球面上の代数的組合せ論入門
An introduction to algebraic combinatorics on a sphere

Hiroshi Nozaki

Aichi University of Education

代数的組合せ論「夏の学校 2014」
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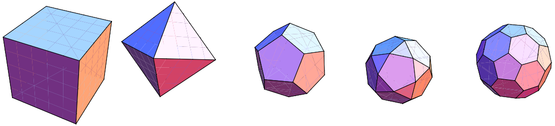
Algebraic Combinatorics

Problem 1

What is a "good" finite set on the unit sphere S^{d-1}?



1. Coding theory (local viewpoint)
   - Spherical u-code
     $\rightarrow$ Kissing number
   - Optimal code
   - s-distance set
2. Design theory (global viewpoint)
   - Spherical t-design

$\rightarrow$ Q-polynomial association scheme

Code on a sphere

X: a finite set on S^{d-1}

$$A(X) := \{\langle x, y \rangle \mid x, y \in X, x \neq y\}.$$

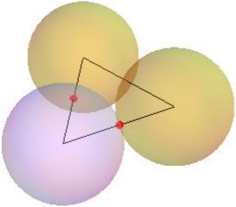
Definition 2

X is called u-code if $A(X) \subset [-1, u]$.

Problem 3

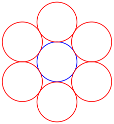
For given $u \in [-1, 1]$ and d, find maximum $|X|$ in u-codes X.

$\frac{1}{2}$-codes $\leftrightarrow$ kissing number problem

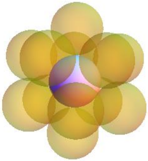


Kissing number on S^{d-1}: $k(d)$

$k(2) = 6$



$k(3) = 12$



- Famous disagreement between Newton and Gregory (1694)
- Proved by Schutte and van der Waerden (1953), Leech (1956), ...

$k(4) = 24$ (24-cell)

- Musin (Annals of Math. 2008), Bachoc–Vallentin (2008,SDP)

$k(8) = 240$ (E_8 root system), LP

$k(24) = 196560$ (Minimum vectors of the Leech lattice), LP

- Odlyzko–Sloane (1979)

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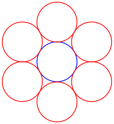

For other dimensions, nobody knows $k(d)$.

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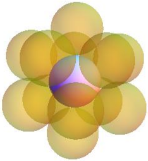

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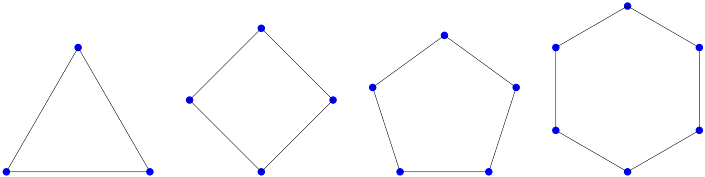
Optimal codes

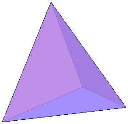
Problem 4

For a given $|X|$ and d, find smallest u such that X is a u-code on S^{d-1}. (optimal code)

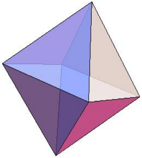


Regular n-gon: Optimal code

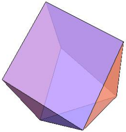
Optimal codes on S^2



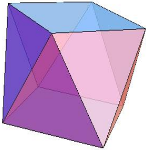
4 points



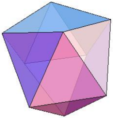
5,6 points



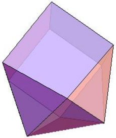
7 points



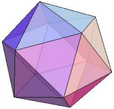
8 points



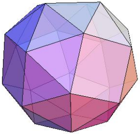
9 points



10 points

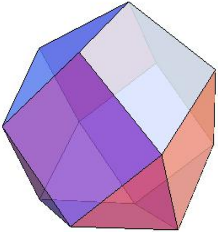


11,12 points



24 points

The strong thirteen spheres problem



13 points
Musin and Tarasov (2012)

