Remarks on generalizations of association schemes and Design theories Part I

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Thm: A tight $2t$-design on X can be considered as a Q-poly. (association) scheme.

X =?

(i): $X =$ a Johnson scheme.

(ii): $X =$ a Q-poly. scheme (Delsarte '73).

(iii): $X =$ a sphere (Delsarte–Goethals–Seidel '77).

(iv): $X =$ a rank one compact symmetric space (Bannai–Hoggar '80's).

Rem: (ii) $\Rightarrow$ (i), (iv) $\Rightarrow$ (iii)

Goal: Understand (ii) and (iv) as examples of one fundamental thorem.

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(ii): $X =$ a Q-poly. scheme.

(iv): $X =$ a rank one compact symmetric space.

Goal: Understand (ii) and (iv) as examples of one fundamental thorem.

What we have to do?

Step 1: Define a generalization of Q-poly. schemes including rank one compact symmetric spaces.

Step 2: Define designs on such generalized schemes.

Step 3: Prove the theorem.

Recent generalizations from other view points

Kuribayashi–Matsuo, "Association schemeoids and Their Categories", to appear in Applied Categorical Structures.

Barg–Skriganov, "Association schemes on general measure spaces and zero-dimensional Abelian groups", arXiv:1310.5359.

Thm: A tight $2t$-design on X can be considered as a Q-poly. (association) scheme.

Plan of this talk:

(1): Prove the theorem for $X =$ a Q-polynomial scheme.

(2): Generalize Q-polynomial schemes and designs on them.

(3): Check a sphere can be considered as such a generalized scheme.

(4): Prove the theorem for such generalized schemes.

(1) : morning, **(2),(3),(4)** : afternoon

§1: For X = a Q-polynomial scheme

I : a $(d+1)$-points set. X : a finite set with $|X| \geq 2$.

$R : X \times X \to I$: a symm. surj. map. $R_i := R^{-1}(i)$.

When is $(X, \{R_i\}_{i \in I})$ a Q-poly. scheme?

$\mathbb{C}^X$: the set of $\mathbb{C}$-valued functions on X.

$\langle f, g \rangle_X := \sum_{x \in X} f(x)\overline{g(x)}$ for $f, g \in \mathbb{C}^X$.

$M(X, \mathbb{C}) := \mathbb{C}^{X \times X} \simeq \mathrm{End}(\mathbb{C}^X)$.

$R^* : \mathbb{C}^I \to M(X, \mathbb{C}) \simeq \mathrm{End}(\mathbb{C}^X)$.

$R : X \times X \to I$: a symm. surj. map. $R_i := R^{-1}(i)$.
$R^* : \mathbb{C}^I \to \mathbb{C}^{X \times X} =: M(X, \mathbb{C}) \simeq \mathrm{End}(\mathbb{C}^X)$.

Fact: $(X, \{R_i\}_{i \in I})$ is a Q-polynomial scheme $\iff$ there exist filterations

$$\{\text{constants}\} = P_0(I) \subset P_1(I) \subset \cdots \subset P_d(I) = \mathbb{C}^I$$
$$\{\text{constants}\} = P_0(X) \subset P_1(X) \subset \cdots \subset P_d(X) = \mathbb{C}^X$$

such that

(i): $P_j(I) \cdot P_k(I) = P_{j+k}(I)$ for j, k with $j + k \le d$.

(ii): $\dim P_1(I) = 2$ ($\iff \dim P_j(I) = j + 1$).

(iii): $R^*(P_j(I)) = \text{Span-}\{\pi_0, \pi_1, \ldots, \pi_j\}$ for each $j = 0, \ldots, d$ where $\pi_j \in \mathrm{End}(\mathbb{C}^X)$ is the orthogonal projection onto $P_j(X)$.

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$$\{\text{constants}\} = P_0(X) \subset P_1(X) \subset \cdots \subset P_d(X) = \mathbb{C}^X$$

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(ii): $\dim P_1(I) = 2$ ($\iff$ $\dim P_j(I) = j+1$).

