \\
&\implies (\frac{1}{n}J)^2 = (\sum_{i=0}^{d} c_i E_i)^2 \\
&\implies \frac{1}{n}J = \sum_{i=0}^{d} c_i^2 E_i \\
&\implies c_i^2 = c_i \implies c_i \in \{0,1\} \implies \frac{1}{n}J = \sum_{c_i=1} E_i
\end{aligned}
$$

Since

$$
1 = \operatorname{rank} J = \operatorname{rank} \sum_{c_i=1} E_i = \operatorname{rank} \sum_{c_i=1} D_i = \sum_{c_i=1} \operatorname{rank} D_i
$$

$\exists ! i,\ \frac{1}{n}J = E_i$. We may assume $\frac{1}{n}J = E_0$.

$$
\begin{aligned}
A_i \circ A_j &= \delta_{ij} A_i, \\
A_i A_j &= \sum_{h=0}^{d} p_{ij}^h A_h, \\
E_i E_j &= \delta_{ij} E_i.
\end{aligned}
$$

$$A_i \circ A_j = \delta_{ij} A_i,$$

$$A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h,$$

$$E_i E_j = \delta_{ij} E_i.$$

To be complete, we need

$$E_i \circ E_j = \frac{1}{n} \sum_{h=0}^{d} q_{ij}^h E_h.$$

$$
\begin{aligned}
A_i \circ A_j &= \delta_{ij} A_i, \\
A_i A_j &= \sum_{h=0}^{d} p_{ij}^h A_h, \\
E_i E_j &= \delta_{ij} E_i.
\end{aligned}
$$

To be complete, we need

$$E_i \circ E_j = \frac{1}{n} \sum_{h=0}^{d} q_{ij}^h E_h.$$

q_{ij}^h are called Krein parameters. It is known that $q_{ij}^h \geq 0$.

$\exists$ nonsingular matrix P:

$$(A_0, A_1, \ldots, A_d) = (E_0, E_1, \ldots, E_d)P$$

P :first eigenmatrix

$Q = nP^{-1}$:second eigenmatrix

$\exists$ nonsingular matrix P:

$$(A_0, A_1, \ldots, A_d) = (E_0, E_1, \ldots, E_d)P$$

P :first eigenmatrix

$Q = nP^{-1}$:second eigenmatrix

$$A_j = \sum_{i=0}^{d} P_{ij} E_i,$$

$$E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i.$$

$\exists$ nonsingular matrix P:

$$(A_0, A_1, \ldots, A_d) = (E_0, E_1, \ldots, E_d)P$$

P :first eigenmatrix

$Q = nP^{-1}$:second eigenmatrix

$$A_j = \sum_{i=0}^{d} P_{ij} E_i,$$

$$E_j = \frac{1}{n} \sum_{i=0}^{d} Q_{ij} A_i.$$

P_{ij} are eigenvalues of A_j, since $A_j E_i = P_{ij} E_i$.