Optimal codes in higher dimensions

dim.	size	$A(X)$	name
n	$n+1$	$-1/n$	simplex
n	$2n$	$-1, 0$	cross polytope
4	10	$-2/3, 1/6$	Petersen graph
4	120	$-1, \pm 1/2, 0, (\pm 1 \pm \sqrt{5})/4$	600-cell
8	240	$-1, \pm 1/2, 0$	E_8 root
7	56	$-1, \pm 1/3$	kissing
6	27	$-1/2, 1/4$	kissing
5	16	$-3/5, 1/5$	kissing
24	196560	$-1, \pm 1/2, \pm 1/4, 0$	Leech lattice
23	4600	$-1, \pm 1/3, 0$	kissing
22	891	$-1/2, -1/8, 1/4$	kissing
23	552	$-1 \pm 1/5$	equiangular lines
22	275	$-1/4, 1/6$	kissing
21	162	$-2/7, 1/7$	kissing
22	100	$-4/11, 1/11$	Higman-Sims
$q\frac{q^3+1}{q+1}$	$(q+1)(q^3+1)$	$-1/q, 1/q^2$	Cameron et al'78

Proved by LP or SDP (SDP: only for Petersen graph)

Homogeneous polynomials

$\mathrm{Hom}_i(\mathbb{R}^d)$ denotes the linear space of homogeneous polynomials of degree i, in d variables $x_1, \ldots, x_d$.

$$\dim \mathrm{Hom}_i(\mathbb{R}^d) = \binom{d+i-1}{i}.$$

(i-combination with repetitions)

$$P_i(\mathbb{R}^d) = \bigoplus_{j=0}^{i} \mathrm{Hom}_j(\mathbb{R}^d).$$

$$\dim P_i(\mathbb{R}^d) = \sum_{j=0}^{i} \binom{d+j-1}{j} = \binom{d+i}{i}.$$

Harmonic polynomials

Laplacian: $\Delta f = \sum_{k=1}^{d} \partial^2 f / \partial x_k^2$ for $f \in \mathrm{Hom}_i(\mathbb{R}^d)$.

$$\Delta : \mathrm{Hom}_i(\mathbb{R}^d) \to \mathrm{Hom}_{i-2}(\mathbb{R}^d) \text{ (linear map)}$$

- $\mathrm{Harm}_i(\mathbb{R}^d) := \mathrm{Ker}\Delta = \{f \mid \Delta f = 0\}$
- An element of $\mathrm{Harm}_i(\mathbb{R}^d)$ is called a harmonic polynomial.

Actually Δ is surjective.

$$\begin{aligned}
\dim \mathrm{Ker}\Delta &= \dim \mathrm{Hom}_i(\mathbb{R}^d) - \dim \mathrm{Im}\Delta \\
\dim \mathrm{Harm}_i(\mathbb{R}^d) &= \dim \mathrm{Hom}_i(\mathbb{R}^d) - \dim \mathrm{Hom}_{i-2}(\mathbb{R}^d) \\
&= \binom{d+i-1}{i} - \binom{d+i-3}{i-2}
\end{aligned}$$

Basic results on harmonic polynomials

Let $r^2 = \sum_{i=1}^{d} x_i^2$.

Theorem 5

1. $$\mathrm{Hom}_i(\mathbb{R}^d) = \mathrm{Harm}_i(\mathbb{R}^d) \oplus r^2 \mathrm{Hom}_{i-2}(\mathbb{R}^d).$$

2. $$\mathrm{Hom}_i(\mathbb{R}^d) = \bigoplus_{j=0}^{\lfloor i/2 \rfloor} r^{2j}\, \mathrm{Harm}_{i-2j}(\mathbb{R}^d).$$

3. $$\mathrm{Harm}_i(\mathbb{R}^d) \perp \mathrm{Harm}_j(\mathbb{R}^d) (i \neq j)$$

   with respect to

   $$\langle\langle f, g \rangle\rangle = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x) g(x) d\mu(x).$$

Polynomials on a sphere

$\mathrm{Hom}_i(S^{d-1}) = \{f|_{S^{d-1}} \mid f \in \mathrm{Hom}_i(\mathbb{R}^d)\}$
$P_i(S^{d-1}) = \{f|_{S^{d-1}} \mid f \in P_i(\mathbb{R}^d)\}$
$\mathrm{Harm}_i(S^{d-1}) = \{f|_{S^{d-1}} \mid f \in \mathrm{Harm}_i(\mathbb{R}^d)\}$

