球面上の代数的組合せ論入門
An introduction to algebraic combinatorics on a sphere

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代数的組合せ論「夏の学校 2014」
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Problem 1

What is a “good” finite set on the unit sphere $S^{d-1}$?

1. Coding theory (local viewpoint)
   - Spherical $u$-code → Kissing number
   - Optimal code
   - $s$-distance set

2. Design theory (global viewpoint)
   - Spherical $t$-design

$\rightarrow Q$-polynomial association scheme
1. Kissing number configurations
2. Optimal spherical codes
3. Spherical harmonics and linear programming method
4. $s$-distance sets
5. Spherical $t$-design
6. Results obtained from parameters $s$ and $t$
$X$: a finite set on $S^{d-1}$

$$A(X) := \{ \langle x, y \rangle \mid x, y \in X, x \neq y \}.$$ 

**Definition 2**

$X$ is called $u$-code if $A(X) \subset [-1, u]$.

**Problem 3**

For given $u \in [-1, 1]$ and $d$, find maximum $|X|$ in $u$-codes $X$.

$\frac{1}{2}$-codes $\leftrightarrow$ kissing number problem
Kissing number on $S^{d-1}$: $k(d)$

$k(2) = 6$

$k(3) = 12$
- Famous disagreement between Newton and Gregory (1694)
- Proved by Schutte and van der Waerden (1953), Leech (1956), ...

$k(4) = 24$ (24-cell)

$k(8) = 240$ ($E_8$ root system), LP

$k(24) = 196560$ (Minimum vectors of the Leech lattice), LP
- Odlyzko–Sloane (1979)
Kissing number on $S^{d-1}$: $k(d)$

$k(2) = 6$

For other dimensions, nobody knows $k(d)$.

$k(3) = 12$

- Famous disagreement between Newton and Gregory (1694)
- Proved by Schutte and van der Waerden (1953), Leech (1956), . . .

$k(4) = 24$ (24-cell)


$k(8) = 240$ ($E_8$ root system), LP

$k(24) = 196560$ (Minimum vectors of the Leech lattice), LP

- Odlyzko–Sloane (1979)
Problem 4

For a given $|X|$ and $d$, find smallest $u$ such that $X$ is a $u$-code on $S^{d-1}$. (optimal code)
Optimal codes on $S^2$
The strong thirteen spheres problem

13 points
Musin and Tarasov (2012)
## Optimal codes in higher dimensions

<table>
<thead>
<tr>
<th>dim.</th>
<th>size</th>
<th>$A(X)$</th>
<th>name</th>
</tr>
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<tbody>
<tr>
<td>$n$</td>
<td>$n + 1$</td>
<td>$-1/n$</td>
<td>simplex</td>
</tr>
<tr>
<td>$n$</td>
<td>$2n$</td>
<td>$-1, 0$</td>
<td>cross polytope</td>
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<tr>
<td>4</td>
<td>10</td>
<td>$-2/3, 1/6$</td>
<td>Petersen graph</td>
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<tr>
<td>4</td>
<td>120</td>
<td>$-1, \pm 1/2, 0, (\pm 1 \pm \sqrt{5})/4$</td>
<td>600-cell</td>
</tr>
<tr>
<td>8</td>
<td>240</td>
<td>$-1, \pm 1/2, 0$</td>
<td>$E_8$ root</td>
</tr>
<tr>
<td>7</td>
<td>56</td>
<td>$-1, \pm 1/3$</td>
<td>kissing</td>
</tr>
<tr>
<td>6</td>
<td>27</td>
<td>$-1/2, 1/4$</td>
<td>kissing</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
<td>$-3/5, 1/5$</td>
<td>kissing</td>
</tr>
<tr>
<td>24</td>
<td>196560</td>
<td>$-1, \pm 1/2, \pm 1/4, 0$</td>
<td>Leech lattice</td>
</tr>
<tr>
<td>23</td>
<td>4600</td>
<td>$-1, \pm 1/3, 0$</td>
<td>kissing</td>
</tr>
<tr>
<td>22</td>
<td>891</td>
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<td>kissing</td>
</tr>
<tr>
<td>23</td>
<td>552</td>
<td>$-1 \pm 1/5$</td>
<td>equiangular lines</td>
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<td>22</td>
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<td>$-1/4, 1/6$</td>
<td>kissing</td>
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<tr>
<td>21</td>
<td>162</td>
<td>$-2/7, 1/7$</td>
<td>kissing</td>
</tr>
<tr>
<td>22</td>
<td>100</td>
<td>$-4/11, 1/11$</td>
<td>Higman-Sims</td>
</tr>
<tr>
<td>$q^{3+1}/q+1$</td>
<td>$(q + 1)(q^3 + 1)$</td>
<td>$-1/q, 1/q^2$</td>
<td>Cameron et al’78</td>
</tr>
</tbody>
</table>