(iii): $R^*(P_j(I)) = \text{Span-}\{\pi_0, \pi_1, \ldots, \pi_j\}$.

Rem:

$\overline{P_j(I)} = P_j(I)$, $\overline{P_j(X)} = P_j(X)$.

$P_j(X) \cdot P_k(X) = P_{j+k}(X)$.

$\mathfrak{A}_X := R^*\mathbb{C}^I = \text{Span-}\{\pi_j \mid j = 0, \ldots, d\}$: the Bose–Mesner algebra.

There exists $i_0 \in I$ such that $R_{i_0} = \Delta := \{(x, x) \mid x \in X\}$.

We fix a Q-poly. scheme X and such a filteration

$$\{\text{constants}\} = P_0(X) \subsetneqq P_1(X) \subsetneqq \cdots \subsetneqq P_d(X) = \mathbb{C}^X$$

Def: $\emptyset \neq Y \subset X$ is a t-design $(1 \leq t \leq d)$

$\overset{\text{def}}{\Longleftrightarrow}$ For each $f \in P_t(X)$,

$$\frac{1}{|X|}\sum_{x \in X} f(x) = \frac{1}{|Y|}\sum_{y \in Y} f(y).$$

Rem: t-designs on a Johnson scheme

$\Leftrightarrow$ Combinatorial t-designs.

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Thm (Fisher's inequality): For any $2t$-design Y on X $(2t \leq d)$,

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Proof: We show the map $P_t(X) \to \mathbb{C}^Y$, $f \mapsto f|_Y$ preserves the natural inner-products up to scalar. In fact, for each $f, g \in P_t(X)$,

$$\begin{aligned}\langle f, g\rangle_X &= \frac{|X|}{|X|} \sum_{x \in X} f(x)\overline{g(x)} \\ &= \frac{|X|}{|Y|} \sum_{y \in Y} f(y)\overline{g(y)} = \frac{|X|}{|Y|} \langle f|_Y, g|_Y\rangle_Y\end{aligned}$$

(Q.E.D.)

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$$= \frac{|X|}{|Y|}\sum_{y\in Y} f(y)\overline{g(y)} = \frac{|X|}{|Y|}\langle f|_Y, g|_Y\rangle_Y$$

(Q.E.D.)

We used $P_t(X) \cdot \overline{P_t(X)} = P_t(X) \cdot P_t(X) = P_{2t}(X)$.

Thm (Fisher's inequality): For any $2t$-design Y on X $(2t \le d)$,

$$|Y| \ge \dim_{\mathbb{C}} P_t(X).$$

$$|Y| = \dim_{\mathbb{C}} P_t(X) \overset{\text{def}}{\longleftrightarrow} Y \text{ is tight.}$$

Ex:

$\Omega = \mathbb{F}_2^3 \setminus \{\mathbf{0}\}$. $X = \binom{\Omega}{3}$: a Johnson scheme.

$Y = \{V \setminus \{\mathbf{0}\} \mid V \subset \mathbb{F}_2^3,\ \text{2-dim. subspace}\} \subset X$.

$\Rightarrow$ Y is a tight 2-design on X with

$$|Y| = 7 = \binom{|\Omega|}{1} = \dim_{\mathbb{C}} P_1(X).$$

Thm (Fisher's inequality): For any $2t$-design Y on X $(2t \leq d)$,

$$|Y| \geq \dim_{\mathbb{C}} P_t(X).$$

$|Y| = \dim_{\mathbb{C}} P_t(X) \overset{\text{def}}{\Longleftrightarrow} Y$ is tight.

Y : a tight $2t$-design on X.
$R^Y := R|_{Y \times Y} : Y \times Y \to I_Y$, where $I_Y := R(Y \times Y)$.
$R_i^Y := (R_i^Y)^{-1}(i)$ for $i \in I_Y$.