Theorem 6

$$\mathrm{Harm}_i(S^{d-1}) \cong \mathrm{Harm}_i(\mathbb{R}^d)$$

$$\mathrm{Hom}_i(\mathbb{R}^d) = \bigoplus_{j=0}^{\lfloor i/2 \rfloor} r^{2j}\,\mathrm{Harm}_{i-2j}(\mathbb{R}^d) \Rightarrow \mathrm{Hom}_i(S^{d-1}) \cong \bigoplus_{j=0}^{\lfloor i/2 \rfloor} \mathrm{Harm}_{i-2j}(\mathbb{R}^d)$$

$$P_i(S^{d-1}) = \sum_{j=0}^{i} \mathrm{Hom}_j(S^{d-1}) \cong \bigoplus_{j=0}^{i} \mathrm{Harm}_j(\mathbb{R}^d)$$

Dimension of $P_i(S^{d-1})$

$$P_i(S^{d-1}) = \bigoplus_{j=0}^{i} \mathrm{Harm}_j(\mathbb{R}^d)$$

$$\begin{aligned}
\dim P_i(S^{d-1}) &= \sum_{j=0}^{i} \dim \mathrm{Harm}_j(\mathbb{R}^d) \\
&= \sum_{j=0}^{i} \left(\binom{d+j-1}{j} - \binom{d+j-3}{j-2}\right) \\
&= \binom{d+i-1}{i} + \binom{d+i-2}{i-1}.
\end{aligned}$$

Gegenbauer polynomials

Gegenbauer polynomials:

$$G_0^{(d)}(t) = 1, \qquad G_1^{(d)}(t) = dt,$$

$$tG_{i-1}^{(d)}(t) = \frac{i}{d+2i-2} G_i^{(d)}(t) + \frac{d+i-4}{d+2i-6} G_{i-2}^{(d)}(t).$$

Gegenbauer polynomials form a sequence of orthogonal polynomials w.r.t.

$$(f,g) = \int_{-1}^{1} f(t)g(t)(1-t^2)^{(d-3)/2} dx$$

Note $G_i^{(d)}(1) = \dim \mathrm{Harm}_i(\mathbb{R}^d)$.

Addition formula

Let $h_i = \dim \mathrm{Harm}_i(\mathbb{R}^d)$.
Let $\{\varphi_{i,1}, \ldots, \varphi_{i,h_i}\}$ be an orthonormal basis of $\mathrm{Harm}_i(\mathbb{R}^d)$ w.r.t. $\langle\langle , \rangle\rangle$.
Let $\langle , \rangle$ be the usual inner product in $\mathbb{R}^d$.

Theorem 7 (Addition formula)

For any $x, y \in S^{d-1}$, we have

$$\sum_{j=0}^{h_i} \varphi_{i,j}(x)\varphi_{i,j}(y) = G_i^{(d)}(\langle x, y\rangle).$$

Positive definiteness of $G_i^{(d)}(t)$

Theorem 8

For arbitrary points $x_1, \ldots, x_n \in S^{d-1}$, and real variables $\xi_1, \ldots, \xi_n$, we have

$$\sum_{i,j=1}^{n} G_k^{(d)}(\langle x_i, x_j \rangle)\xi_i \xi_j \geq 0,$$

or equivalently $(G_k^d(\langle x_i, x_j \rangle))_{i,j}$ is positive semidefinite.

Proof:

$$\begin{aligned}(G_k^{(d)}(\langle x_i, x_j \rangle))_{i,j} &= (\sum_{l=0}^{h_k} \varphi_{k,l}(x_i)\varphi_{k,l}(x_j))_{i,j} \\ &= (\varphi_{k,l}(x_i))_{i,l}{}^t(\varphi_{k,l}(x_i))_{i,l} \succeq 0\end{aligned}$$

Corollary: $\sum_{i,j=1}^{n} G_k^{(d)}(\langle x_i, x_j \rangle) \geq 0$.

Linear programming bound, Delsarte method

Theorem 9 (Delsarte, Goethals and Seidel (1977))

Let X be a subset in S^{d-1}. Suppose there exists a polynomial $g(t) = \sum_{i \geq 0} g_i G_i^{(d)}(t)$ s.t.