Proved by LP or SDP (SDP: only for Petersen graph)
Homogeneous polynomials

$\text{Hom}_i(\mathbb{R}^d)$ denotes the linear space of homogeneous polynomials of degree $i$, in $d$ variables $x_1, \ldots, x_d$.

$$\dim \text{Hom}_i(\mathbb{R}^d) = \binom{d+i-1}{i}.$$  

($i$-combination with repetitions)

$$P_i(\mathbb{R}^d) = \bigoplus_{j=0}^{i} \text{Hom}_j(\mathbb{R}^d).$$

$$\dim P_i(\mathbb{R}^d) = \sum_{j=0}^{i} \binom{d+j-1}{j} = \binom{d+i}{i}.$$
Harmonic polynomials

Laplacian: \( \Delta f = \sum_{k=1}^{d} \frac{\partial^2 f}{\partial x_k^2} \) for \( f \in \text{Hom}_i(\mathbb{R}^d) \).

\[ \Delta : \text{Hom}_i(\mathbb{R}^d) \to \text{Hom}_{i-2}(\mathbb{R}^d) \] (linear map)

- \( \text{Harm}_i(\mathbb{R}^d) := \ker \Delta = \{ f \mid \Delta f = 0 \} \)
- An element of \( \text{Harm}_i(\mathbb{R}^d) \) is called a harmonic polynomial.

Actually \( \Delta \) is surjective.

\[
\dim \ker \Delta = \dim \text{Hom}_i(\mathbb{R}^d) - \dim \text{Im} \Delta
\]
\[
\dim \text{Harm}_i(\mathbb{R}^d) = \dim \text{Hom}_i(\mathbb{R}^d) - \dim \text{Hom}_{i-2}(\mathbb{R}^d)
= \binom{d+i-1}{i} - \binom{d+i-3}{i-2}
\]
Basic results on harmonic polynomials

Let \( r^2 = \sum_{i=1}^{d} x_i^2 \).

**Theorem 5**

1

\[
\text{Hom}_i(\mathbb{R}^d) = \text{Harm}_i(\mathbb{R}^d) \oplus r^2 \text{Hom}_{i-2}(\mathbb{R}^d).
\]

2

\[
\text{Hom}_i(\mathbb{R}^d) = \bigoplus_{j=0}^{[i/2]} r^{2j} \text{Harm}_{i-2j}(\mathbb{R}^d).
\]

3

\[
\text{Harm}_i(\mathbb{R}^d) \perp \text{Harm}_j(\mathbb{R}^d) (i \neq j)
\]

*with respect to*

\[
\langle\langle f, g \rangle\rangle = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x)g(x)d\mu(x).
\]
Polynomials on a sphere

\[ \text{Hom}_i(S^{d-1}) = \{ f|_{S^{d-1}} \mid f \in \text{Hom}_i(\mathbb{R}^d) \} \]
\[ P_i(S^{d-1}) = \{ f|_{S^{d-1}} \mid f \in P_i(\mathbb{R}^d) \} \]
\[ \text{Harm}_i(S^{d-1}) = \{ f|_{S^{d-1}} \mid f \in \text{Harm}_i(\mathbb{R}^d) \} \]

