Delsarte 理論入門
Introduction to Delsarte Theory

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What is Delsarte Theory?

- Philippe Delsarte,
  An algebraic approach to the association schemes of coding theory,
  Philips Res. Rep. Suppl. No. 10 (1973).

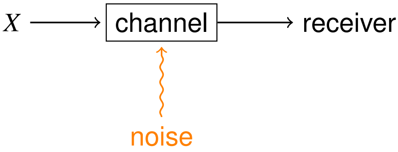
- ... studies codes and designs within the unifying framework of association schemes;
- ... has been playing a central role in Algebraic Combinatorics;
- ... is still important. Applications include extremal set theory and finite geometry.

What is Delsarte Theory?

- Keywords:
  - codes
  - designs
  - linear programming bound
  - P-polynomial and Q-polynomial properties
  - 4 fundamental parameters (including minimum distance and strength)
  - duality (in the case of translation association schemes)

Coding theory (in a very general form)

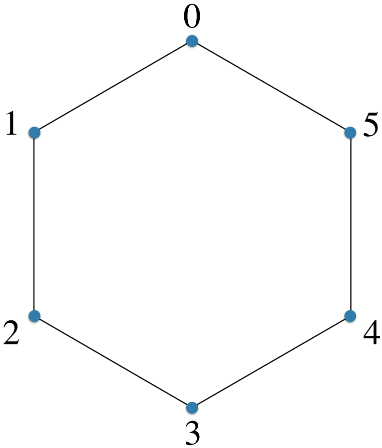
- X : a finite set $=$ a set of codewords



- $x \sim y \overset{\text{def}}{\iff} x$ and y can be "confused" $\quad (x, y \in X)$

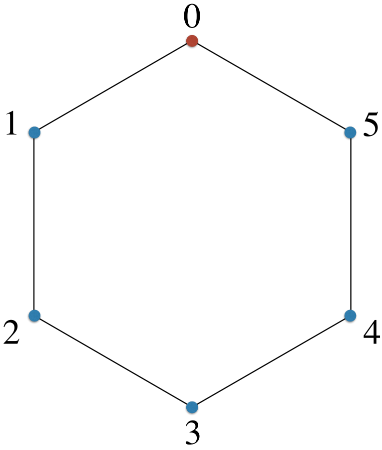
- Find a large subset C of X which can be sent without confusion !!

Coding theory: an example



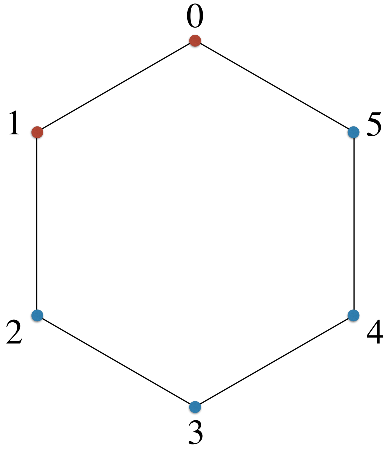
Coding theory: an example

- $C = \{0\}$



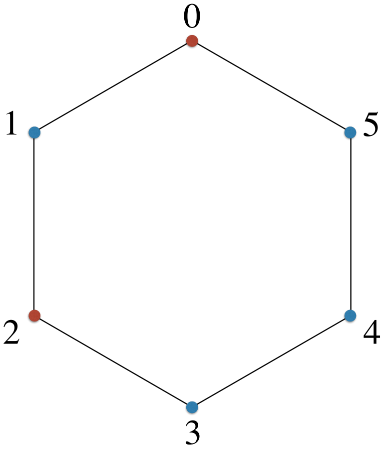
Coding theory: an example

- $C = \{0, 1\}$



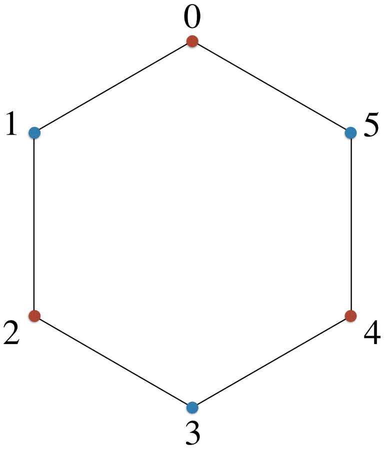
Coding theory: an example

- $C = \{0, 2\}$



Coding theory: an example

- $C = \{0, 2, 4\}$

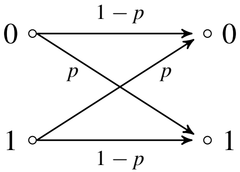


- Find the independence number of the graph !!