Thm (Delsarte '73): $(Y, \{R_i^Y\}_{i \in I_Y})$ is a Q-poly. scheme.

Thm: A tight $2t$-design Y on X is a Q-poly. scheme.

Ex:

$\Omega = \mathbb{F}_2^3 \setminus \{\mathbf{0}\}$. $X = \binom{\Omega}{3}$: a Johnson scheme.

$R : X \times X \to \{0, 1, 2, 3\}$, $(x_1, x_2) \mapsto |x_1 \setminus x_2|$.

$Y = \{V \setminus \{\mathbf{0}\} \mid V \subset \mathbb{F}_2^3$, 2-dim. subspace$\} \subset X$.

$\Rightarrow Y$ is a tight 2-design on X with $|Y| = 7$.

$I_Y := R(Y \times Y) = \{0, 2\}$.

$R^Y(y_1, y_2) = 2 \iff y_1 \neq y_2$ for $y_1, y_2 \in Y$.

$\Rightarrow Y \simeq K_7$ as Q-poly. schemes.

Thm: A tight $2t$-design Y on X is a Q-poly. scheme ($t \geq 1$).

Proof:

$R^Y := R|_{Y \times Y} : Y \times Y \to I_Y$, where $I_Y := R(Y \times Y)$.

$d_Y := |I_Y| - 1$.

$P_j(I_Y) := P_j(I)|_{I_Y}$, $P_j(Y) := P_j(X)|_Y$.

$R^Y := R|_{Y\times Y} : Y \times Y \to I_Y$, where $I_Y := R(Y \times Y)$.

$d_Y := |I_Y| - 1$.

$P_j(I_Y) := P_j(I)|_{I_Y}$, $P_j(Y) := P_j(X)|_Y$.

Obs:

$P_j(I_Y) \cdot P_k(I_Y) = P_{j+k}(I_Y)$, $P_j(Y) \cdot P_k(Y) = P_{j+k}(Y)$.

$\{\text{const.}\} = P_0(I_Y) \subsetneqq \cdots \subsetneqq P_{d_Y}(I_Y) = \mathbb{C}^{I_Y}$.

$\{\text{const.}\} = P_0(Y) \subsetneqq \cdots \subsetneqq P_t(Y) = \mathbb{C}^Y$ ($\because$ the tightness of Y).

$P_t(X) \to P_t(Y)$, $f \mapsto f|_Y$ is an isometry.

$\dim_{\mathbb{C}} P_1(I_Y) = 2$.

Obs:

$P_j(I_Y) \cdot P_k(I_Y) = P_{j+k}(I_Y)$, $P_j(Y) \cdot P_k(Y) = P_{j+k}(Y)$.

$\{\text{const.}\} = P_0(I_Y) \subsetneqq \cdots \subsetneqq P_{d_Y}(I_Y) = \mathbb{C}^{I_Y}$.

$\{\text{const.}\} = P_0(Y) \subsetneqq \cdots \subsetneqq P_t(Y) = \mathbb{C}^Y$.

$P_t(X) \to P_t(Y)$, $f \mapsto f|_Y$ is an isometry.

$\dim_{\mathbb{C}} P_1(I_Y) = 2$.

It is enough to show that $\pi_j^Y := \pi_j|_{Y \times Y} \in M(Y, \mathbb{C}) \simeq \mathrm{End}(\mathbb{C}^Y)$ is the orthogonal projection onto $P_j(Y)$ for $j = 0, \ldots, t$ and $d_Y := |I_Y| - 1 \le t$ ($\Rightarrow d_Y = t$).

Goal: $\pi_j^Y := \pi_j|_{Y \times Y} \in M(Y, \mathbb{C}) \simeq \mathrm{End}(\mathbb{C}^Y)$ is the orthogonal projection onto $P_j(Y)$ for $j = 0, \ldots, t$, and $d_Y := |I_Y| - 1 \le t$.