Theorem 6

\[ \text{Harm}_i(S^{d-1}) \cong \text{Harm}_i(\mathbb{R}^d) \]

\[ \text{Hom}_i(\mathbb{R}^d) = \bigoplus_{j=0}^{\lfloor i/2 \rfloor} r^{2j} \text{Harm}_{i-2j}(\mathbb{R}^d) \Rightarrow \text{Hom}_i(S^{d-1}) \cong \bigoplus_{j=0}^{\lfloor i/2 \rfloor} \text{Harm}_{i-2j}(\mathbb{R}^d) \]

\[ P_i(S^{d-1}) = \sum_{j=0}^{i} \text{Hom}_j(S^{d-1}) \cong \bigoplus_{j=0}^{i} \text{Harm}_j(\mathbb{R}^d) \]
Dimension of $P_i(S^{d-1})$

$$P_i(S^{d-1}) = \bigoplus_{j=0}^{i} \text{Harm}_j(\mathbb{R}^d)$$

$$\dim P_i(S^{d-1}) = \sum_{j=0}^{i} \dim \text{Harm}_j(\mathbb{R}^d)$$

$$= \sum_{j=0}^{i} \left( \binom{d+j-1}{j} - \binom{d+j-3}{j-2} \right)$$

$$= \binom{d+i-1}{i} + \binom{d+i-2}{i-1}.$$
Gegenbauer polynomials:

\[ G_0^{(d)}(t) = 1, \quad G_1^{(d)}(t) = dt, \]

\[ tG_{i-1}^{(d)}(t) = \frac{i}{d + 2i - 2} G_i^{(d)}(t) + \frac{d + i - 4}{d + 2i - 6} G_{i-2}^{(d)}(t). \]

Gegenbauer polynomials form a sequence of orthogonal polynomials w.r.t.

\[ (f, g) = \int_{-1}^{1} f(t)g(t)(1 - t^2)^{(d-3)/2} dx \]

Note \( G_i^{(d)}(1) = \dim \text{Harm}_i(\mathbb{R}^d). \)
Let $h_i = \dim \text{Harm}_i(\mathbb{R}^d)$.
Let $\{\varphi_{i,1}, \ldots, \varphi_{i,h_i}\}$ be an orthonormal basis of $\text{Harm}_i(\mathbb{R}^d)$ w.r.t. $\langle\langle , \rangle\rangle$.
Let $\langle , \rangle$ be the usual inner product in $\mathbb{R}^d$.

**Theorem 7 (Addition formula)**

*For any $x, y \in S^{d-1}$, we have*

$$
\sum_{j=0}^{h_i} \varphi_{i,j}(x) \varphi_{i,j}(y) = G_i^{(d)}(\langle x, y \rangle).
$$
Positive definiteness of $G_{i}^{(d)}(t)$

**Theorem 8**

For arbitrary points $x_1, \ldots, x_n \in S^{d-1}$, and real variables $\xi_1, \ldots, \xi_n$, we have

$$
\sum_{i,j=1}^{n} G_{k}^{(d)}(\langle x_i, x_j \rangle) \xi_i \xi_j \geq 0,
$$

or equivalently $(G_{k}^{d}(\langle x_i, x_j \rangle))_{i,j}$ is positive semidefinite.

**Proof:**

$$(G_{k}^{(d)}(\langle x_i, x_j \rangle))_{i,j} = (\sum_{l=0}^{h} \varphi_{k,l}(x_i) \varphi_{k,l}(x_j))_{i,j} = (\varphi_{k,l}(x_i))_{i,l}^t (\varphi_{k,l}(x_i))_{i,l} \succeq 0$$

**Corollary:** $\sum_{i,j=1}^{n} G_{k}^{(d)}(\langle x_i, x_j \rangle) \geq 0.$
Theorem 9 (Delsarte, Goethals and Seidel (1977))

Let $X$ be a subset in $S^{d-1}$. Suppose there exists a polynomial $g(t) = \sum_{i \geq 0} g_i G_i^{(d)}(t)$ s.t.

- $g(1) > 0$, $g(\alpha) \leq 0$ for any $\alpha \in A(X)$,
- $g_0 > 0$, and $g_i \geq 0$ for any $i$.