a good upper bound on

Coding theory: another example

- $H(6,2) = (X, \{R_i\}_{i=0}^{6})$: the binary Hamming scheme of class 6
- $X = \{0,1\}^6$
- Consider the binary symmetric channel with error probability $p \ll 1$



Coding theory: another example

Example

- Let us send 000000.
- $\mathrm{Prob}[\,000000 \longmapsto 100000\,] = p(1-p)^5$
- $\mathrm{Prob}[\,1 \text{ error occurs}\,] = 6 \times p(1-p)^5$
- $\mathrm{Prob}[\,000000 \longmapsto 110000\,] = p^2(1-p)^4$
- $\mathrm{Prob}[\,2 \text{ errors occur}\,] = \binom{6}{2} \times p^2(1-p)^4 = 15 \times p^2(1-p)^4$
- $\mathrm{Prob}[\,3 \text{ errors occur}\,] = \binom{6}{3} \times p^3(1-p)^3 = 20 \times p^3(1-p)^3$

- $\underline{20 \times p^3(1-p)^3} \ll 15 \times p^2(1-p)^4 \ll 6 \times p(1-p)^5$

Ignore this possibility, i.e., at most 2 errors occur !!

Coding theory: another example

- We assume that at most 2 errors occur.

Example

- 000000 and 100000 can be confused.
- 000000 and 110000 can be confused.
- 000000 and 111000 can be confused. Indeed:

$$000000 \longmapsto 100000 \longleftarrow\!\shortmid 111000$$

- 000000 and 111100 can still be confused. Indeed:

$$000000 \longmapsto 110000 \longleftarrow\!\shortmid 111100$$

- 000000 and 111110 (or 111111) can not be confused.

- $x \sim y \iff 0 < \partial(x, y) < 5$ ← the Hamming distance
- The edge set of our graph is $R_1 \cup R_2 \cup R_3 \cup R_4$.

Notation

- $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$: a symmetric association scheme
- $A_0, A_1, \ldots, A_d$: the adjacency matrices

  $A_i^{\mathsf{T}} = A_i$
- $\mathfrak{A} = \langle A_0, A_1, \ldots, A_d \rangle$: the Bose–Mesner algebra (over $\mathbb{R}$)
- $E_0, E_1, \ldots, E_d$: the primitive idempotents

  $\overline{E_i} = E_i^{\mathsf{T}} = E_i$
- P, Q : the first and second eigenmatrices, i.e.,

$$A_i = \sum_{j=0}^{d} P_{j,i} E_j, \quad E_i = \frac{1}{|X|} \sum_{j=0}^{d} Q_{j,i} A_j$$

M-codes

- $M \subseteq \{1, 2, \ldots, d\}$
- $C \subseteq X$: an M-code

$\overset{\text{def}}{\iff}$ C : an independent set of the graph $(X, \bigcup_{i \in M} R_i)$

$\iff$ $(C \times C) \cap \bigcup_{i \in M} R_i = \emptyset$

$\iff$ $(C \times C) \cap R_i = \emptyset$ for $\forall i \in M$

A review

- B : a (real) $n \times n$ matrix
- $\boldsymbol{u} \in \mathbb{R}^n$: an n-dimensional column vector
- $\boldsymbol{u}^{\mathsf{T}} B \boldsymbol{u} = \sum_{i,j=1}^{n} B_{i,j} \boldsymbol{u}_i \boldsymbol{u}_j$

M-codes (continued)

- $C \subseteq X$
- $\chi \in \mathbb{R}^X$: the (column) characteristic vector of C, i.e.,

$$\chi_x = \begin{cases} 1 & \text{if } x \in C \\ 0 & \text{if } x \notin C \end{cases} \qquad (x \in X)$$

- $\chi^{\mathsf{T}} A_i \chi = \sum_{x,y \in X} (A_i)_{x,y}\, \chi_x \chi_y = \left|(C \times C) \cap R_i\right|$

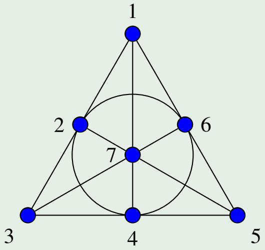
- $C \subseteq X$: an M-code
  $\iff (C \times C) \cap R_i = \emptyset$ for $\forall i \in M$
  $\iff \chi^{\mathsf{T}} A_i \chi = 0$ for $\forall i \in M$

t-(v, d, λ) designs

- Ω : a finite set with $|\Omega| = v$
- $\binom{\Omega}{k}$: the set of k-subsets of Ω
- $D \subseteq \binom{\Omega}{d}$: a t-(v, d, λ) design

$$\overset{\text{def}}{\Longleftrightarrow} \quad \forall z \in \binom{\Omega}{t} \quad \left|\{x \in D : z \subseteq x\}\right| = \lambda$$

Example



a 2-$(7, 3, 1)$ design

- Given t, v, d, find a small t-(v, d, λ) design !!

$$\binom{v}{t}\lambda = \binom{d}{t}|D|$$

t-(v, d, λ) designs

- $J(v,d) = (X, \{R_i\}_{i=0}^d)$: the Johnson scheme
- $X = \binom{\Omega}{d}$
- $D \subseteq X$
- $\chi \in \mathbb{R}^X$: the characteristic vector of D

Theorem (Delsarte, 1973)

- *D : a t-(v, d, λ) design (for some λ)*

  $\iff \chi^{\mathsf{T}} E_i \chi = 0$ *for* $\forall i \in \{1, 2, \ldots, t\}$

M-designs

More generally:

- $M \subseteq \{1, 2, \ldots, d\}$
- $D \subseteq X$: an M-design $\overset{\text{def}}{\iff}$ $\chi^{\mathsf{T}} E_i \chi = 0$ for $\forall i \in M$

Remark

- $C \subseteq X$: an M-code $\iff$ $\chi^{\mathsf{T}} A_i \chi = 0$ for $\forall i \in M$

M-designs

Theorem (Delsarte, 1973)

- *Suppose* $\mathfrak{X} = H(d, q)$.
- D *: a* $\{1, 2, \ldots, t\}$*-design*

  $\iff$ D *: an orthogonal array of strength* t

Remark

- Many more concepts of t-designs can be viewed as Delsarte M-designs in some association schemes.
- See, e.g., [7, Section 8].

A review

- B : a real symmetric $n \times n$ matrix
- B : positive semidefinite

$$\overset{\text{def}}{\iff} \quad \boldsymbol{u}^{\mathsf{T}} B \, \boldsymbol{u} = \sum_{i,j=1}^{n} B_{i,j} \, \boldsymbol{u}_i \boldsymbol{u}_j \geqslant 0 \ \text{ for } \ \forall \boldsymbol{u} \in \mathbb{R}^n$$

- $\eta_1, \eta_2, \ldots, \eta_n \in \mathbb{R}$: the eigenvalues of B
- B : positive semidefinite $\iff \eta_1 \geqslant 0, \ \eta_2 \geqslant 0, \ldots, \eta_n \geqslant 0$

Proof.

- $\exists U$: an orthogonal matrix (i.e., $U^{-1} = U^{\mathsf{T}}$) s.t.

$$U^{\mathsf{T}} B \, U = \mathrm{Diag}\,(\eta_1, \eta_2, \ldots, \eta_n)$$

- Set $\boldsymbol{v} = U^{\mathsf{T}} \boldsymbol{u} = U^{-1} \boldsymbol{u}$.
- $\boldsymbol{u}^{\mathsf{T}} B \, \boldsymbol{u} = (U\boldsymbol{v})^{\mathsf{T}} B \, (U\boldsymbol{v}) = \boldsymbol{v}^{\mathsf{T}} (U^{\mathsf{T}} B \, U) \boldsymbol{v} = \sum_{i=1}^{n} \eta_i \, \boldsymbol{v}_i^2$ □

The linear programming bound for M-codes

- $C \subseteq X$: an M-code
- $\chi \in \mathbb{R}^X$: the characteristic vector of C
- $\chi^{\mathsf{T}} A_i \chi = \sum_{x,y \in X} (A_i)_{x,y}\, \chi_x \chi_y = \big|(C \times C) \cap R_i\big| \geqslant 0$

- $i = 0 \implies \chi^{\mathsf{T}} A_0 \chi = |C| \quad (\because\ A_0 = I)$
- $i \in M \implies \chi^{\mathsf{T}} A_i \chi = 0$
- $\chi^{\mathsf{T}} J\, \chi = \sum_{x,y \in X} J_{x,y}\, \chi_x \chi_y = |C \times C| = |C|^2$

The linear programming bound for M-codes

- E_i : a real symmetric matrix
- $E_i^2 = E_i \iff E_i(E_i - I) = 0$

  $\iff$ every eigenvalue is 0 or 1

  $\implies E_i$: positive semidefinite

- $\chi^{\mathsf{T}} E_i \chi \geqslant 0$

The linear programming bound for M-codes

$\chi^{\mathsf{T}} A_0 \chi$	$\chi^{\mathsf{T}} A_i \chi$ $i \in M$	$\chi^{\mathsf{T}} A_i \chi$ $i \in \{1,2,\ldots,d\} \backslash M$	$\chi^{\mathsf{T}} J \chi$	$\chi^{\mathsf{T}} E_i \chi$ $i \in \{1,2,\ldots,d\}$
$\lvert C \rvert$	0	$\geqslant 0$	$\lvert C \rvert^2$	$\geqslant 0$

- Set $e_i := \dfrac{\chi^{\mathsf{T}} A_i \chi}{|C|}$. $\boldsymbol{e} = (e_0, e_1, \ldots, e_d)$: the inner distribution of C
- $\chi^{\mathsf{T}} J \chi = \chi^{\mathsf{T}} \left(\sum_{i=0}^{d} A_i \right) \chi = |C| \sum_{i=0}^{d} e_i$
- $\chi^{\mathsf{T}} E_i \chi = \chi^{\mathsf{T}} \left(\frac{1}{|X|} \sum_{j=0}^{d} Q_{j,i} A_j \right) \chi = \frac{|C|}{|X|} \sum_{j=0}^{d} Q_{j,i}\, e_j$

e_0	e_i $i \in M$	e_i $i \in \{1,2,\ldots,d\} \backslash M$	$\sum_{i=0}^{d} e_i$	$\sum_{j=0}^{d} Q_{j,i}\, e_j$ $i \in \{1,2,\ldots,d\}$
1	0	$\geqslant 0$	$\lvert C \rvert$	$\geqslant 0$

The linear programming bound for M-codes

- View the e_i as real variables !!

Theorem (Delsarte, 1973)

- *Consider the following linear programming problem:*

$$\texttt{maximize} \quad \vartheta = \sum_{i=0}^{d} e_i$$

$\texttt{subject to}$
- $e_0 = 1$
- $e_i = 0 \ \texttt{for} \ i \in M$
- $e_i \geqslant 0 \ \texttt{for} \ i \in \{1, 2, \ldots, d\} \backslash M$
- $\sum_{j=0}^{d} Q_{j,i}\, e_j \geqslant 0 \ \texttt{for} \ i \in \{1, 2, \ldots, d\}$

- *If C is an M-code, then $|C| \leqslant \vartheta$.*

The linear programming bound for M-codes

Remark

- Linear programming problems can be solved by the simplex method.

Example

- Suppose $\mathfrak{X} = H(16, 2)$, $M = \{1, 2, 3, 4, 5\}$.
- Then $\vartheta = 256$.
- This is attained by the Nordstrom–Robinson code.

Remarks on the linear programming bound

- Delsarte's linear programming bound, combined with the duality of linear programming, provides us with the most powerful method for bounding the sizes of general M-codes in association schemes.
- See, e.g., [6].
- Similarly, Delsarte formulated the linear programming (lower) bound for the sizes of M-designs.

P-polynomial association schemes

Definition (Delsarte, 1973)

- $\mathfrak{X}$: P-polynomial w.r.t. the ordering $\{R_i\}_{i=0}^{d}$

  $\overset{\text{def}}{\Longleftrightarrow}$ $\exists v_0(t), v_1(t), \ldots, v_d(t) \in \mathbb{R}[t]$, $\exists \theta_0, \theta_1, \ldots, \theta_d \in \mathbb{R}$ s.t.

  - $\deg v_i(t) = i \quad (0 \leqslant i \leqslant d)$
  - $P_{j,i} = v_i(\theta_j) \quad (0 \leqslant i,j \leqslant d)$

Remark

- By the orthogonality relation of P, the $v_i(t)$ form a system of orthogonal polynomials:

$$\sum_{\ell=0}^{d} v_i(\theta_\ell)\, v_j(\theta_\ell)\, m_\ell = \delta_{i,j} \cdot |X|\, k_i \quad (0 \leqslant i,j \leqslant d)$$

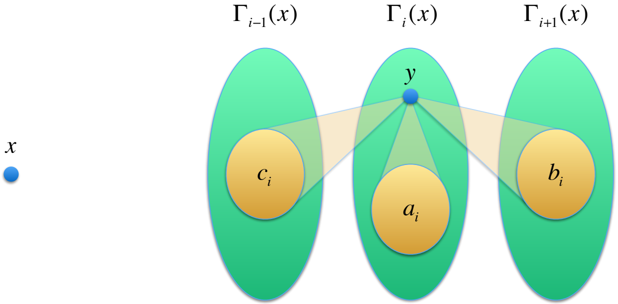
P-polynomial association schemes

Example

- The Hamming scheme $H(d, q)$
- The Johnson scheme $J(v, d)$

Distance-regular graphs

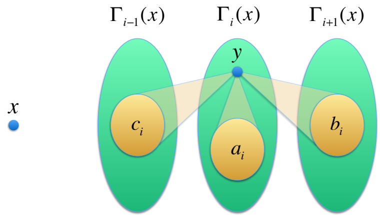
- $\Gamma = (X, R)$: a distance-regular graph with diameter d



Distance-regular graphs

- ∂ : the path-length distance
- A_i : the i^{th} distance matrix:

$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{otherwise} \end{cases}$$



- $A_0 = I$
- $A_0 + A_1 + \cdots + A_d = J$
- $A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}$ where $A_{-1} = A_{d+1} = 0$

$(A_1 A_i)_{y,x} = |\Gamma_1(y) \cap \Gamma_i(x)|$

Distance-regular graphs $\Longrightarrow$ P-polynomial schemes

- $A_0 = I$
- $A_0 + A_1 + \cdots + A_d = J$
- $A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1}$ where $A_{-1} = A_{d+1} = 0$

- $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$: a symmetric association scheme, where

$$R_i := \{(x,y) : \partial(x,y) = i\} \quad (0 \leqslant i \leqslant d)$$

- Set $v_0(t) = 1$, $v_1(t) = t$, and

$$t\, v_i(t) = b_{i-1}\, v_{i-1}(x) + a_i\, v_i(x) + c_{i+1}\, v_{i+1}(t) \quad (1 \leqslant i \leqslant d-1).$$

- $A_i = v_i(A_1) \quad \Longrightarrow \quad P_{j,i} = v_i(\theta_j)$ where $\theta_j := p_1(j)$

- $\mathfrak{X}$: P-polynomial w.r.t. the ordering $\{R_i\}_{i=0}^d$

P-polynomial schemes $\Longrightarrow$ distance-regular graphs

Remark

- Conversely, if a symmetric association scheme $\mathfrak{X}$ is P-polynomial w.r.t. the ordering $\{R_i\}_{i=0}^{d}$ then the graph $\Gamma = (X, R_1)$ is distance-regular.
- This follows from the three-term recurrence relation for a system of orthogonal polynomials.

Codes in P-polynomial schemes

- Suppose $\mathfrak{X}$ is P-polynomial w.r.t. the ordering $\{R_i\}_{i=0}^d$.
- $C \subseteq X \quad (1 < |C| < |X|)$
- $\chi \in \mathbb{R}^X$: the characteristic vector of C

- $\delta := \min\{i \neq 0 : \chi^{\mathsf{T}} A_i \chi \neq 0\}$: the minimum distance of C
  $= \max\left\{i \neq 0 : \ C \text{ is a } \{1, 2, \ldots, i-1\}\text{-code}\right\}$
- $s^* := \left|\{i \neq 0 : \chi^{\mathsf{T}} E_i \chi \neq 0\}\right|$: the dual degree of C

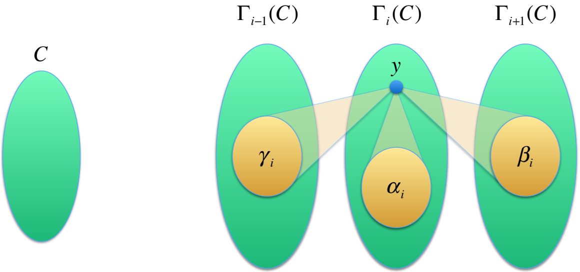
????

Theorem (Delsarte, 1973)

- $\delta \leqslant 2s^* + 1$;
- *If* $\delta \geqslant 2s^* - 1$ *then* C *is completely regular.*

Codes in P-polynomial schemes

- Complete regularity of C is illustrated as follows:



- $\rho := \max\{\partial(x, C) : x \in X\}$: the covering radius of C
- $\chi_i \in \mathbb{R}^X$: the characteristic vector of $\Gamma_i(C)$ $(0 \leqslant i \leqslant \rho)$

- $A_1 \chi_i = \beta_{i-1} \chi_{i-1} + \alpha_i \chi_i + \gamma_{i+1} \chi_{i+1}$ where $\chi_{-1} = \chi_{\rho+1} = 0$

The role of s^*: the outer distribution of C

- $s^* = \left|\{i \neq 0 : \chi^{\mathsf{T}} E_i \chi \neq 0\}\right|$ $\longleftarrow$ computable from the inner distribution

Remark

- $\chi^{\mathsf{T}} E_0 \chi = \dfrac{\chi^{\mathsf{T}} J \chi}{|X|} > 0$
- $\chi^{\mathsf{T}} E_i \chi = \chi^{\mathsf{T}} (E_i)^2 \chi = (E_i \chi)^{\mathsf{T}} (E_i \chi) = ||E_i \chi||^2$, so that

$$s^* = \left|\{i \neq 0 : E_i \chi \neq 0\}\right|.$$

The role of s^*: the outer distribution of C

- $B = [\,A_0\chi,\, A_1\chi, \ldots,\, A_d\chi\,]$: the outer distribution of C:

$$B_{x,i} = (A_i\chi)_x = \sum_{y\in X} (A_i)_{x,y}\, \chi_y = \left|\Gamma_i(x)\cap C\right| \quad (x\in X,\ 0\leqslant i\leqslant d)$$

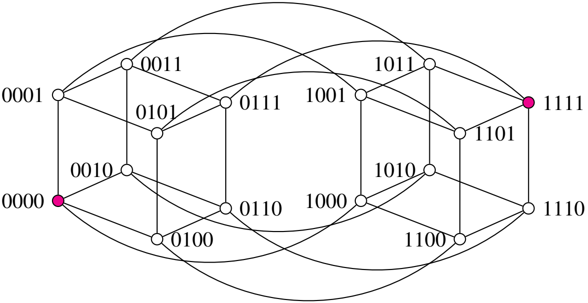
- Recall $E_i\chi = \dfrac{1}{|X|}\displaystyle\sum_{j=0}^{d} Q_{j,i}\, A_j\chi$.

- $\dfrac{1}{|X|}BQ = [\,E_0\chi,\, E_1\chi, \ldots,\, E_d\chi\,]$

- $\operatorname{rank} B = \operatorname{rank} BQ = s^* + 1$

The role of s^*: the outer distribution of C

- As an example, suppose $\mathfrak{X} = H(4,2)$ and $C = \{0000, 1111\}$:



- $\delta = 4,\ \ \rho = 2,\ \ s^* = 2 \implies \delta \geqslant 2s^* - 1$

  covering radius (ρ), minimum distance (δ), dual degree (s^*)

- Some of the rows of B:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix} \begin{matrix} \cdots\cdots\ 0000^{\text{th}} \text{ row} \\ \cdots\cdots\ 0001^{\text{th}} \text{ row} \\ \cdots\cdots\ 0110^{\text{th}} \text{ row} \end{matrix}$$

$B_{x,i} = |\Gamma_i(x) \cap C|$

The role of s^*: the outer distribution of C

- Some of the rows of B:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix} \begin{matrix} \cdots\cdots\ 0000^{\text{th}} \text{ row} \\ \cdots\cdots\ 0001^{\text{th}} \text{ row} \\ \cdots\cdots\ 0110^{\text{th}} \text{ row} \end{matrix}$$

ρ^{th} column

Observation

- $A_0\chi, A_1\chi, \ldots, A_\rho\chi$: linearly independent

Theorem (Delsarte, 1973)

- $\rho \leqslant s^*$

hard to compute in general

Proof.

- $\dim\langle A_0\chi, A_1\chi, \ldots, A_d\chi\rangle = \operatorname{rank} B = s^* + 1$ □

Perfect codes and Lloyd polynomials

Theorem (Delsarte, 1973)

- $\delta \leqslant 2s^* + 1$;
- *If* $\delta \geqslant 2s^* - 1$ *then* C *is completely regular.*

- $U_r(x) := \{y \in X : \partial(x, y) \leqslant r\}$: the "ball" of radius r centered at x
- C : perfect $\overset{\text{def}}{\Longleftrightarrow}$ $X = \coprod_{x \in C} U_\rho(x)$
- $L_r(t) := v_0(t) + v_1(t) + \cdots + v_r(t)$: the Lloyd polynomial of degree r

Theorem (Delsarte, 1973; "Lloyd Theorem")

- $\delta = 2s^* + 1 \iff C$ *: perfect* $\implies \rho = s^*$ *and* $L_\rho(t)$ *has* ρ *simple roots in* $\{\theta_0, \theta_1, \ldots, \theta_d\}$.

an extremely strong condition !!

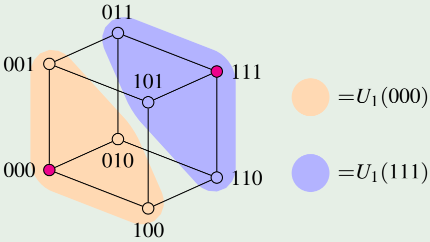
Perfect codes and Lloyd polynomials

Theorem (Delsarte, 1973; “Lloyd Theorem”)

- $\delta = 2s^* + 1 \iff C$: *perfect* $\implies \rho = s^*$ *and* $L_\rho(t)$ *has* ρ *simple roots in* $\{\theta_0, \theta_1, \ldots, \theta_d\}$.

Example

- $\mathfrak{X} = H(3,2)$
- $C = \{000, 111\}$
- $L_1(t) = 1 + t$ (with $1 = v_0(t)$ and $t = v_1(t)$)
- $\{\theta_0, \theta_1, \theta_2, \theta_3\} = \{3, 1, -1, -3\}$



Q-polynomial association schemes

Definition (Delsarte, 1973)

- $\mathfrak{X}$: Q-polynomial w.r.t. the ordering $\{E_i\}_{i=0}^{d}$

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists v_0^*(t), v_1^*(t), \dots, v_d^*(t) \in \mathbb{R}[t]$, $\exists \theta_0^*, \theta_1^*, \dots, \theta_d^* \in \mathbb{R}$ s.t.

  - $\deg v_i^*(t) = i \quad (0 \leqslant i \leqslant d)$
  - $Q_{j,i} = v_i^*(\theta_j^*) \quad (0 \leqslant i,j \leqslant d)$

Remark

- By the orthogonality relation of Q, the $v_i^*(t)$ form a system of orthogonal polynomials:

$$\sum_{\ell=0}^{d} v_i^*(\theta_\ell^*)\, v_j^*(\theta_\ell^*)\, k_\ell = \delta_{i,j} \cdot |X|\, m_i \quad (0 \leqslant i,j \leqslant d)$$

Q-polynomial association schemes

Example

- The Hamming scheme $H(d, q)$
- The Johnson scheme $J(v, d)$

Designs in Q-polynomial schemes

- Suppose $\mathfrak{X}$ is Q-polynomial w.r.t. the ordering $\{E_i\}_{i=0}^d$.
- $D \subseteq X \quad (1 < |D| < |X|)$
- $\chi \in \mathbb{R}^X$: the characteristic vector of D

- $\tau := \min\{i \neq 0 : \chi^{\mathsf{T}} E_i \chi \neq 0\} - 1$: the (maximum) strength of D
  $= \max\left\{i : D \text{ is a } \{1, 2, \ldots, i\}\text{-design}\right\}$
- $s := \left|\{i \neq 0 : \chi^{\mathsf{T}} A_i \chi \neq 0\}\right|$: the degree of D

Theorem (Delsarte, 1973)

- $\tau \leqslant 2s$;
- *If* $\tau \geqslant 2s - 2$ *then* $\left(D, \left\{(D \times D) \cap R_i\right\}_{i=0}^d\right)$ *is a* Q*-polynomial scheme (called a subscheme).*

remove empty relations

Tight designs and Wilson polynomials

Theorem (Delsarte, 1973)

- $\tau \leqslant 2s$;
- *If* $\tau \geqslant 2s - 2$ *then* $\left(D, \left\{(D \times D) \cap R_i\right\}_{i=0}^{d}\right)$ *is a* Q*-polynomial subscheme.*

- In general, $|D| \geqslant m_0 + m_1 + \cdots + m_{\lfloor \tau/2 \rfloor}$ (the Fisher type bound).
- D : tight $\overset{\text{def}}{\Longleftrightarrow}$ $|D| = m_0 + m_1 + \cdots + m_{\lfloor \tau/2 \rfloor}$
- $W_r(t) := v_0^*(t) + v_1^*(t) + \cdots + v_r^*(t)$: the Wilson polynomial of degree r

Theorem (Delsarte, 1973)

- $\tau = 2s \iff D$ *: tight* $\implies W_s(t)$ *has* s *simple roots in* $\{\theta_0^*, \theta_1^*, \ldots, \theta_d^*\}$.

Translation association schemes

- Suppose $\mathfrak{X} = (X, \{R_i\}_{i=0}^d)$ is a translation scheme on the abelian group X.
- $\widehat{\mathfrak{X}} = (\widehat{X}, \{S_i\}_{i=0}^d)$: the dual of $\mathfrak{X}$, where $\widehat{X}$: the character group of X

Remark

- $\mathfrak{X}$: P-polynomial $\iff$ $\widehat{\mathfrak{X}}$: Q-polynomial
- $\mathfrak{X}$: Q-polynomial $\iff$ $\widehat{\mathfrak{X}}$: P-polynomial

Translation association schemes

- $C \leqslant X$: a subgroup of X
- $C^{\perp} := \{ f \in \widehat{X} : f(x) = 1 \text{ for } \forall x \in C \} \leqslant \widehat{X}$

Theorem (Delsarte, 1973)

- *Let* $M \subseteq \{1, 2, \ldots, d\}$.
- C *: an* M*-code* $\iff$ $C^{\perp}$ *: an* M*-design*
- C *: an* M*-design* $\iff$ $C^{\perp}$ *: an* M*-code*

- In particular, if $\mathfrak{X}$ is P-polynomial and / or Q-polynomial, then

$$\delta(C) = \tau(C^{\perp}) + 1, \qquad s^*(C) = s(C^{\perp}),$$
$$\tau(C) = \delta(C^{\perp}) - 1, \qquad s(C) = s^*(C^{\perp}).$$

strength

degree

minimum distance

dual degree

Further reading

- Classification problem of P- & Q-polynomial schemes
  - See, e.g., [1, Chapter III], [2, Chapters 8,9], [4, Section 5].
  - The Terwilliger algebra; cf. [9]
  - Orthogonal polynomials, Leonard pairs, tridiagonal pairs; cf. [10]
- Two more fundamental parameters (width w, dual width w^*) [3]
- The semidefinite programming bound [8]

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