- *$g(1) > 0$, $g(\alpha) \leq 0$ for any $\alpha \in A(X)$,*
- *$g_0 > 0$, and $g_i \geq 0$ for any i.*

Then

$$|X| \leq \frac{g(1)}{g_0}.$$

Proof of LP bound

Proof: $n_\alpha = |\{(x,y) \in X \times X \mid \langle x,y\rangle = \alpha\}|$

$$\sum_{x,y\in X} g(\langle x,y\rangle) = \sum_{x,y\in X}\sum_{i\geq 0} g_i G_i^{(d)}(\langle x,y\rangle)$$

$$|X|g(1) \geq |X|g(1) + \sum_{\alpha\in A(X)} n_\alpha g(\alpha)$$

$$= |X|^2 g_0 + \sum_{i\geq 1} g_i \sum_{x,y\in X} G_i^{(d)}(\langle x,y\rangle) \geq |X|^2 g_0$$

$$|X| \leq \frac{g(1)}{g_0}. \qquad \square$$

Equality holds $\Leftrightarrow$
$g(\alpha) = 0$ and $g_i \sum_{x,y\in X} G_i^{(d)}(\langle x,y\rangle) = 0$ for any $1 \leq i \leq \deg g$.

Linear programming bound for spherical sets.

$A(X) = \{\alpha_1, \ldots, \alpha_s\}$, $n_i = \frac{1}{X}|\{(x,y) \in X \times X \mid \langle x, y\rangle = \alpha_i\}|$

- $L = -(G_j^{(d)}(\alpha_i))_{1 \le i \le s, 1 \le j \le r}$,
- $b = (g_0, \ldots, g_0)$, $c = (G_1^{(d)}(1), \ldots, G_r^{(d)}(1))$,
- $x = (g_1, \ldots, g_r)$, $y = (n_1, \ldots, n_s)$.

$$\text{maximize} \quad yb^T \quad \textsf{subject to} \quad yL \le c, y \ge 0$$

Dual linear problem:

$$\text{minimize} \quad cx^T \quad \textsf{subject to} \quad Lx^T \ge b, x \ge 0$$

M: Maximum, m: Minimum

$$g_0(|X|-1) = g_0 \sum_{i=1}^{s} n_i \le M = m \le \sum_{i=1}^{r} g_i G_i^{(d)}(1) = g(1) - g_0$$

Application of LP bound for kissing numbers

$X \subset S^7$: E_8 root system
$|X| = 240$. $A(X) = \{1/2, 0, -1/2, -1\}$.

We want to find a polynomial $g(t) = \sum_{i \geq 0} g_i G_i^{(d)}(t)$ such that

- $g(1) > 0$, $g(\alpha) \leq 0$ for any $\alpha \in [-1, 1/2]$,
- $g_0 > 0$, and $g_i \geq 0$ for any i,
- $g(1)/g_0 < 241$.

Actually

$$g(t) = (t+1)(t+\frac{1}{2})^2 t^2 (t-\frac{1}{2})$$

satisfies the condition. ($g(1)/g_0 = 240$)
Therefore X is a kissing number configuration.
($k(24)$: same method)

Application of LP bound for optimal codes

$X \subset S^7$: E_8 root system
$|X| = 240$. $A(X) = \{1/2, 0, -1/2, -1\}$.

X attains the LP bound from

$$g(t) = (t+1)(t+\frac{1}{2})^2 t^2 (t-\frac{1}{2}),$$

where $g(1)/g_0 = 240$.

If there exists $Y \subset S^7$ such that $|Y| = 240$ and $A(Y) \subset [-1, 1/2)$.
Y also attains the same LP bound. Thus $A(Y) = \{-1, -1/2, 0\}$.

We perturb Y continuously to another spherical α-code with $0 < \alpha < 1/2$.
$g(t)$ must have the root α, a contradiction.

What concept is closely related to LP bound?

X attains the LP bound from $g(t) = \sum_{i \geq 0} g_i G_i^{(d)}(t)$.

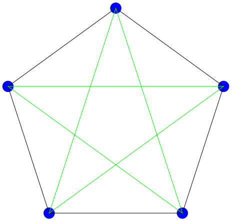
$\Rightarrow$

- $g(\alpha) = 0$ for any $\alpha \in A(X)$.
  $\rightarrow X$ has few distances. (s-distance set)
- $g_i \sum_{x,y \in X} G_i^{(d)}(\langle x, y \rangle) = 0$ for any $1 \leq i \leq \deg g$.
  $\rightarrow \sum_{x,y \in X} G_i^{(d)}(\langle x, y \rangle) = 0$ for any $1 \leq i \leq t$.
  (spherical t-design)

Spherical s-distance set

Definition 10 (s-distance set)

X is called an s-distance set if $|A(X)| = s$.