Then

$$|X| \leq \frac{g(1)}{g_0}.$$
Proof of LP bound

Proof: \( n_\alpha = |\{(x, y) \in X \times X \mid \langle x, y \rangle = \alpha\}| \)

\[
\sum_{x,y\in X} g(\langle x, y \rangle) = \sum_{x,y\in X} \sum_{i \geq 0} g_i G_i^{(d)}(\langle x, y \rangle)
\]

\[
|X|g(1) \geq |X|g(1) + \sum_{\alpha \in A(X)} n_\alpha g(\alpha)
\]

\[
= |X|^2 g_0 + \sum_{i \geq 1} g_i \sum_{x,y \in X} G_i^{(d)}(\langle x, y \rangle) \geq |X|^2 g_0
\]

\[
|X| \leq \frac{g(1)}{g_0}.
\]

Equality holds \( \iff \)

\( g(\alpha) = 0 \) and \( g_i \sum_{x,y \in X} G_i^{(d)}(\langle x, y \rangle) = 0 \) for any \( 1 \leq i \leq \deg g \).
Linear programming bound for spherical sets.

\[ A(X) = \{\alpha_1, \ldots, \alpha_s\}, \quad n_i = \frac{1}{|X|} \left| \{(x, y) \in X \times X \mid \langle x, y \rangle = \alpha_i\} \right| \]

- \( L = -(G^{(d)}_j(\alpha_i))_{1 \leq i \leq s, 1 \leq j \leq r} \),
- \( b = (g_0, \ldots, g_0) \), \( c = (G^{(d)}_1(1), \ldots, G^{(d)}_r(1)) \),
- \( x = (g_1, \ldots, g_r) \), \( y = (n_1, \ldots, n_s) \).

maximize \( yb^T \) subject to \( yL \leq c, y \geq 0 \)

Dual linear problem:

minimize \( cx^T \) subject to \( Lx^T \geq b, x \geq 0 \)

\( M \): Maximum, \( m \): Minimum

\[ g_0(|X| - 1) = g_0 \sum_{i=1}^{s} n_i \leq M = m \leq \sum_{i=1}^{r} g_i G^{(d)}_i(1) = g(1) - g_0 \]
Application of LP bound for kissing numbers

\[ X \subset S^7: \ E_8 \text{ root system} \]
\[ |X| = 240. \ A(X) = \{1/2, 0, -1/2, -1\}. \]

We want to find a polynomial \( g(t) = \sum_{i \geq 0} g_i G_i^{(d)}(t) \) such that

- \( g(1) > 0, \ g(\alpha) \leq 0 \) for any \( \alpha \in [-1, 1/2] \),
- \( g_0 > 0, \) and \( g_i \geq 0 \) for any \( i \),
- \( g(1)/g_0 < 241. \)

Actually

\[
g(t) = (t + 1)(t + \frac{1}{2})^2 t^2 (t - \frac{1}{2})
\]

satisfies the condition. \( (g(1)/g_0 = 240) \)

Therefore \( X \) is a kissing number configuration.

\( (k(24): \text{same method}) \)
Application of LP bound for optimal codes

\[ X \subset S^7: \ E_8 \text{ root system} \]
\[ |X| = 240. \ A(X) = \{1/2, 0, -1/2, -1\}. \]

\[ X \text{ attains the LP bound from} \]
\[ g(t) = (t + 1)(t + \frac{1}{2})^2t^2(t - \frac{1}{2}), \]

where \( g(1)/g_0 = 240. \)

If there exists \( Y \subset S^7 \) such that \( |Y| = 240 \) and \( A(Y) \subset [-1, 1/2) \), \( Y \) also attains the same LP bound. Thus \( A(Y) = \{-1, -1/2, 0\} \).

We perturb \( Y \) continuously to another spherical \( \alpha \)-code with \( 0 < \alpha < 1/2 \).
\( g(t) \) must have the root \( \alpha \), a contradiction.
What concept is closely related to LP bound?

$X$ attains the LP bound from $g(t) = \sum_{i \geq 0} g_i G_i^{(d)}(t)$.