Step 1: π_j^Y is the orthogonal projection onto $P_j(Y)$ (up to scalar).

Step 2: $\pi_t^Y(y_1, y_2) = 0$ for $y_1, y_2 \in Y$ with $y_1 \neq y_2$ ($\Rightarrow$ $I_Y \setminus \{i_0\}$ are zeros of a function in $P_t(I)$).

Step 3: The number of zeros of any function in $P_k(I)$ on $I \le k$ for each $k = 0, \ldots, d$.

Step 1: π_j^Y is the orthogonal projection onto $P_j(Y)$.

Fact(the reproducing kernel): $Z = X$ or Y. Let $e_1^Z, \ldots, e_m^Z \in P_j(Z)$ be an o.n.b. Then $K \in M(Z, \mathbb{C}) \simeq \mathrm{End}(\mathbb{C}^Z)$ defined by

$$K(z_1, z_2) := \sum_{k=1}^{m} e_k^Z(z_1)\overline{e_k^Z(z_2)}$$

gives the orthogonal projection onto $P_j(Z)$.

$P_j(X) \to P_j(Y), f \mapsto f|_Y$: isometry

$\Rightarrow$ **Step 1 can be proved!**

Step 2: $\pi_t^Y(y_1, y_2) = 0$ for $y_1, y_2 \in Y$ with $y_1 \neq y_2$.

Fix $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since $\pi_t^Y = \mathrm{id}_{\mathbb{C}^Y}$, we have $\pi_t^Y(y_1, y_2) = 0$.

Step 2 is completed!

Step 3: The number of zeros of any function in $P_k(I)$ on $I \leq k$ for each $k = 0, \ldots, d$.

$P_1(I) = \mathbb{C}\{\varpi\} + \{\text{const.}\}$ since $\dim_{\mathbb{C}} P_1(I_Y) = 2 \Rightarrow$ $P_k(I) = \text{Span-}\{\varpi^l \mid l = 0, \ldots, k\}$.

Lem: $\varpi : I \to \mathbb{C}$: injective

Proof of Lemma:

$\varpi(i) = \varpi(i')$ for $i, i' \in I$

$\Rightarrow F(i) = F(i')$ for any $F \in \mathbb{C}^I = \text{Span-}\{\varpi^l \mid l = 0, \ldots, d\}$.

(Q.E.D.)

Step 3: The number of zeros of any function in $P_k(I)$ on $I \leq k$ for each $k = 0, \ldots, d$.

$P_1(I) = \mathbb{C}\{\varpi\} + \{\text{const.}\}$ since $\dim_{\mathbb{C}} P_1(I_Y) = 2 \Rightarrow$ $P_k(I) = \text{Span-}\{\varpi^l \mid l = 0, \ldots, k\}$.

Lem: $\varpi : I \to \mathbb{C}$: injective

For each $a \in I$, we put $\varpi_a \in P_1(I) \backslash P_0(I)$ with $\varpi_a(a) = 0$ (unique).

By the division of "polynomials", we have

Lem: $F \in \mathbb{C}^I$ and $\{a_1, \ldots, a_m\}$ = the zeros of F. Then $F = c \cdot \varpi_{a_1} \cdot \cdots \cdot \varpi_{a_m} \in P_m(I)$ for $c \in \mathbb{C}$.

$\Rightarrow$ **Step 3 can be proved!**

Goal: $\pi_j^Y := \pi_j|_{Y\times Y} \in M(Y,\mathbb{C}) \simeq \mathrm{End}(\mathbb{C}^Y)$ is the orthogonal projection onto $P_j(Y)$ for $j = 0, \ldots, t$, and $d_Y := |I_Y| - 1 \leq t$.

We obtained the theorem below.

Thm: A tight $2t$-design Y on X is a Q-poly. scheme.

On the afternoon session: We will generalize the thorem for **compact Hausdorff Q-polynomial schemes!**

Thank you!