Introduction to Association Schemes
Part II

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Algebraic Combinatorics Summer School
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Definition

A coherent configuration

is a pair $(X, \{R_i\}_{i=0}^d)$ where $\{R_i\}_{i=0}^d$ is a partition of $X \times X$

(i) $\{(x, x) \mid x \in X\}$ is a union of some R_i's

(ii) $\forall i \in \{0, 1, \ldots, d\}$, $\exists i'$, $R_i^\top = R_{i'}$, where

$$R_i{}^\top = \{(x, y) \in X \times X \mid (y, x) \in R_i\}.$$

(iii) $\forall h, i, j \in \{0, 1, \ldots, d\}$,

$$|\{z \in X \mid (x, z) \in R_i,\ (z, y) \in R_j\}| = \text{constant}$$

independent of $(x, y) \in R_h$. This constant is denoted by p_{ij}^h. These constants are called the intersection numbers.

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$$\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle = \langle E_0, E_1, \ldots, E_d \rangle.$$

$$
\begin{aligned}
\boldsymbol{A}_i \circ \boldsymbol{A}_j &= \delta_{ij}\boldsymbol{A}_i, \\
\boldsymbol{A}_i\boldsymbol{A}_j &= \sum_{h=0}^{d} p_{ij}^{h}\boldsymbol{A}_h, \\
\boldsymbol{E}_i\boldsymbol{E}_j &= \delta_{ij}\boldsymbol{E}_i, \\
\boldsymbol{E}_i \circ \boldsymbol{E}_j &= \frac{1}{n}\sum_{h=0}^{d} q_{ij}^{h}\boldsymbol{E}_h, \\
\boldsymbol{A}_j &= \sum_{i=0}^{d} P_{ij}\boldsymbol{E}_i, \\
\boldsymbol{E}_j &= \frac{1}{n}\sum_{i=0}^{d} Q_{ij}\boldsymbol{A}_i, \\
\boldsymbol{E}_0 &= \frac{1}{n}\boldsymbol{J}.
\end{aligned}
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$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

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$$E_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad E_2 = \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

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$$A_1^2 = 2A_0 + 2A_2, \quad A_1 A_2 = A_1, \quad A_2^2 = A_0,$$

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$$P = Q = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & -2 & 1 \end{pmatrix}.$$

$$A_j = \sum_{i=0}^{d} P_{ij} E_i \implies A_j E_0 = P_{0j} E_0 \implies A_j J = P_{0j} J.$$

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$$m_i = \text{rank}\, E_i = \text{tr}\, E_i \quad (\text{since } E_i^2 - E_i = 0)$$

$$= \text{tr}(E_i \circ I) = \text{tr}(\frac{1}{n} Q_{0i} I) = Q_{0i}$$

$$A_j = \sum_{i=0}^{d} P_{ij} E_i \implies A_j E_0 = P_{0j} E_0 \implies A_j J = P_{0j} J.$$

$k_j = P_{0j}$ is the degree (valency) of the graph (X, R_j).

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= "multiplicity" of the i-th common eigenvalue

Lemma

$$\frac{P_{jh}}{k_h} = \frac{Q_{hj}}{m_j}.$$

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Proof.

Compute $\mathrm{tr}\, A_h E_j$ in two ways:

$$\begin{aligned} &= \mathrm{tr}\left(\sum_{i=0}^{d} P_{ih} E_i\right) E_j \\ &= \mathrm{tr}\, P_{jh} E_j \\ &= P_{jh} m_j. \end{aligned} \qquad \begin{aligned} &= A_h \left(\frac{1}{n}\,\mathrm{tr} \sum_{i=0}^{d} Q_{ij} A_i\right) \\ &= \frac{1}{n} \sum_{i=0}^{d} Q_{ij}\, \mathrm{tr}(A_h A_i) \\ &= \frac{1}{n} \sum_{i=0}^{d} Q_{ij} n k_i \delta_{hi} \\ &= Q_{hj} k_h. \end{aligned}$$

Note

$$\mathrm{tr}(A_h A_i) = \mathrm{tr}(A_h A_i^{\top}) = n k_i \delta_{hi}$$

□

By Lemma, we have the orthogonality relations:

$$n\delta_{ij} = (PQ)_{ij} = \sum_{h=0}^{d} P_{ih} Q_{hj} = \frac{1}{m_i} \sum_{h=0}^{d} Q_{hi} Q_{hj} k_h$$

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Spherical representation

$$F_j = \frac{n}{m_j} E_j = \sum_{i=0}^{d} \frac{Q_{ij}}{Q_{0j}} A_i \in \langle A_0, \ldots, A_d \rangle$$

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$\{b_x \mid x \in X\} \subset S^{m_j - 1}$ unit sphere

- Complete graphs. $\boldsymbol{\mathcal{A}} = \langle \boldsymbol{A_0} = \boldsymbol{I}, \boldsymbol{A_1} = \boldsymbol{J} - \boldsymbol{I} \rangle$.

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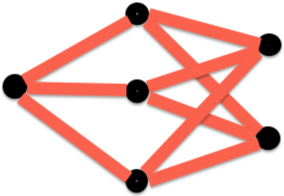
- Complete bipartite graph $K_{m,m}$.
  $\mathcal{A} = \langle A_0 = I, A_1 = A, A_2 = J - A - I \rangle$.



$(m = 3)$.

- Polygons. $\boldsymbol{A} = \boldsymbol{C} + \boldsymbol{C}^\top$, where

$$\boldsymbol{C} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & & \mathbf{0} \\ & \ddots & \ddots & \\ & & \ddots & \mathbf{1} \\ \mathbf{1} & & & \mathbf{0} \end{bmatrix}$$

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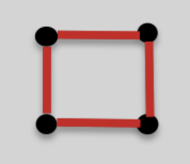
$$C = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}$$

$$\mathcal{A} = \begin{cases} \langle I, C + C^\top, C^2 + (C^\top)^2, \ldots, C^m + (C^\top)^m \rangle \\ \quad ((2m+1)\text{-gon}), \\ \langle I, C + C^\top, C^2 + (C^\top)^2, \ldots, C^m \rangle \\ \quad (2m\text{-gon}). \end{cases}$$

 $(2m = 4)$.

- Let F be a finite set with $q \geq 2$ elements, and set $X = F^d$. Then X is a metric space with respect to the Hamming distance

$$d_H(x, y) = |\{i \mid i \in [d],\ x_i \neq y_i\}|,$$

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Then $(X, \{R_i\}_{i=0}^d)$ is a symmetric association scheme called the Hamming scheme $H(d, q)$, where

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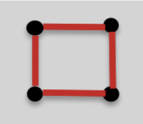
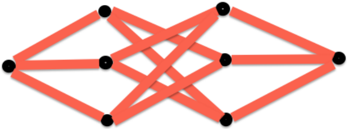


$H(2, 2)$



$H(3, 2)$

- Let Ω be a finite set with v elements, and let X be the collection of all d-element subsets of Ω. Define

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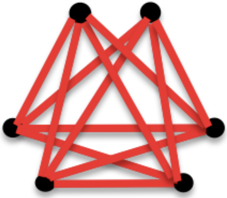

$J(4, 2)$ has $\binom{4}{2} = 6$ vertices.

- Let G be a finite group of order n. Define $(A_g)_{xy} = \delta_{x,gy}$. Then $A_g A_h = A_{gh}$.

$$\mathcal{A} = \langle A_g \mid g \in G \rangle$$

is a coherent algebra, defining an association scheme.

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$$\begin{aligned} \text{commutative} &\iff G \text{ is abelian,} \\ \text{symmetric} &\iff g^2 = 1 \ (\forall g \in G). \end{aligned}$$

- If H is a subgroup of $\mathrm{Aut}\, G$, with orbits $S_0 = \{1\}, S_1, \cdots S_d$, then

$$\mathcal{A}' = \langle \sum_{g \in S_i} A_g \mid 0 \leq i \leq d \rangle$$

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- $G = \mathbb{F}_p^m$ is an abelian group. Any subgroup H of $\mathbf{Aut}\,G = GL(m, \mathbb{F}_p)$ gives rise to a commutative association scheme.
- $G = \mathbb{F}_{p^m}$ is an abelian group. Any subgroup H of $\mathbb{F}_{p^m}^\times$ containing -1 gives rise to a symmetric association scheme.

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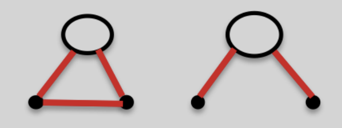
- Let $G = S_v$, the symmetric group. It acts on $\binom{[v]}{d}$, the collection of all d-elements subsets. This action gives the Johnson scheme $J(v, d)$ by taking R_i's to be orbits.
- The primitive idempotents E_i is the projection onto the irreducible S_v-module corresponding to the partition $(v - i, i) \vdash v$.

- Let Γ be a connected regular graph of diameter 2. If $\exists \lambda, \mu$ such that for any distinct vertices x, y,

$$\#\text{common neighbors of } x, y = \begin{cases} \lambda & \text{if } x \sim y, \\ \mu & \text{otherwise} \end{cases}$$

then Γ is called a strongly regular graph. Let A be the adjacency matrix. $\mathcal{A} = \langle I, A, J - I - A \rangle$.



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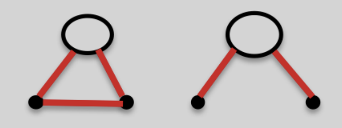
- $H(2, q)$, $J(v, 2)$, $K_{m,m}$



λ μ

Let $(\mathcal{P},\mathcal{B})$ be a $\mathbf{2}$-$(\boldsymbol{v},\boldsymbol{k},\boldsymbol{\lambda})$ design:

$$|\mathcal{P}| = \boldsymbol{v}, \qquad \mathcal{B} \subset \binom{\mathcal{P}}{\boldsymbol{k}},$$

$$\forall \boldsymbol{\alpha},\beta \in \mathcal{P},\, \boldsymbol{\alpha} \neq \beta,\ \#\{B \in \mathcal{B} \mid \boldsymbol{\alpha},\beta \in B\} = \boldsymbol{\lambda}.$$

Let $(\mathcal{P}, \mathcal{B})$ be a 2-(v, k, λ) design:

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$(\mathcal{P}, \mathcal{B})$ is quasi-symmetric if $\exists x, y$ with $x < y$ such that

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$\binom{\mathcal{P}}{k}$ = vertices of $J(v, k)$, so

$$\mathcal{B} \subset J(v, k)$$

association scheme $\subset$ association scheme

A Steiner system $(\mathcal{P}, \mathcal{B})$ is a 2-$(v, k, 1)$ design.

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Summary

- graphs leading to association schemes
- primitive idempotents and spherical representations
- finite groups and actions
- some important subsets of association schemes