$\Rightarrow$

- $g(\alpha) = 0$ for any $\alpha \in A(X)$.
  $\rightarrow X$ has few distances. ($s$-distance set)

- $g_i \sum_{x,y \in X} G_i^{(d)}(\langle x, y \rangle) = 0$ for any $1 \leq i \leq \deg g$.
  $\rightarrow \sum_{x,y \in X} G_i^{(d)}(\langle x, y \rangle) = 0$ for any $1 \leq i \leq t$.
  (spherical $t$-design)
Definition 10 ($s$-distance set)

$X$ is called an $s$-distance set if $|A(X)| = s$.

Problem 11

For given $s$ and $d$, find largest $|X|$ in $s$-distance sets $X \subset S^{d-1}$. (maximum distance set)
Theorem 12 (Delsarte-Goethals-Seidel (1977))

1. If $X \subset S^{d-1}$ is an $s$-distance set, then we have

$$|X| \leq \binom{d + s - 1}{s} + \binom{d + s - 2}{s - 1}.$$ 

2. If $X$ is an antipodal $s$-distance set ($X = -X$), then we have

$$|X| \leq 2 \binom{d + s - 2}{s - 1}.$$ 

$X$ is called a tight spherical $s$-distance set if equality holds.
Proof of the absolute bound for $s$-distance sets

**Proof**  
$X$: $s$-distance set in $S^{d-1}$

For each $x \in X$,

$$f_x(\xi) = \prod_{\alpha \in A(X)} \frac{\langle x, \xi \rangle - \alpha}{1 - \alpha}.$$  

- $f_x \in P_s(S^{d-1})$.
- For $y \in X$,

$$f_x(y) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

$$\sum_{x \in X} c_x f_x(\xi) = 0 \Rightarrow \xi = y \in X, \text{ then } c_y = 0.$$  

$\{f_x\}_{x \in X}$ are linearly independent.

$$|X| \leq \dim P_s(S^{d-1}) = \binom{d + s - 1}{s} + \binom{d + s - 2}{s - 1}. \square$$
Maximum distance sets on $S^1$

1-distance set  2-distance set  3-distance set  4-distance set
Regular $(2s + 1)$-gon $\Leftrightarrow$ Maximum $s$-distance set

2-distance set  3-distance set  4-distance set
Regular $2s$-gon $\Leftrightarrow$ Maximum antipodal $s$-distance set
Maximum 2-distance sets on $S^2$
Maximum distance sets on $S^{d-1}$

Maximum 3-distance set on $S^2$
(Shinohara, arXiv:1309.2047)

Maximum 2-distance set on $S^{d-1}$:

<table>
<thead>
<tr>
<th>$d$</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8 \cdots 21</th>
<th>22</th>
<th>23</th>
<th>24 \cdots 93 (d \neq 46, 78)</th>
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<tr>
<td>$</td>
<td>X</td>
<td>$</td>
<td>10</td>
<td>16</td>
<td>27</td>
<td>28</td>
<td>$\frac{d(d+1)}{2}$</td>
<td>275</td>
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</table>

Theorem 13 (Musin and N. (2010))

1. A maximum 3-distance set on $S^7$ has 120 points [subsets of the $E_8$ root system]

2. A maximum 3-distance set on $S^{21}$ has 2025 points [subset of the minimum vectors of the Leech lattice]
Main tools to determine maximum distance sets

- Linear programming bound, or semidefinite programming bound
- Harmonic absolute bound
Theorem 14 (N. and Shinohara (2010))

Let $X$ be an $s$-distance set in $S^{d-1}$. Let

$$\prod_{\alpha \in X} (t - \alpha) = \sum_{i=1}^{s} g_i G^{(d)}_i (t).$$

Then we have

$$|X| \leq \sum_{i: g_i > 0} h_i,$$

where $h_i = \dim \text{Harm}_i(\mathbb{R}^d) = \binom{d+i-1}{i} - \binom{d+i-3}{i-2}$.