2-distance set on S^1

Problem 11

For given s and d, find largest $|X|$ in s-distance sets $X \subset S^{d-1}$. (maximum distance set)

Absolute bound on distance sets

Theorem 12 (Delsarte-Goethals-Seidel (1977))

1. *If $X \subset S^{d-1}$ is an s-distance set, then we have*

$$|X| \leq \binom{d+s-1}{s} + \binom{d+s-2}{s-1}.$$

2. *If X is an antipodal s-distance set ($X = -X$), then we have*

$$|X| \leq 2\binom{d+s-2}{s-1}.$$

X is called a tight spherical s-distance set if equality holds.

Proof of the absolute bound for s-distance sets

Proof X: s-distance set in S^{d-1}
For each $x \in X$,

$$f_x(\xi) = \prod_{\alpha \in A(X)} \frac{\langle x, \xi \rangle - \alpha}{1 - \alpha}.$$

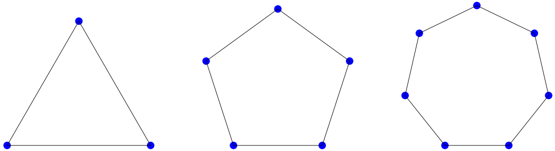
- $f_x \in P_s(S^{d-1})$.
- For $y \in X$,

$$f_x(y) = \begin{cases} 1 \text{ if } x = y, \\ 0 \text{ if } x \neq y. \end{cases}$$

$\sum_{x \in X} c_x f_x(\xi) = 0 \Rightarrow \xi = y \in X$, then $c_y = 0$.
$\{f_x\}_{x \in X}$ are linearly independent.

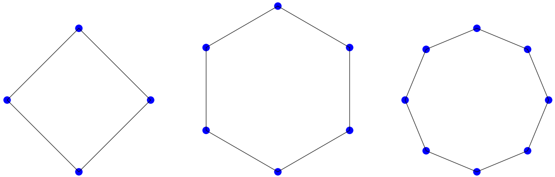
$$|X| \leq \dim P_s(S^{d-1}) = \binom{d+s-1}{s} + \binom{d+s-2}{s-1}. \square$$

Maximum distance sets on S^1



1-distance set   2-distance set   3-distance set

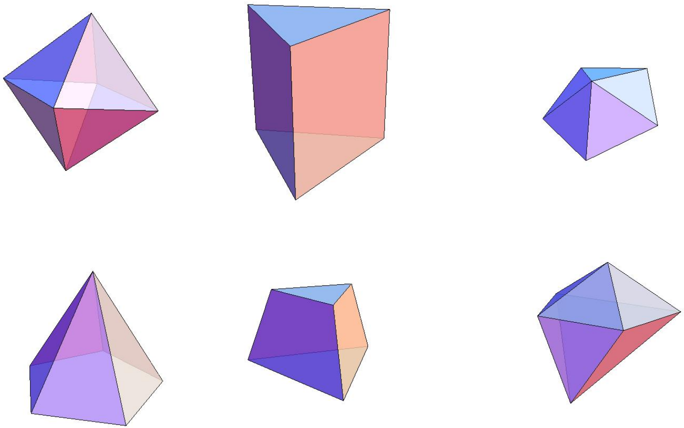
Regular $(2s+1)$-gon $\Leftrightarrow$ Maximum s-distance set



2-distance set   3-distance set   4-distance set

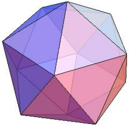
Regular $2s$-gon $\Leftrightarrow$ Maximum antipodal s-distance set

Maximum 2-distance sets on S^2



Maximum 2-distance set on S^2

Maximum distance sets on S^{d-1}



Maximum 3-distance set on S^2
(Shinohara,arXiv:1309.2047)

Maximum 2-distance set on S^{d-1}:

d	4	5	6	7	$8 \cdots 21$	22	23	$24 \cdots 93 (d \neq 46, 78)$
$\lvert X \rvert$	10	16	27	28	$\frac{d(d+1)}{2}$	275	276	$\frac{d(d+1)}{2}$

Theorem 13 (Musin and N. (2010))

1. *A maximum 3-distance set on S^7 has 120 points [subsets of the E_8 root system]*
2. *A maximum 3-distance set on S^{21} has 2025 points [subset of the minimum vectors of the Leech lattice]*

Main tools to determine maximum distance sets

- Linear programming bound, or semidefinite programming bound
- Harmonic absolute bound
- Generalization of the Larman–Rogers–Seidel theorem.

Harmonic absolute bound

Theorem 14 (N. and Shinohara (2010))

Let X be an s-distance set in S^{d-1}. Let

$$\prod_{\alpha \in X} (t - \alpha) = \sum_{i=1}^{s} g_i G_i^{(d)}(t).$$

Then we have

$$|X| \leq \sum_{i : g_i > 0} h_i,$$

where $h_i = \dim \mathrm{Harm}_i(\mathbb{R}^d) = \binom{d+i-1}{i} - \binom{d+i-3}{i-2}$.

- Musin (2009) proved the bound for $s = 2$ and $g_1 \leq 0$.
- $\sum_{i=0}^{s} h_i = \binom{d+s-1}{s} + \binom{d+s-2}{s-1}$(absolute bound)

LRS type theorem

Theorem 15 (N. (2010))

X: an s-distance set in S^{d-1} with $s \geq 2$, and $A(X) = \{\alpha_1, \alpha_2, \dots, \alpha_s\}$.
For each $i = 1, 2, \dots, s$, we define

$$K_i = \prod_{j=1,2,\dots,s, j \neq i} \frac{1-\alpha_j}{\alpha_i - \alpha_j}.$$

If $|X| \geq 2 \dim P_{s-1}(S^{d-1})$, then K_i is an integer. Moreover $|K_i|$ is bounded above by some function of d and s.

- Larman, Rogers, and Seidel (1977) proved it for $s = 2$.
- $\sum_{i=1}^{s} K_i = 1$
- $\alpha_1, \dots, \alpha_{s-1}$ are determined by $K_1, \dots, K_{s-1}$, α_s.

Spherical t-design

Let X be a finite subset on the unit sphere S^{d-1}.

Definition 16 (Spherical t-design, Delsarte-Goethals-Seidel (1977))

X is called a spherical t-design in S^{d-1} $\Leftrightarrow$

$$\frac{1}{|X|}\sum_{x\in X} f(x) = \frac{1}{|S^{d-1}|}\int_{S^{d-1}} f(x)d\mu(x)$$

for any $f(x) \in P_t(S^{d-1})$.

1. t-design $\Rightarrow$ $(t-1)$-design
2. X, Y: t-design $(X \cap Y = \emptyset)$ $\Rightarrow$ $X \cup Y$: t-design

Equivalent condition of spherical design

Theorem 17

$X \subset S^{d-1}$. The following are equivalent.

1. *X is a spherical t-design.*
2. *For each $f \in \mathrm{Harm}_i(\mathbb{R}^d)$ and any $1 \leq i \leq t$, we have*

$$\sum_{x \in X} f(x) = 0.$$

3. *For each $1 \leq i \leq t$, we have*

$$\sum_{x,y \in X} G_i^{(d)}(\langle x, y \rangle) = 0,$$

where $G_i^{(d)}$ is the Gegenbauer polynomial of degree i.

Proof of the theorem of equivalent conditions

$(1) \Leftrightarrow (2)$: $f \in P_t(S^{d-1})$ can be expressed by

$$f = c_0 + \sum_{i=1}^{t} \varphi_i, \text{ where } \varphi_i \in \mathrm{Harm}_i(\mathbb{R}^d).$$

Then

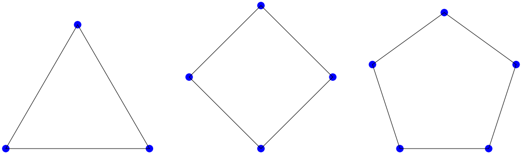
$$\frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x) d\mu(x) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} (c_0 + \sum_{i=1}^{t} \varphi_i(x)) d\mu(x) = c_0,$$

$$\frac{1}{|X|} \sum_{x \in X} f(x) = c_0 + \frac{1}{|X|} \sum_{x \in X} \sum_{i=1}^{t} \varphi_i(x).$$

$(2) \Leftrightarrow (3)$:

$$\sum_{x,y \in X} G_i^{(d)}(\langle x, y \rangle) = \sum_{x,y \in X} \sum_{j=0}^{h_i} \varphi_{i,j}(x) \varphi_{i,j}(y) = \sum_{j=0}^{h_i} (\sum_{x \in X} \varphi_{i,j}(x))^2$$

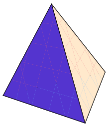
Spherical t-designs on S^1



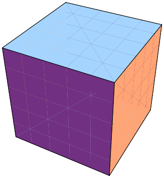
2-design $\quad$ 3-design $\quad$ 4-design $\quad \cdots$ regular n-gon

$\cdots$ $(n-1)$-design

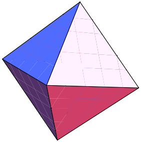
Regular polyhedron



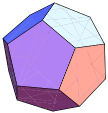
spherical 2-design
4 points



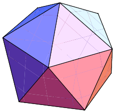
spherical 3-design
8 points



spherical 3-design
6 points

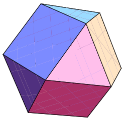


spherical 5-design
20 points

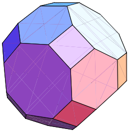


spherical 5-design
12 points

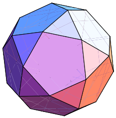
Semi-regular polyhedron



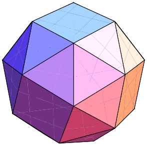
spherical 3-design
12 points



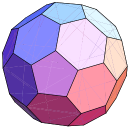
spherical 3-design
48 points



spherical 5-design
30 points



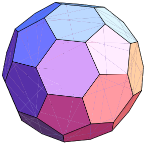
spherical 3-design
24 points



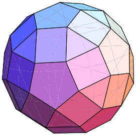
spherical 5-design
60 points

Spherical 9-design on S^2

Remark that the following are NOT semi-regular polyhedrons.



spherical 9-design
60 points
angles corresponding edges
are
$20.5424°$ or $24.8207°$



spherical 9-design
60 points
angles corresponding edges
are
$24.2511°$ or $28.3728°$

(Goethals and Seidel, The football, (1981))

Absolute bound for spherical design

Theorem 18 (Delsarte-Goethals-Seidel (1977))

1. *If X is a spherical $2e$-design on S^{d-1}, then we have*

$$|X| \geq \binom{d+e-1}{e} + \binom{d+e-2}{e-1}.$$

2. *If X is a spherical $(2e-1)$-design on S^{d-1}, then we have*

$$|X| \geq 2\binom{d+e-2}{e-1}.$$

X is called a tight spherical design if equality holds.

LP bound for spherical design

Theorem 19 (Delsarte, Goethals and Seidel (1977))

Let X be a spherical t-design in S^{d-1}. Suppose there exists a polynomial $g(x) = \sum_{i \geq 0} g_i G_i^{(d)}(x)$ s.t.

- $g(1) > 0$, $g(\alpha) \geq 0$ *for any* $\alpha \in [-1, 1]$,
- $g_0 > 0$, *and* $g_i \leq 0$ *for any* $i > t$.

Then

$$|X| \geq \frac{g(1)}{g_0}.$$

Proof of LP bound for design

Proof: $n_\alpha = |\{(x,y) \in X \times X \mid \langle x,y\rangle = \alpha\}|$

$$\sum_{x,y\in X} g(\langle x,y\rangle) = \sum_{x,y\in X}\sum_{i\geq 0} g_i G_i^{(d)}(\langle x,y\rangle)$$

$$|X|g(1) \leq |X|g(1) + \sum_{\alpha\in A(X)} n_\alpha g(\alpha)$$

$$= |X|^2 g_0 + \sum_{i>t} g_i \sum_{x,y\in X} G_i^{(d)}(\langle x,y\rangle) \leq |X|^2 g_0$$

$$|X| \geq \frac{g(1)}{g_0}. \qquad \square$$

Equality holds $\Leftrightarrow$
$g(\alpha) = 0$ and $g_i \sum_{x,y\in X} G_i^{(d)}(\langle x,y\rangle) = 0$ for any $t+1 \leq i \leq \deg g$.

Proof of the absolute bound for design

Proof for $2e$-designs: Use LP method.

$$g(x) = (\sum_{i=0}^{e} G_i^{(d)}(x))^2 = \sum_{i=0}^{2e} g_i G_i^{(d)}(x).$$

Then $g_0 = \sum_{i=0}^{e} G_i^{(d)}(1) > 0$, $g_i = 0$ for $i > t$, and $g(x) \geq 0$ for $-1 \leq x \leq 1$.

$$\begin{aligned} |X| \geq \frac{g(1)}{g_0} &= \sum_{i=0}^{e} G_i^{(d)}(1) = \sum_{i=0}^{e} \dim \mathrm{Harm}_i(\mathbb{R}^d) \\ &= \binom{d+e-1}{e} + \binom{d+e-2}{e-1}. \end{aligned}$$

Classification of tight spherical designs

Theorem 20 (Bannai–Damerell (1979,1980))

If a tight t-design on S^{d-1} for $d \geq 3$ exists, then $t \leq 5$ or $t = 7, 11$

$t = 2, 3, 11$: classified, $t = 4, 5, 7$: open.

dim.	size	t	$A(X)$	name
n	$n+1$	2	$-1/n$	simplex
n	$2n$	3	$-1, 0$	cross polytope
8	240	7	$-1, \pm 1/2, 0$	E_8 root
7	56	5	$-1, \pm 1/3$	kissing
6	27	4	$-1/2, 1/4$	kissing
24	196560	11	$-1, \pm 1/2, \pm 1/4, 0$	Leech lattice
23	4600	7	$-1, \pm 1/3, 0$	kissing
23	552	5	$-1 \pm 1/5$	equiangular lines
22	275	4	$-1/4, 1/6$	kissing

Existence and construction for spherical designs

Theorem 21 (Seymour-Zaslavsky (1984))

There exists a spherical t-design on S^d for any d and t.

Theorem 22 (Bondarenko, Radchenko, and Viazovska (Annals of Math. (2013)))

For each $N \geq c_d t^d$, there exists a spherical t-design in S^d consisting of N points, where c_d is a constant depending only on d.

Problem 23

Give a explicit construction of a spherical t-design for any d and t.

For S^2, Kuperberg (2005) gives a certain explicit construction.

Parameters s and t

X: spherical t-design and s-distance set

- $t \leq 2s$. If $X = -X$, then $t \leq 2s - 1$.
- $t = 2s$ or ($t = 2s - 1$ and $X = -X$)
  $\Leftrightarrow$ X: tight spherical design.
- $t \geq s - 1 \Rightarrow X$: distance invariant
- $t \geq 2s - 2$ or ($t \geq 2s - 3$ and $X = -X$)
  $\Rightarrow$ X has the structure of a Q-polynomial scheme.
- $t \geq 2s - 1$
  $\Rightarrow$ X is an optimal code (Levenshtein (1992)).

Problem 24

Classify spherical codes satisfying $t \geq 2s - 1$ or $t \geq 2s - 2$.

Bounds on s-distance t-design

X: s-distance set and $2e$-design on S^{d-1}

$$\binom{d+e-1}{e} + \binom{d+e-2}{e-1} \leq |X| \leq \binom{d+s-1}{s} + \binom{d+s-2}{s-1}$$

X: tight s-distance set $\Leftrightarrow$ X: tight $2s$-design (DGS(1977)).
We say X has strength t if X is a t-design but not a $(t+1)$-design

- Strength $2s \Leftrightarrow |X| = \binom{d+s-1}{s} + \binom{d+s-2}{s-1}$
- Strength $2s-1 \Rightarrow |X| \leq \binom{d+s-1}{s} + \binom{d+s-2}{s-1} - 1$
- Strength $2s-2 \Rightarrow |X| \leq$??

Theorem 25 (Cameron-Goethals-Seidel (1978), Neumaier (1981))

X: 2-distance set with strength 2.
Then $|X| \leq \binom{d+1}{2} (=$ above bound $- d)$.

New bound

Theorem 26 (N. and Suda (2011))

X: s-distance set with strength $2s-2$. Then

$$|X| \leq \binom{d+s-1}{s} + \binom{d+s-4}{s-3}$$
$$= \dim P_s(S^{d-1}) - \dim \mathrm{Harm}_{s-1}(R^d).$$

X: antipodal s-distance set (s: odd) with strength $2s-5$. Then

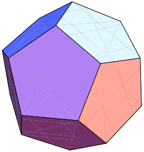
$$|X| \leq 2\binom{d+s-2}{s-1} - 2(\binom{d+s-4}{s-3} - \binom{d+s-6}{s-5})$$

Examples attaining the bound

- 2025-point 3-distance set on S^{21} with strength 4 (Maximum spherical 3-distance set)

Antipodal set:

- Dodecahedron: 20-point 5-distance set with strength 5



Summary

- Kissing number configuration, optimal code, spherical t-design, spherical s-distance set.
- Linear programming method, spherical harmonics.
- $t \geq 2s - 2 \rightarrow$ association scheme, orthogonal polynomial.

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[2] E. Bannai, E. Bannai, A survey on spherical designs and algebraic combinatorics on spheres, *European J. Combin.* 30 (2009), 1392–1425.

[3] P. Delsarte, J.M. Goethals, and J.J. Seidel, Spherical codes and designs, *Geom. Dedicata* 6 (1977), no. 3, 363–388.