Remarks on generalizations of association schemes and Design theories Part II

Takayuki OKUDA

Hiroshima University

18 June, 2014

Summer School 2014 (Sendai)

Thm: A tight $2t$-design on X can be considered as a Q-poly. (association) scheme.

Plan of this talk:

(1): Prove the theorem for $X =$ a Q-polynomial scheme.

(2): Generalize Q-polynomial schemes and designs on them.

(3): Check a sphere can be considered as such a generalized scheme.

(4): Prove the theorem for such generalized schemes.

(1) : morning, **(2),(3),(4)** : afternoon

§2: Q-poly. scheme str. on a compact Hausdorff space

I, X : Second countable cpt Hausdorff spaces (with $|X| \geq 2$).

$C(I), C(X)$: The spaces of conti. $\mathbb{C}$-functions with the cpt-open topology.

$\mu : C(X) \to \mathbb{R}$, $f \mapsto \int_X f d\mu$: a Radon measure with $\int_X |f|^2 d\mu = 0 \iff f \equiv 0$.

$|X| := \mu(X) (< \infty)$.

$R : X \times X \to I$: a symm. surj. conti. map.

$R_i := R^{-1}(i)$.

When is $(X, \{R_i\}_{i \in I})$ a Q-poly. scheme?

I, X : Second countable cpt Hausdorff spaces.

$R : X \times X \to I$: a symm. surj. conti. map.

$R_i := R^{-1}(i)$.

When is $(X, \{R_i\}_{i \in I})$ a Q-poly. scheme?

$\langle f, g \rangle_X := \int_{x \in X} f(x)\overline{g(x)} d\mu(x)$ for $f, g \in C(X)$.

$M(X, \mathbb{C}) := C(X \times X) \subset \mathrm{End}(C(X))$ by

$$(Af)(z) := \int_{x \in X} A(z, x) f(x) d\mu(x)$$

for $A \in M(X, \mathbb{C})$, $f \in C(X)$ and $z \in X$.

$R^* : C(I) \to M(X, \mathbb{C}) \subset \mathrm{End}(C(X))$.

Def: $(X, \{R_i\}_{i \in I})$ is a cpt. Hausdorff Q-polynomial scheme $\overset{\text{def}}{\longleftrightarrow}$ there exist sequences of finite dimensional subspaces

$$\{\text{constants}\} = P_0(I) \subset \cdots \subset P_j(I) \subset \cdots \subset C(I)$$
$$\{\text{constants}\} = P_0(X) \subset \cdots \subset P_j(X) \subset \cdots \subset C(X)$$

such that

(0): $\bigcup_{j=0}^{\infty} P_j(I), \bigcup_{j=0}^{\infty} P_j(X)$ are dense in $C(I)$, $C(X)$.

(i): $P_j(I) \cdot P_k(I) = P_{j+k}(I)$ for j, k.

(ii): $\dim_{\mathbb{C}} P_1(I) = 2$ ($\Rightarrow \dim_{\mathbb{C}} P_j(I) \le j + 1$).

(iii): $R^*(P_j(I)) = \text{Span-}\{\pi_0, \pi_1, \ldots, \pi_j\}$ for each j where $\pi_j \in M(X, \mathbb{C})$ is the reproducing kernel of $P_j(X)$.

(iv): There exists $i_0 \in I$ such that $R_{i_0} = \Delta := \{(x, x) \mid x \in X\}$.

$$\{\text{constants}\} = P_0(I) \subset \cdots \subset P_j(I) \subset \cdots \subset C(I)$$

$$\{\text{constants}\} = P_0(X) \subset \cdots \subset P_j(X) \subset \cdots \subset C(X)$$

(0):$\bigcup_{j=0}^{\infty} P_j(I), \bigcup_{j=0}^{\infty} P_j(X)$ are dense in $C(I)$, $C(X)$.

(i):$P_j(I) \cdot P_k(I) = P_{j+k}(I)$.

(ii):$\dim_{\mathbb{C}} P_1(I) = 2$.

(iii):$R^*(P_j(I)) =$ Span-$\{\pi_0, \pi_1, \ldots, \pi_j\}$.

(iv):${}^{\exists} i_0 \in I$ such that $R_{i_0} = \Delta$.

(0), (i), (ii), (iii) $\Rightarrow$ (iv) ??

Compact Hausdorff Q-poly. schemes

$\rightsquigarrow$ **Polynomial spaces, Delsarte spaces** ??

$$\bigcup_{j=0}^{\infty} P_j(I) \overset{\text{dense}}{\subset} C(I), \quad \bigcup_{j=0}^{\infty} P_j(X) \overset{\text{dense}}{\subset} C(X)$$

(i): $P_j(I) \cdot P_k(I) = P_{j+k}(I)$.

(ii): $\dim_{\mathbb{C}} P_1(I) = 2$ ($\Rightarrow \dim_{\mathbb{C}} P_j(I) \leq j+1$).

(iii): $R^*(P_j(I)) = \text{Span-}\{\pi_0, \pi_1, \ldots, \pi_j\}$.

(iv): ${}^{\exists}i_0 \in I$ such that $R_{i_0} = \Delta$.

Rem:

$\overline{P_j(I)} = P_j(I)$, $\overline{P_j(X)} = P_j(X)$.

$P_j(X) \cdot P_k(X) = P_{j+k}(X)$.

$$\bigcup_{j=0}^{\infty} P_j(I) \overset{\text{dense}}{\subset} C(I), \quad \bigcup_{j=0}^{\infty} P_j(X) \overset{\text{dense}}{\subset} C(X)$$

(i):$P_j(I) \cdot P_k(I) = P_{j+k}(I)$.

(ii):$\dim_{\mathbb{C}} P_1(I) = 2$ ($\Rightarrow \dim_{\mathbb{C}} P_j(I) \leq j + 1$).

(iii):$R^*(P_j(I)) =$ Span-$\{\pi_0, \pi_1, \ldots, \pi_j\}$.

(iv):$^\exists i_0 \in I$ such that $R_{i_0} = \Delta$.

Ex 1: A Q-poly. scheme $(X, \{R_i\}_I)$ ($|I| = d + 1$) is a CH Q-poly. scheme with

$$\mu : C(X) = \mathbb{C}^X \to \mathbb{C}, \; f \mapsto \sum_{x \in X} f(x),$$

$P_{d+m}(I) := \mathbb{C}^I = C(I)$ and $P_{d+m}(X) := \mathbb{C}^X = C(X)$.

$$\bigcup_{j=0}^{\infty} P_j(I) \overset{\text{dense}}{\subset} C(I), \quad \bigcup_{j=0}^{\infty} P_j(X) \overset{\text{dense}}{\subset} C(X)$$

(i): $P_j(I) \cdot P_k(I) = P_{j+k}(I)$.

(ii): $\dim_{\mathbb{C}} P_1(I) = 2$ ($\Rightarrow \dim_{\mathbb{C}} P_j(I) \leq j+1$).

(iii): $R^*(P_j(I)) = \text{Span-}\{\pi_0, \pi_1, \ldots, \pi_j\}$.

(iv): ${}^{\exists}i_0 \in I$ such that $R_{i_0} = \Delta$.

Ex 2:

$I := [-1,1] \subset \mathbb{R}$. $S^{N-1} := \{x \in \mathbb{R}^N \mid \|x\| = 1\}$.

μ : A spherical measure.

$R : S^{N-1} \times S^{N-1} \to I, \quad (x,y) \mapsto \langle x, y \rangle_{\mathbb{R}^N}$.

$(S^{N-1}, \{R_i\}_{i \in I})$ is a cpt. Hausdorff Q-poly. scheme.

(i):$P_j(I) \cdot P_k(I) = P_{j+k}(I)$.

(ii):$\dim_{\mathbb{C}} P_1(I) = 2$ ($\Rightarrow \dim_{\mathbb{C}} P_j(I) \leq j+1$).

(iii):$R^*(P_j(I)) =$ Span-$\{\pi_0, \pi_1, \ldots, \pi_j\}$.

(iv):$^{\exists} i_0 \in I$ such that $R_{i_0} = \Delta$.

$(S^{N-1}, \{R_i\}_{i \in I})$ is a cpt. Hausdorff Q-poly. scheme.

In fact,

$P_j(I) :=$ {polynomials on I of degree $\leq j\} \subset C(I)$.

$P_j(S^{N-1}) := \{f|_{S^{N-1}} \mid f \in P_j(\mathbb{R}^N)\} \subset C(S^N)$.

satisfies the definition.

Rem: $G_k^{(N)} \in P_k(I)$: Gegenbauer poly. of degree k

$\Rightarrow R^*(\sum_{k=0}^{j} G_k^{(N)}) = \pi_j$ (by the addition formula).

We fix a CH Q-poly. scheme X and such a filteration

$$\{\text{constants}\} = P_0(X) \subset \cdots \subset P_j(X) \subset \cdots \subset C(X)$$

Def: $\emptyset \neq Y \overset{\text{finite}}{\subset} X$ is a t-design ($t \geq 1$)

$\overset{\text{def}}{\longleftrightarrow}$ For each $f \in P_t(X)$,

$$\frac{1}{|X|}\int_X f d\mu = \frac{1}{|Y|}\sum_{y \in Y} f(y).$$

Rem: t-designs on S^{N-1}

$\Leftrightarrow$ Spherical t-designs.

Def: $\emptyset \neq Y \overset{\text{finite}}{\subset} X$ is a t-design ($t \geq 1$)

$\overset{\text{def}}{\longleftrightarrow}$ For each $f \in P_t(X)$,

$$\frac{1}{|X|}\int_X f d\mu = \frac{1}{|Y|}\sum_{y \in Y} f(y).$$

Rem: t-designs on S^{N-1}

$\Leftrightarrow$ Spherical t-designs.

Thm (Fisher's inequality): For any $2t$-design Y on X,

$$|Y| \geq \dim_{\mathbb{C}} P_t(X).$$

Thm (Fisher's inequality): For any $2t$-design Y on X,

$$|Y| \geq \dim_{\mathbb{C}} P_t(X).$$

Proof: We show the map $P_t(X) \to \mathbb{C}^Y$, $f \mapsto f|_Y$ preserves the natural inner-products up to scalar. In fact, for each $f, g \in P_t(X)$,

$$\begin{aligned} \langle f, g \rangle_X &= \frac{|X|}{|X|} \int_{x \in X} f(x)\overline{g(x)} d\mu(x) \\ &= \frac{|X|}{|Y|} \sum_{y \in Y} f(y)\overline{g(y)} = \frac{|X|}{|Y|} \langle f|_Y, g|_Y \rangle_Y. \end{aligned}$$

(Q.E.D.)

Thm (Fisher's inequality): For any $2t$-design Y on X,

$$|Y| \geq \dim_{\mathbb{C}} P_t(X).$$

Proof: For each $f, g \in P_t(X)$,

$$\langle f, g \rangle_X = \frac{|X|}{|X|} \int_{x \in X} f(x)\overline{g(x)} d\mu(x)$$

$$= \frac{|X|}{|Y|} \sum_{y \in Y} f(y)\overline{g(y)} = \frac{|X|}{|Y|} \langle f|_Y, g|_Y \rangle_Y$$

(Q.E.D.)

We used $P_t(X) \cdot \overline{P_t(X)} = P_t(X) \cdot P_t(X) = P_{2t}(X)$.

Thm (Fisher's inequality): For any $2t$-design Y on X,

$$|Y| \geq \dim_{\mathbb{C}} P_t(X).$$

$$|Y| = \dim_{\mathbb{C}} P_t(X) \overset{\text{def}}{\longleftrightarrow} Y \text{ is tight.}$$

Ex:

$S^2 := \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$.

$Y = \{\text{vertices of regular tetrahedron}\}$ ($|Y| = 4$).

Y is a tight 2-design on X with

$$|Y| = 4 = \dim_{\mathbb{C}} P_1(S^2) = \dim \mathrm{Harm}_0(\mathbb{R}^3) + \dim \mathrm{Harm}_1(\mathbb{R}^3).$$

Thm (Fisher's inequality): For any $2t$-design Y on X,

$$|Y| \geq \dim_{\mathbb{C}} P_t(X).$$

$|Y| = \dim_{\mathbb{C}} P_t(X) \overset{\text{def}}{\longleftrightarrow} Y$ is tight.

Y : a tight $2t$-design on X.
$R^Y := R|_{Y\times Y} : Y \times Y \to I_Y$, where $I_Y := R(Y \times Y)$.
$R_i^Y := (R_i^Y)^{-1}(i)$ for $i \in I_Y$.

Thm: $(Y, \{R_i^Y\}_{i\in I_Y})$ is a Q-poly. scheme.

X = a cpt Hausdorff Q-poly. scheme.

Thm: A tight $2t$-design Y on X is a Q-poly. scheme.

Ex:

$S^2 := \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$.

$R : S^2 \times S^2 \to [-1, 1],\ (x_1, x_2) \mapsto \langle x_1, x_2 \rangle_{\mathbb{R}^3}$.

$Y = \{$vertices of regular tetrahedron$\}$ ($|Y| = 4$).

Y is a tight 2-design on X with

$$|Y| = 4 = \dim_{\mathbb{C}} P_1(S^2).$$

$|I_Y| := |R|_{Y \times Y}| = 2 \Rightarrow Y \simeq K_4$ as Q-poly. schemes.

Thm: A tight $2t$-design Y on X is a Q-poly. scheme ($t \geq 1$).

Proof:

$R^Y := R|_{Y\times Y} : Y \times Y \to I_Y$, where $I_Y := R(Y \times Y)$.

$d_Y := |I_Y| - 1$.

$P_j(I_Y) := P_j(I)|_{I_Y}$, $P_j(Y) := P_j(X)|_Y$.

Obs:

$P_j(I_Y) \cdot P_k(I_Y) = P_{j+k}(I_Y)$, $P_j(Y) \cdot P_k(Y) = P_{j+k}(Y)$.

$\{\text{const.}\} = P_0(I_Y) \subset \cdots \subset P_{d_Y}(I_Y) = \mathbb{C}^{I_Y}$.

$\{\text{const.}\} = P_0(Y) \subset \cdots \subset P_t(Y) = \mathbb{C}^Y$.

$P_t(X) \to P_t(Y)$, $f \mapsto f|_Y$ is an isometry (up to scalar).

$R^Y := R|_{Y\times Y} : Y \times Y \to I_Y$. $d_Y := |I_Y| - 1$.

$P_j(I_Y) := P_j(I)|_{I_Y}$, $P_j(Y) := P_j(X)|_Y$.

Obs:

$P_j(I_Y)\cdot P_k(I_Y) = P_{j+k}(I_Y)$, $P_j(Y)\cdot P_k(Y) = P_{j+k}(Y)$.

$\{\text{const.}\} = P_0(I_Y) \subsetneqq \cdots \subsetneqq P_{d_Y}(I_Y) = \mathbb{C}^{I_Y}$.

$\{\text{const.}\} = P_0(Y) \subsetneqq \cdots \subsetneqq P_t(Y) = \mathbb{C}^Y$.

$P_t(X) \to P_t(Y)$, $f \mapsto f|_Y$ is an isometry.

$\dim_{\mathbb{C}} P_1(I_Y) = 2$.

It is enough to show that $\pi_j^Y := \pi_j|_{Y\times Y} \in M(Y, \mathbb{C})$ is the reproducing kernel of $P_j(Y)$ for $j = 0, \ldots, t$ and $d_Y := |I_Y| - 1 \le t$ ($\Rightarrow d_Y = t$).

Goal: $\pi_j^Y := \pi_j|_{Y\times Y} \in M(Y,\mathbb{C})$ is the reproducing kernel of $P_j(Y)$ for $j = 0,\dots,t$ (up to scalar), and $d_Y := |I_Y| - 1 \le t$.

Step 1: π_j^Y is the reproducing kernel of $P_j(Y)$ (up to scalar).

Step 2: $\pi_t^Y(y_1, y_2) = 0$ for $y_1, y_2 \in Y$ with $y_1 \ne y_2$ ($\Rightarrow I_Y \setminus \{i_0\}$ are zeros of a function in $P_t(I)$).

Step 3: The number of zeros of any function in $P_k(I)$ on $I \le k$ for each $k = 0,\dots,d$.

Step 1: π_j^Y is the reproducing kernel of $P_j(Y)$.

Def. of the reproducing kernel: $Z = X$ or Y. Let $e_1^Z, \ldots, e_m^Z \in P_j(Z)$ be an o.n.b. Then the reproducing kernel $K \in M(Z, \mathbb{C})$ defined by

$$K(z_1, z_2) := \sum_{k=1}^{m} e_k^Z(z_1)\overline{e_k^Z(z_2)}.$$

This gives the orthogonal projection onto $P_j(Z)$.

$P_j(X) \to P_j(Y), f \mapsto f|_Y$: isometry

$\Rightarrow$ **Step 1 can be proved!**

Step 2: $\pi_t^Y(y_1, y_2) = 0$ for $y_1, y_2 \in Y$ with $y_1 \neq y_2$.

Fix $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since $\pi_t^Y = \mathrm{id}_{\mathbb{C}^Y}$, we have $\pi_t^Y(y_1, y_2) = 0$.

Step 2 is completed!

Step 3: The number of zeros of any function in $P_k(I)$ on $I \leq k$ for each $k = 0, \ldots, d$.

$P_1(I) = \mathbb{C}\{\varpi\} + \{\text{const.}\}$ since $\dim_{\mathbb{C}} P_1(I_Y) = 2 \Rightarrow P_k(I) = \text{Span-}\{\varpi^l \mid l = 0, \ldots, k\}$.

Lem (cf. the Stone–Weierstrass theorem): $\varpi : I \to \mathbb{C}$: injective

Proof of Lemma:

$\varpi(i) = \varpi(i')$ for $i, i' \in I$

$\Rightarrow F(i) = F(i')$ for any $F \in C(I) \overset{\text{dense}}{\supset} \text{Span-}\{\varpi^l \mid l = 0, \ldots, \}$.

(Q.E.D.)

Step 3: The number of zeros of any function in $P_k(I)$ on $I \leq k$ for each $k = 0, \ldots, d$.

$P_1(I) = \mathbb{C}\{\varpi\} + \{\text{const.}\}$ since $\dim_{\mathbb{C}} P_1(I_Y) = 2 \Rightarrow P_k(I) = \text{Span-}\{\varpi^l \mid l = 0, \ldots, k\}$.

Lem: $\varpi : I \to \mathbb{C}$: injective

For each $a \in I$, we put $\varpi_a \in P_1(I) \backslash P_0(I)$ with $\varpi_a(a) = 0$ (unique).

By the division of "polynomials", we have

Lem: $F \in \mathbb{C}^I$ and $\{a_1, \ldots, a_m\}$ = the zeros of F. Then $F = c \cdot \varpi_{a_1} \cdot \cdots \cdot \varpi_{a_m} \in P_m(I)$ for $c \in \mathbb{C}$.

$\Rightarrow$ **Step 3 can be proved!**

Goal: $\pi_j^Y := \pi_j|_{Y\times Y} \in M(Y,\mathbb{C}) \simeq \mathrm{End}(\mathbb{C}^Y)$ is the reproducing kernel of $P_j(Y)$ for $j = 0,\ldots,t$, and $d_Y := |I_Y| - 1 \le t$.

We obtained the theorem below.

Thm: A tight $2t$-design Y on X is a Q-poly. scheme for a CH Q-poly. scheme X.

§3: Future works

What is the best definition of compact Hausdorff Q-poly. scheme?

Does there exists CH Q-poly. scehems with no group action?

How about higher rank cases? P-polynomial cases?

Can we define non-compact association schemes and its designs?

$\Gamma\backslash G/K$: Compact Clifford–Klein form.

Lattice $\subset \mathbb{R}^N \supset$ Euclidean t-design.

Zeros of a modular form $\subset$ Hyperbolic plane

Fiber of $G/K \to \Gamma\backslash G/K$.

Thank you!