- Musin (2009) proved the bound for $s = 2$ and $g_1 \leq 0$.
- $\sum_{i=0}^{s} h_i = \binom{d+s-1}{s} + \binom{d+s-2}{s-1}$ (absolute bound)
Theorem 15 (N. (2010))

\( X \): an \( s \)-distance set in \( S^{d-1} \) with \( s \geq 2 \), and
\( A(X) = \{\alpha_1, \alpha_2, \ldots, \alpha_s\} \).
For each \( i = 1, 2, \ldots, s \), we define

\[
K_i = \prod_{j=1,2,\ldots,s,j\neq i} \frac{1 - \alpha_j}{\alpha_i - \alpha_j}.
\]

If \( |X| \geq 2 \dim P_{s-1}(S^{d-1}) \), then \( K_i \) is an integer. Moreover \( |K_i| \) is bounded above by some function of \( d \) and \( s \).

- Larman, Rogers, and Seidel (1977) proved it for \( s = 2 \).
- \( \sum_{i=1}^{s} K_i = 1 \)
- \( \alpha_1, \ldots, \alpha_{s-1} \) are determined by \( K_1, \ldots, K_{s-1}, \alpha_s \).
Let $X$ be a finite subset on the unit sphere $S^{d-1}$.

**Definition 16 (Spherical $t$-design, Delsarte-Goethals-Seidel (1977))**

$X$ is called a spherical $t$-design in $S^{d-1}$ ⇔

$$
\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x) \, d\mu(x)
$$

for any $f(x) \in P_t(S^{d-1})$.

1. $t$-design $\Rightarrow$ $(t - 1)$-design
2. $X, Y$: $t$-design ($X \cap Y = \emptyset$) $\Rightarrow$ $X \cup Y$: $t$-design
Theorem 17

$X \subset S^{d-1}$. The following are equivalent.

1. $X$ is a spherical $t$-design.

2. For each $f \in \text{Harm}_i(\mathbb{R}^d)$ and any $1 \leq i \leq t$, we have
   \[ \sum_{x \in X} f(x) = 0. \]

3. For each $1 \leq i \leq t$, we have
   \[ \sum_{x,y \in X} G_i^{(d)}(\langle x, y \rangle) = 0, \]
   where $G_i^{(d)}$ is the Gegenbauer polynomial of degree $i$. 

Equivalent condition of spherical design
Proof of the theorem of equivalent conditions

(1) $\Leftrightarrow$ (2): $f \in P_t(S^{d-1})$ can be expressed by

$$f = c_0 + \sum_{i=1}^{t} \varphi_i, \text{ where } \varphi_i \in \text{Harm}_i(\mathbb{R}^d).$$

Then

$$\frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x) d\mu(x) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} (c_0 + \sum_{i=1}^{t} \varphi_i(x)) d\mu(x) = c_0,$$

$$\frac{1}{|X|} \sum_{x \in X} f(x) = c_0 + \frac{1}{|X|} \sum_{x \in X} \sum_{i=1}^{t} \varphi_i(x).$$

(2) $\Leftrightarrow$ (3):

$$\sum_{x,y \in X} G_i^{(d)}(\langle x, y \rangle) = \sum_{x,y \in X} \sum_{j=0}^{h_i} \varphi_{i,j}(x) \varphi_{i,j}(y) = \sum_{j=0}^{h_i} (\sum_{x \in X} \varphi_{i,j}(x))^2$$
Spherical $t$-designs on $S^1$

2-design  
3-design  
4-design  
\cdots \quad \text{regular $n$-gon}  
\cdots \quad (n - 1)$-design
Regular polyhedron

spherical 2-design
  4 points

spherical 3-design
  8 points

spherical 3-design
  6 points

spherical 5-design
  20 points

spherical 5-design
  12 points
Semi-regular polyhedron

spherical 3-design
12 points

spherical 3-design
48 points

spherical 5-design
30 points

spherical 3-design
24 points

spherical 5-design
60 points
Remark that the following are NOT semi-regular polyhedrons.

Spherical 9-design on $S^2$

- spherical 9-design
  - 60 points
  - angles corresponding edges are
    - $20.5424^\circ$ or $24.8207^\circ$
  - (Goethals and Seidel, The football, (1981))

- spherical 9-design
  - 60 points
  - angles corresponding edges are
    - $24.2511^\circ$ or $28.3728^\circ$
Theorem 18 (Delsarte-Goethals-Seidel (1977))

1. If $X$ is a spherical $2e$-design on $S^{d-1}$, then we have

$$|X| \geq \binom{d + e - 1}{e} + \binom{d + e - 2}{e - 1}.$$ 

2. If $X$ is a spherical $(2e - 1)$-design on $S^{d-1}$, then we have

$$|X| \geq 2 \binom{d + e - 2}{e - 1}.$$ 

$X$ is called a **tight** spherical design if equality holds.
Theorem 19 (Delsarte, Goethals and Seidel (1977))

Let $X$ be a spherical $t$-design in $S^{d-1}$. Suppose there exists a polynomial $g(x) = \sum_{i \geq 0} g_i G_i^{(d)}(x)$ s.t.

- $g(1) > 0$, $g(\alpha) \geq 0$ for any $\alpha \in [-1, 1]$,
- $g_0 > 0$, and $g_i \leq 0$ for any $i > t$.

Then

$$|X| \geq \frac{g(1)}{g_0}.$$
Proof of LP bound for design

Proof: \[ n_\alpha = |\{(x, y) \in X \times X \mid \langle x, y \rangle = \alpha\}| \]

\[
\sum_{x, y \in X} g(\langle x, y \rangle) = \sum_{x, y \in X} \sum_{i \geq 0} g_i G_i^{(d)}(\langle x, y \rangle) 
\]

\[
|X| g(1) \leq |X| g(1) + \sum_{\alpha \in A(X)} n_\alpha g(\alpha) 
\]

\[
= |X|^2 g_0 + \sum_{i > t} g_i \sum_{x, y \in X} G_i^{(d)}(\langle x, y \rangle) \leq |X|^2 g_0 
\]

\[
|X| \geq \frac{g(1)}{g_0}. \quad \square
\]

Equality holds \(\Leftrightarrow\)

\[ g(\alpha) = 0 \text{ and } g_i \sum_{x, y \in X} G_i^{(d)}(\langle x, y \rangle) = 0 \text{ for any } t + 1 \leq i \leq \deg g. \]
Proof of the absolute bound for design

**Proof for 2e-designs**: Use LP method.

\[
g(x) = \left( \sum_{i=0}^{e} G_{i}^{(d)}(x) \right)^2 = \sum_{i=0}^{2e} g_{i} G_{i}^{(d)}(x).
\]

Then \( g_0 = \sum_{i=0}^{e} G_{i}^{(d)}(1) > 0, \ g_i = 0 \) for \( i > t \), and \( g(x) \geq 0 \) for \( -1 \leq x \leq 1 \).

\[
|X| \geq \frac{g(1)}{g_0} = \sum_{i=0}^{e} G_{i}^{(d)}(1) = \sum_{i=0}^{e} \dim \text{Harm}_i(\mathbb{R}^d)
\]

\[
= \binom{d+e-1}{e} + \binom{d+e-2}{e-1}.
\]
Classification of tight spherical designs

Theorem 20 (Bannai–Damerell (1979,1980))

If a tight $t$-design on $S^{d-1}$ for $d \geq 3$ exists, then $t \leq 5$ or $t = 7, 11$

$t = 2, 3, 11$: classified, $t = 4, 5, 7$: open.

<table>
<thead>
<tr>
<th>dim.</th>
<th>size</th>
<th>t</th>
<th>$A(X)$</th>
<th>name</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$n+1$</td>
<td>2</td>
<td>$-1/n$</td>
<td>simplex</td>
</tr>
<tr>
<td>$n$</td>
<td>$2n$</td>
<td>3</td>
<td>$-1,0$</td>
<td>cross polytope</td>
</tr>
<tr>
<td>8</td>
<td>240</td>
<td>7</td>
<td>$-1,\pm1/2,0$</td>
<td>$E_8$ root</td>
</tr>
<tr>
<td>7</td>
<td>56</td>
<td>5</td>
<td>$-1,\pm1/3$</td>
<td>kissing</td>
</tr>
<tr>
<td>6</td>
<td>27</td>
<td>4</td>
<td>$-1/2,1/4$</td>
<td>kissing</td>
</tr>
<tr>
<td>24</td>
<td>196560</td>
<td>11</td>
<td>$-1,\pm1/2,\pm1/4,0$</td>
<td>Leech lattice</td>
</tr>
<tr>
<td>23</td>
<td>4600</td>
<td>7</td>
<td>$-1,\pm1/3,0$</td>
<td>kissing</td>
</tr>
<tr>
<td>23</td>
<td>552</td>
<td>5</td>
<td>$-1 \pm 1/5$</td>
<td>equiangular lines</td>
</tr>
<tr>
<td>22</td>
<td>275</td>
<td>4</td>
<td>$-1/4,1/6$</td>
<td>kissing</td>
</tr>
</tbody>
</table>
Existence and construction for spherical designs

**Theorem 21 (Seymour-Zaslavsky (1984))**

There exists a spherical $t$-design on $S^d$ for any $d$ and $t$.

**Theorem 22 (Bondarenko, Radchenko, and Viazovska (Annals of Math. (2013)))**

For each $N \geq c_d t^d$, there exists a spherical $t$-design in $S^d$ consisting of $N$ points, where $c_d$ is a constant depending only on $d$.

**Problem 23**

Give a explicit construction of a spherical $t$-design for any $d$ and $t$.

For $S^2$, Kuperberg (2005) gives a certain explicit construction.
Parameters $s$ and $t$

$X$: spherical $t$-design and $s$-distance set

- $t \leq 2s$. If $X = -X$, then $t \leq 2s - 1$.
- $t = 2s$ or $(t = 2s - 1$ and $X = -X)$
  $\iff X$: tight spherical design.
- $t \geq s - 1 \Rightarrow X$: distance invariant
- $t \geq 2s - 2$ or $(t \geq 2s - 3$ and $X = -X)$
  $\Rightarrow X$ has the structure of a $Q$-polynomial scheme.
- $t \geq 2s - 1$
  $\Rightarrow X$ is an optimal code (Levenshtein (1992)).

Problem 24

Classify spherical codes satisfying $t \geq 2s - 1$ or $t \geq 2s - 2$. 
Bounds on $s$-distance $t$-design

$X$: $s$-distance set and $2e$-design on $S^{d-1}$

\[
\binom{d + e - 1}{e} + \binom{d + e - 2}{e - 1} \leq |X| \leq \binom{d + s - 1}{s} + \binom{d + s - 2}{s - 1}
\]

$X$: tight $s$-distance set $\iff X$: tight $2s$-design (DGS(1977)).

We say $X$ has strength $t$ if $X$ is a $t$-design but not a $(t + 1)$-design

- Strength $2s \iff |X| = \binom{d+s-1}{s} + \binom{d+s-2}{s-1}$
- Strength $2s - 1 \implies |X| \leq \binom{d+s-1}{s} + \binom{d+s-2}{s-1} - 1$
- Strength $2s - 2 \implies |X| \leq ??$

Theorem 25 (Cameron-Goethals-Seidel (1978), Neumaier (1981))

$X$: 2-distance set with strength 2.

Then $|X| \leq \binom{d+1}{2}$ (\(= \text{above bound} - d\)).
Theorem 26 (N. and Suda (2011))

**X**: $s$-distance set with strength $2s - 2$. Then

$$|X| \leq \binom{d + s - 1}{s} + \binom{d + s - 4}{s - 3} = \dim P_s(S^{d-1}) - \dim \text{Harm}_{s-1}(R^d).$$

**X**: antipodal $s$-distance set ($s$: odd) with strength $2s - 5$. Then

$$|X| \leq 2\binom{d + s - 2}{s - 1} - 2\left(\binom{d + s - 4}{s - 3} - \binom{d + s - 6}{s - 5}\right)$$
Examples attaining the bound

- 2025-point 3-distance set on $S^{21}$ with strength 4 (Maximum spherical 3-distance set)

Antipodal set:
- Dodecahedron: 20-point 5-distance set with strength 5
Summary

- Kissing number configuration, optimal code, spherical $t$-design, spherical $s$-distance set.
- Linear programming method, spherical harmonics.
- $t \geq 2s - 2 \rightarrow$ association scheme, orthogonal polynomial.

References:

