INSTABILITY OF STANDING WAVES FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH POTENTIALS

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Abstract. We study the instability of standing waves $e^{i\omega t} \phi_\omega(x)$ for a nonlinear Schrödinger equation with an attractive power nonlinearity $|u|^{p-1}u$ and a potential $V(x)$ in $\mathbb{R}^n$. Here, $\omega > 0$ and $\phi_\omega(x)$ is a minimal action solution of the stationary problem. Under suitable assumptions on $V(x)$, we show that if $p > 1+4/n$, $e^{i\omega t} \phi_\omega(x)$ is unstable for sufficiently large $\omega$. For example, our theorem covers a harmonic potential $V(x) = |x|^2$, to which the arguments in the previous papers [2], [14] and [19] are not directly applicable. As another application, we also prove a similar result for a nonlinear Schrödinger equation with a constant magnetic field.

1. Introduction and main result

In this paper, we consider the instability of standing wave solutions for the nonlinear Schrödinger equations with a real valued potential $V(x)$:

$$i\partial_t u = -\Delta u + V(x)u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R}^{1+n},$$

where $1 < p < 2^* - 1$. Here, we put $2^* = \infty$ if $n = 1, 2$, and $2^* = 2n/(n-2)$ if $n \geq 3$.

When $V(x) \equiv 0$, (1.1) arises in various physical contexts such as nonlinear optics and plasma physics (see, e.g., [7, 26, 29]). The nonlinearity enters due to the effect of changes in the field intensity on the wave propagation characteristics of the medium. The potential $V(x)$ can be thought of as modeling inhomogeneities in the medium. In [23], Equation (1.1) with a bounded potential $V(x)$ is studied as a model proposed to describe the local dynamics at a nucleation site. Equation (1.1) with a harmonic potential $V(x) = |x|^2$ is known as a model to describe the Bose-Einstein condensate.

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with attractive inter-particle interactions under a magnetic trap (see, e.g., [1, 13, 27]). By a standing wave, we mean a solution of (1.1) of the form $u_\omega(t, x) = e^{i\omega t}\phi_\omega(x)$, where $\omega \in \mathbb{R}$, and $\phi_\omega(x)$ is a minimal action solution of

$$-\Delta \phi + \omega \phi + V(x)\phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^n \quad (1.2)$$

(see Definition 1.1 below). The main purpose of this paper is to show that under suitable assumptions on $V(x)$ and $p > 1 + 4/n$, the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is unstable for sufficiently large $\omega > 0$ (see Theorem 1.1 below). As an application, in Section 4, we prove a similar result for a nonlinear Schrödinger equation (4.1) with a constant magnetic field.

Many authors have been studying the problem of stability and instability of standing waves for nonlinear Schrödinger equations (see, e.g., [2, 6, 9, 14, 15, 19, 23, 24, 28, 30]). We recall some known results. First, we consider the case $V(x) \equiv 0$. For any $\omega > 0$, there exists a unique positive radial solution $\psi_\omega(x)$ of

$$-\Delta \psi + \omega \psi - |\psi|^{p-1}\psi = 0, \quad x \in \mathbb{R}^n \quad (1.3)$$

in $H^1(\mathbb{R}^n)$ (see [16] for the uniqueness), and the standing wave solution $e^{i\omega t}\psi_\omega(x)$ of (1.1) with $V(x) \equiv 0$ is stable for any $\omega > 0$ if $p < 1 + 4/n$, and unstable for any $\omega > 0$ if $p \geq 1 + 4/n$ (see [2, 6, 28]). Meanwhile, Rose and Weinstein [23] showed that when $-\Delta + V(x)$ has the first eigenvalue $\lambda_1$, the standing wave solution $e^{i\omega t}\phi_\omega(x)$ of (1.1) is stable for $\omega$ such that $\omega > -\lambda_1$ and sufficiently close to $-\lambda_1$, even if $p \geq 1 + 4/n$ (see also [10]).

In this paper, for potential $V(x)$, we assume the following (V0)–(V2).

(V0) There exist real valued functions $V_1(x)$ and $V_2(x)$ such that $V(x) = V_1(x) + V_2(x)$.

(V1.1) $V_1(x) \in C^2(\mathbb{R}^n)$ and there exist positive constants $m$ and $C$ such that $0 \leq V_1(x) \leq C(1 + |x|^m)$ on $\mathbb{R}^n$.

(V1.2) There exists $C_\alpha > 0$ such that $|x^\alpha \partial_x \phi_1 V_1(x)| \leq C_\alpha (1 + V_1(x))$ on $\mathbb{R}^n$ for $|\alpha| \leq 2$.

(V2) There exists $\tilde{q}$ such that $\tilde{q} \geq 1$, $q > n/2$ and $x^\alpha \partial_x \phi_1 V_2(x) \in L^q(\mathbb{R}^n) + L^{\infty}(\mathbb{R}^n)$ for $|\alpha| \leq 2$.

Example. (i) (Harmonic potentials) For $c_1, \cdots, c_n \in \mathbb{R}$, $\sum_{j=1}^n c_j^2 x_j^2$ satisfies (V1.1) and (V1.2).

(ii) For $c \in \mathbb{R}$ and $0 < a < \min\{2, n\}$, $c|x|^{-a}$ satisfies (V2).

(iii) (V2) is satisfied if $U(x) \in C^2(\mathbb{R}^n)$ satisfies $|\partial_x^\alpha U(x)| \leq C_\alpha (x)^{-|\alpha|}$ for $|\alpha| \leq 2$.

(iv) $1 + \sin x_1$ satisfies (V1.1), but does not satisfy (V1.2) nor (V2).

We define a real Hilbert space $X$ by

$$X := \{v \in H^1(\mathbb{R}^n, \mathbb{C}) : V_1(x)|v(x)|^2 \in L^1(\mathbb{R}^n)\}$$
with the inner product

\[(v, w)_X := \text{Re} \int_{\mathbb{R}^n} (v(x)\bar{w}(x) + \nabla v(x) \cdot \nabla \bar{w}(x) + V_1(x)v(x)\bar{w}(x))dx.\]

The norm of \(X\) is denoted by \(\| \cdot \|_X\). Let \(G\) be a closed subgroup of \(O(n)\) such that \(V_1(x)\) and \(V_2(x)\) are invariant under \(G\), i.e., \(V_j(gx) = V_j(x)\) for \(g \in G, x \in \mathbb{R}^n\) and \(j = 1, 2\). We define a closed subspace \(X_G\) of \(X\) by

\[X_G := \{v \in X : v(gx) = v(x), g \in G, x \in \mathbb{R}^n\}.\]

We note that \(X_G = X\) if \(G = \{\text{Id (identity matrix)}\}\), and \(X_G = X_{rad}\) if \(G = O(n)\), where \(X_{rad} = \{v \in X : v(x) = v(|x|), x \in \mathbb{R}^n\}\). Moreover, we define the energy functional \(E\) on \(X_G\) by

\[E(v) := \frac{1}{2} \|\nabla v\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^n} V(x)|v(x)|^2dx - \frac{1}{p+1} \|v\|_{p+1}^{p+1},\]

where \(\| \cdot \|_r\) stands for the norm of \(L^r(\mathbb{R}^n)\). We remark that by the assumptions (V2) and \(1 < p < 2^* - 1\), the functional \(E\) is well-defined on \(X_G\). We assume that the time local well-posedness for the Cauchy problem to (1.1) in \(X_G\), the conservation of energy and \(L^2(\mathbb{R}^n)\)-norm, and the virial identity hold.

**Assumption (A1).** For any \(u_0 \in X_G\), there exist \(T = T(||u_0||_X) > 0\) and a unique solution \(u(t) \in C([0, T], X_G)\) of (1.1) with \(u(0) = u_0\) satisfying

\[E(u(t)) = E(u_0), \quad \|u(t)\|_2^2 = \|u_0\|_2^2, \quad t \in [0, T].\]

In addition, if \(u_0 \in X_G\) satisfies \(|x|u_0 \in L^2(\mathbb{R}^n)\), then the virial identity

\[\frac{d^2}{dt^2} \|xu(t)\|_2^2 = 8P(u(t))\]  

holds for \(t \in [0, T]\), where

\[P(v) := \|\nabla v\|_2^2 - \frac{1}{2} \int_{\mathbb{R}^n} x \cdot \nabla V(x)|v(x)|^2dx - \frac{n(p-1)}{2(p+1)} \|v\|_{p+1}^{p+1}.\]

**Remark 1.1.** The assumption (A1) is verified, if \(V(x)\) satisfies the following (A1.1)–(A1.2) with (V0) (see Section 6.4, Theorem 9.2.5 and Remark 9.2.9 of [4]).

(A1.1) \(V_1(x) \in C^\infty(\mathbb{R}^n), V_1(x) \geq 0 \in \mathbb{R}^n, \partial^\alpha_x V_1(x) \in L^\infty(\mathbb{R}^n)\) for \(|\alpha| \geq 2, \)

and there exists \(C > 0\) such that \(|x \cdot \nabla V_1(x)| \leq C(|x|^2 + V_1(x))\) in \(\mathbb{R}^n,\)

(A1.2) \(V_2(x) \in L^{q_0}(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)\) for some \(q_0 \geq 1, q_0 > n/2\) and \(x \cdot \nabla V_2(x) \in L^{q_1}(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)\) for some \(q_1 \geq 1, q_1 > n/2,\)

Next, we consider the stationary problem (1.2).
We note that $M$ and $\phi$ satisfy Assumption (A2). Moreover, for any $(i) \in \mathbb{N}$, there exists $\delta > 0$ such that for any $\phi \in M$, the solution $\phi$ satisfies $I_\omega(\phi) = 0$. Thus, by the definition of $M$, we have $S_\omega(\phi) = S_\omega(v)$. That is, $\phi \in M$ is a minimal action solution of (1.2).

We also assume the existence of minimal action solutions of (1.2) for large $\omega$.

**Assumption (A2).** There exists $\omega_0 \in (0, \infty)$ such that $M_{G, \omega}$ is not empty and $M_{G, \omega} \subset \{v \in X_G : |x|v(x) \in L^2(\mathbb{R}^n)\}$ for any $\omega \in (\omega_0, \infty)$.

**Remark 1.3.** If $V(x) \in C(\mathbb{R}^n)$ satisfies $\lim_{|x|\to \infty} V(x) = +\infty$, it is easy to see that $M_{G, \omega}$ is not empty for sufficiently large $\omega$, since the embedding $X_G \subset L^r(\mathbb{R}^n)$ is compact for $2 \leq r < 2^*$. However, in general, we may need some additional assumptions related to the concentration compactness principle (see, e.g., [17, 18, 22]). The assumption $M_{G, \omega} \subset \{v \in X_G : |x|v(x) \in L^2(\mathbb{R}^n)\}$ is needed to use the virial identity (1.4) in the proof of Proposition 1.1 below.

**Definition 1.1.** We define two functionals on $X_G$:

$$S_\omega(v) := E(v) + \frac{\omega}{2} \|v\|^2_2 \quad \text{(action)},$$

$$I_\omega(v) := \|\nabla v\|^2_2 + \omega \|v\|^2_2 + \int_{\mathbb{R}^n} V(x)|v(x)|^2 dx - \|v\|^{p+1}_{p+1}.$$  

Let $M_{G, \omega}$ be the set of all minimizers for

$$\inf\{S_\omega(v) : v \in X_G \setminus \{0\}, I_\omega(v) = 0\}. \quad (1.6)$$

**Remark 1.2.** (i) We note that $P(v) = \partial_\lambda S_\omega(v^\lambda)|_{\lambda=1}, I_\omega(v) = \partial_\lambda S_\omega(\lambda v)|_{\lambda=1},$ where $v^\lambda(x) := \lambda^{n/2} v(\lambda x)$ for $\lambda > 0$.

(ii) Let $\phi_\omega \in M_{G, \omega}$. There exists a Lagrange multiplier $\Lambda \in \mathbb{R}$ such that $S'_\omega(\phi_\omega) = \Lambda I'_\omega(\phi_\omega)$. Taking the pairing of this equation with $\phi$, we obtain $\langle S'_\omega(\phi_\omega), \phi \rangle = \Lambda \langle I'_\omega(\phi_\omega), \phi \rangle$. Since $\langle S'_\omega(\phi_\omega), \phi \rangle = I_\omega(\phi_\omega) = 0$ and $\langle I'_\omega(\phi_\omega), \phi \rangle = -(p-1)\|\phi_\omega\|^{p+1}_{p+1} < 0$, we have $\Lambda = 0$. Namely, $\phi_\omega$ satisfies (1.2). Moreover, for any $v \in X_G \setminus \{0\}$ satisfying $S'_\omega(v) = 0$, we have $I_\omega(v) = 0$. Thus, by the definition of $M_{G, \omega}$, we have $S_\omega(\phi_\omega) \leq S_\omega(v)$. That is, $\phi_\omega \in M_{G, \omega}$ is a minimal action solution of (1.2).

**Definition 1.2.** Let $T_V$ be the maximal linear subspace of $\mathbb{R}^n$ contained in $\{y \in \mathbb{R}^n : V(x + y) = V(x), x \in \mathbb{R}^n\}$, and for $\phi_\omega \in M_{G, \omega}$, we put

$$N_\delta(\phi_\omega) := \left\{ v \in X_G : \inf\{\|v - e^{it}\phi_\omega(\cdot + y)\|_X : \theta \in \mathbb{R}, y \in T_V\} < \delta \right\}.$$  

We say that a standing wave solution $e^{it}\phi_\omega(x)$ of (1.1) is stable in $X_G$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $u_0 \in N_\delta(\phi_\omega)$, the solution $u(t)$ of (1.1) with $u(0) = u_0$ satisfies $u(t) \in N_\varepsilon(\phi_\omega)$ for any $t \geq 0$. Otherwise, $e^{it}\phi_\omega(x)$ is said to be unstable in $X_G$. 

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Remark 1.4. Let $n = 3$, $c > 0$ and $V(x) = c(x_1^2 + x_2^2)$. In this case, we have $T_V = \{(0, 0, z) \in \mathbb{R}^3 : z \in \mathbb{R}\}$. This example will be used in Section 4.

Our main result in this paper is the following.

Theorem 1.1. Assume $(V0)$–$(V2)$, $(A1)$ and $(A2)$. Let $1 + 4/n < p < 2^* - 1$ and $\phi_\omega(x) \in \mathcal{M}_{G, \omega}$. Then there exists $\omega_* = \omega_*(n, p) \in (\omega_0, \infty)$ such that the standing wave solution $e^{i\omega t} \phi_\omega(x)$ of (1.1) is unstable in $X_G$ for any $\omega \in (\omega_*, \infty)$.

By the general theory in Grillakis, Shatah and Strauss [14, 15], under some assumptions on the spectrum of a linearized operator, the standing wave solution $e^{i\omega t} \phi_\omega(x)$ is stable (resp. unstable) if the function $\|\phi_\omega\|^2_2$ is strictly increasing (resp. decreasing) at $\omega = \omega_1$. In the case $V(x) \equiv 0$, by the scaling $\psi_\omega(x) = \omega^{1/(p-1)} \psi_1(\sqrt{\omega} x)$, it is easy to check the increase and decrease of $\|\psi_\omega\|^2_2$. However, it seems difficult to check this property of $\|\phi_\omega\|^2_2$ for general $V(x)$. So, for the proof of Theorem 1.1, we use the following sufficient condition for instability, which is a modification of Theorem 3 in [20] (see also [10, 11, 24]).

Proposition 1.1. Assume $(V0)$–$(V2)$, $(A1)$ and $(A2)$. Let $1 < p < 2^* - 1$ and $\phi_\omega(x) \in \mathcal{M}_{G, \omega}$. If $\partial_\lambda^2 E(\phi_\omega^\lambda)\big|_{\lambda=1} < 0$, then the standing wave solution $e^{i\omega t} \phi_\omega(x)$ of (1.1) is unstable in $X_G$. Here, $v^\lambda(x) := \lambda^{n/2} v(\lambda x)$ for $\lambda > 0$.

For a bounded potential $V(x)$ as in Example (iii), Rose and Weinstein [23] studied by numerical simulations that if $\omega$ is sufficiently large and $p > 1 + 4/n$, then $\|\phi_\omega\|^2_2$ would decrease. We can affirm that this numerical result is correct with mathematical precision from Theorem 1.1. Moreover, for the nonlinear Schrödinger equation (4.1) with a constant magnetic field, Gonçalves Ribeiro [11] showed that if $\omega > 0$ and $p_0(3) := 1 + 4/3 + (4\sqrt{10} - 8)/9 < 5$, then the standing wave solution $e^{i\omega t} \phi_\omega(x)$ of (4.1) is unstable in $H^1_{A,0}(\mathbb{R}^3)$ (see Section 4). Recently, in [10], one of the authors studied the case of $V(x) = |x|^2$ and proved that if $\omega > 0$ and $p \geq p_0(n) := (n^2 + 4 + 4\sqrt{n^2 + 1})/n^2$, then the standing wave solution $e^{i\omega t} \phi_\omega(x)$ of (1.1) is unstable. Here, we note that $1 + 4/n < p_0(n) < 2^* - 1$, so that Theorem 1.1 also gives an improvement of the results in [10] and [11].

This paper is organized as follows. In Section 2, we prove Theorem 1.1 using Proposition 1.1. The variational characterization of $\phi_\omega(x) \in \mathcal{M}_{G, \omega}$ and the rescaled function $\tilde{\phi}_\omega(x)$ defined by $\phi_\omega(x) = \omega^{1/(p-1)} \tilde{\phi}_\omega(\sqrt{\omega} x)$ play an important role in the proof of Theorem 1.1 (see Lemma 2.1). In Section 3, we give the proof of Proposition 1.1 following that of Theorem 3 in [20]. In Section 4, as an application of Theorem 1.1, we study the nonlinear
Schrödinger equation (4.1) with a constant magnetic field, and improve the result in Gonçalves Ribeiro [11].

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 using Proposition 1.1, which will be proved in Section 3. By simple computations, we have

\[
E(v^\lambda) = \frac{\lambda^2}{2} \|\nabla v\|^2 + \frac{1}{2} \int_{\mathbb{R}^n} V(x) |v(x)|^2 dx - \frac{\lambda^{n(p-1)/2}}{p+1} \|v\|_{p+1}^{p+1},
\]

\[
\partial_\lambda^2 E(v^\lambda) |_{\lambda=1} = \|\nabla v\|^2 + \frac{1}{2} \int_{\mathbb{R}^n} \left\{ 2x \cdot \nabla V(x) + \sum_{j,k=1}^n x_j x_k \partial_j \partial_k V(x) \right\} |v(x)|^2 dx
\]

\[
- \frac{n(p-1)}{2(p+1)} \left\{ \frac{n(p-1)}{2} - 1 \right\} \|v\|_{p+1}^{p+1}.
\]

Since \(P(\phi_\omega) = \partial_\lambda S_\omega (\phi^\lambda_\omega)|_{\lambda=1} = 0\) (see (1.5) and Remark 1.2), if we put

\[
V^*(x) = 3x \cdot \nabla V(x) + \sum_{j,k=1}^n x_j x_k \partial_j \partial_k V(x),
\]

then we have

\[
\partial_\lambda^2 E(\phi^\lambda_\omega)|_{\lambda=1} = \frac{1}{2} \int_{\mathbb{R}^n} V^*(x) |\phi_\omega(x)|^2 dx - \frac{n(p-1)}{2(p+1)} \left\{ \frac{n(p-1)}{2} - 2 \right\} \|\phi_\omega\|_{p+1}^{p+1}.
\]

Thus, we see that the condition \(\partial_\lambda^2 E(\phi^\lambda_\omega)|_{\lambda=1} < 0\) is equivalent to

\[
\frac{\int_{\mathbb{R}^n} V^*(x) |\phi_\omega(x)|^2 dx}{\|\phi_\omega\|_{p+1}^{p+1}} < \frac{n(p-1)\{n(p-1) - 4\}}{2(p+1)}.
\]

We remark that the right hand side of (2.2) is a positive constant by the assumption \(p > 1 + 4/n\) in Theorem 1.1. In what follows, we will show that the left hand side of (2.2) converges to 0 as \(\omega \to \infty\). To this end, we rescale \(\phi_\omega(x) \in \mathcal{M}_{G,\omega}\) as follows:

\[
\phi_\omega(x) = \omega^{1/(p-1)} \tilde{\phi}_\omega(\sqrt{\omega} x), \quad \omega \in (\omega_0, \infty).
\]

Then, the rescaled function \(\tilde{\phi}_\omega(x)\) satisfies

\[
-\Delta \phi + \phi + \omega^{-1} V\left(\frac{x}{\sqrt{\omega}}\right) \phi - |\phi|^{p-1} \phi = 0, \quad x \in \mathbb{R}^n.
\]
Moreover, since we have
\[ \frac{\int_{\mathbb{R}^n} V^*(x)|\tilde{\phi}_\omega(x)|^2 dx}{\|\tilde{\phi}_\omega\|_{p+1}^{p+1}} = \frac{\omega^{-1} \int_{\mathbb{R}^n} V^*(x/\sqrt{\omega})|\tilde{\phi}_\omega(x)|^2 dx}{\|\tilde{\phi}_\omega\|_{p+1}^{p+1}}, \]
it suffices to prove
\[ \lim_{\omega \to \infty} \frac{\omega^{-1} \int_{\mathbb{R}^n} V^*(x/\sqrt{\omega})|\tilde{\phi}_\omega(x)|^2 dx}{\|\tilde{\phi}_\omega\|_{p+1}^{p+1}} = 0. \tag{2.5} \]

When \( \omega \to \infty \), the term \( \omega^{-1}V(x/\sqrt{\omega})\phi \) in (2.4) disappears formally, and we expect that \( \tilde{\phi}_\omega(x) \) may converge to the unique positive radial solution \( \psi_1(x) \) of (1.3) with \( \omega = 1 \) in some sense. Since the standing wave solution \( e^{it\psi_1(x)} \) of (1.1) with \( V(x) \equiv 0 \) is unstable in \( H^1(\mathbb{R}^n) \) when \( p > 1 + 4/n \), we expect that the standing wave solution \( e^{iwt\tilde{\phi}_\omega(x)} \) of (1.1) may be also unstable in \( X_G \) when \( p > 1 + 4/n \) and \( \omega \) is sufficiently large. This is the reason why we introduce the rescaled function \( \tilde{\phi}_\omega(x) \) to prove (2.5). In what follows, we justify this formal argument. First, we put
\[ I_\omega(v) := \|\nabla v\|_2^2 + \|v\|_2^2 + \omega^{-1} \int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right)|v(x)|^2 dx - \|v\|_{p+1}^{p+1}, \]
\[ I_0^0(v) := \|\nabla v\|_2^2 + \|v\|_2^2 - \|v\|_{p+1}^{p+1}. \]

The following Lemma 2.1 is a key to show (2.5).

**Lemma 2.1.** Let \( 1 < p < 2^* - 1 \) and \( \phi_\omega \in M_{G, \omega} \) for large \( \omega \). Assume (V0), (V1.1) and \( V_2(x) \in L^q(\mathbb{R}^n) + \mathcal{L}^\infty(\mathbb{R}^n) \) for some \( q \) such that \( q > n/2 \) and \( q \geq 1 \). Let \( \tilde{\phi}_\omega(x) \) be the rescaled function defined by (2.3), and \( \psi_1(x) \) be the unique positive radial solution of (1.3) with \( \omega = 1 \) in \( H^1(\mathbb{R}^n) \). Then, we have

(i) \( \lim_{\omega \to \infty} \|\tilde{\phi}_\omega\|_{p+1}^{p+1} = \|\psi_1\|_{p+1}^{p+1} \),

(ii) \( \lim_{\omega \to \infty} I_\omega^0(\tilde{\phi}_\omega) = 0 \),

(iii) \( \lim_{\omega \to \infty} \|\tilde{\phi}_\omega\|_{H^1}^2 = \|\psi_1\|_{H^1}^2 \),

(iv) \( \lim_{\omega \to \infty} \omega^{-1} \int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right)|\tilde{\phi}_\omega(x)|^2 dx = 0 \).

We prepare one lemma to prove Lemma 2.1.

**Lemma 2.2.** Let \( U(x) \in L^q(\mathbb{R}^n) + \mathcal{L}^\infty(\mathbb{R}^n) \) for some \( q \) such that \( q > n/2 \) and \( q \geq 1 \). Then, there exists a constant \( C > 0 \) such that
\[ \left| \int_{\mathbb{R}^n} U(x)|v(x)|^2 dx \right| \leq C\|U\|_{L^q + \mathcal{L}^\infty} \|v\|_{H^1}^2, \quad v \in H^1(\mathbb{R}^n). \]
Lemma 2.2 is easily proved by the Hölder and the Gagliardo-Nirenberg inequalities. So, we omit the proof.

**Proof of Lemma 2.1.** First of all, we note that $\tilde{\phi}(x)$ is a minimizer of 

$$
\inf \left\{ \|v\|_{p+1}^{p+1} : v \in X_G \setminus \{0\}, \hat{I}_\omega(v) \leq 0 \right\},
$$

and $\psi(\xi)$ is a minimizer of 

$$
\inf \left\{ \|v\|_{p+1}^{p+1} : v \in H^1(\mathbb{R}^n) \setminus \{0\}, I_0^1(v) \leq 0 \right\},
$$

(see Lemma 3.1). In order to prove (i), we show that for any $\omega$ we have 

$$
\omega \leq \omega_0 \leq \omega_1.
$$

Indeed, from (V1.1) and Lemma 2.2, we have 

$$
1 \mu \|\psi\|_{p+1}^{p+1} \leq \|\tilde{\phi}\|_{p+1}^{p+1} \leq \mu^p \|\psi\|_{p+1}^{p+1}, \quad \omega \in (\omega(\mu), \infty).
$$

Since $\mu > 1$ is arbitrary, we conclude (i). First, from $I_0^1(\psi_1) = 0$, we have 

$$
\mu^{-2} \hat{I}_\omega(\mu \psi_1) = - (\mu^{p-1} - 1) \|\psi_1\|_{p+1}^{p+1} + \omega^{-1} \int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right) \|\psi_1(x)\|^2 dx.
$$

Since $\psi_1(x)$ has an exponential decay at infinity (see, e.g., [3, Lemma 2]), we have 

$$
\lim_{\omega \to \infty} \omega^{-1} \int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right) |\psi_1(x)|^2 dx = 0. \quad (2.7)
$$

Indeed, from (V1.1) and Lemma 2.2, we have 

$$
\omega^{-1} \int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right) |\psi_1(x)|^2 dx 
\leq \omega^{-1} \int_{\mathbb{R}^n} V_0\left(\frac{x}{\sqrt{\omega}}\right) |\psi_1(x)|^2 dx + \omega^{-1} \int_{\mathbb{R}^n} V_2\left(\frac{x}{\sqrt{\omega}}\right) |\psi_1(x)|^2 dx
\leq \omega^{-1} C \int_{\mathbb{R}^n} (1 + \omega^{-m/2} |x|^m) |\psi_1(x)|^2 dx + C(\omega^{-\theta(q)} + \omega^{-1}) \|V_2\|_{L^1 + L^\infty} \|\psi_1\|_{H^1}^2,
$$

where $\theta(q) := 1 - n/2q$. Therefore we obtain (2.7) since $|x|^m |\psi_1(x)|^2 \in L^1(\mathbb{R}^n)$ and $q > n/2$. Thus, for any $\mu > 1$, there exists $\omega(\mu) \in (\omega_0, \infty)$ such that $I_\omega(\mu \psi_1) < 0$ for any $\omega \in (\omega(\mu), \infty)$. Next, from $I_\omega(\tilde{\phi}_\omega) = 0$, we have 

$$
\mu^{-2} I_0^1(\mu \tilde{\phi}_\omega) = - (\mu^{p-1} - 1) \|\tilde{\phi}_\omega\|_{p+1}^{p+1} + \omega^{-1} \int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\phi}_\omega(x)|^2 dx
\leq - (\mu^{p-1} - 1) \|\tilde{\phi}_\omega\|_{p+1}^{p+1} + \omega^{-1} \int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\phi}_\omega(x)|^2 dx,
$$

where $\tilde{\phi}_\omega(x)$ is a minimizer of 

$$
\inf \left\{ \|v\|_{p+1}^{p+1} : v \in X_G \setminus \{0\}, \hat{I}_\omega(v) \leq 0 \right\}.
$$
where $V_-(x) = \max\{-V(x), 0\}$. From the assumptions (V0) and (V1.1), we have $V_\in L^q(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ with $q \geq 1$ and $q > n/2$. Thus, by Lemma 2.2, there exists $C > 0$ such that

$$\omega^{-1} \int_{\mathbb{R}^n} V_-(\frac{x}{\sqrt{\omega}}) |\phi_{\omega}(x)|^2 dx \leq C(\omega^{-\theta(q)} + \omega^{-1}) \|V_\|_{L^q + L^\infty} \|\phi_{\omega}\|_{H^1}^2.$$ 

Note that $\theta(q) \in (0, 1]$ since $q > n/2$. Moreover, from $I_\omega(\tilde{\phi_{\omega}}) = 0$, we have

$$\|\tilde{\phi_{\omega}}\|_{H^1}^2 \leq \|\tilde{\phi_{\omega}}\|_{H^1}^2 \leq C(\omega^{-\theta(q)} + \omega^{-1}) \|V_\|_{L^q + L^\infty} \|\tilde{\phi_{\omega}}\|_{H^1}^2,$$

which implies

$$\left(1 - C(\omega^{-\theta(q)} + \omega^{-1}) \|V_\|_{L^q + L^\infty}\right)\|\tilde{\phi_{\omega}}\|_{H^1}^2 \leq \|\tilde{\phi_{\omega}}\|_{H^1}^2.$$ 

Thus, we have

$$\mu^{-2} I_\omega^*(\mu \tilde{\phi_{\omega}}) \leq -\left(\mu^{p-1} - 1 - \frac{C(\omega^{-\theta(q)} + \omega^{-1}) \|V_\|_{L^q + L^\infty}}{1 - C(\omega^{-\theta(q)} + \omega^{-1}) \|V_\|_{L^q + L^\infty}}\right)\|\tilde{\phi_{\omega}}\|_{H^1}^2.$$ 

(2.8)

Therefore, for any $\mu > 1$, there exists $\omega_2(\mu) \in (\omega_0, \infty)$ such that $I_\omega^*(\mu \tilde{\phi_{\omega}}) < 0$ for any $\omega \in (\omega_2(\mu), \infty)$. Hence, we conclude (i).

Secondly, we show (ii). By (2.8) with $\mu = 1$ and (i), we have

$$\limsup_{\omega \to \infty} I_\omega^*(\tilde{\phi_{\omega}}) \leq 0.$$ 

Moreover, for any $\omega \in (\omega_0, \infty)$ there exists $\mu(\omega) > 0$ such that $I_\omega^*(\mu(\omega) \tilde{\phi_{\omega}}) = 0$. Thus, we have

$$\|\psi_1\|_{H^1}^2 \leq \|\mu(\omega) \tilde{\phi_{\omega}}\|_{H^1}^2 = \mu(\omega) \|\tilde{\phi_{\omega}}\|_{H^1}^2,$$

which together with (i) implies that

$$\liminf_{\omega \to \infty} \mu(\omega) \geq \liminf_{\omega \to \infty} \|\psi_1\|_{H^1}^2 = 1.$$ 

From $I_\omega^*(\mu(\omega) \tilde{\phi_{\omega}}) = 0$ and (i), we have

$$\liminf_{\omega \to \infty} I_\omega^*(\tilde{\phi_{\omega}}) = \liminf_{\omega \to \infty} (\mu(\omega) - 1) \|\tilde{\phi_{\omega}}\|_{H^1}^2 \geq 0.$$ 

Hence, we conclude (ii).

Next, from (i), (ii) and $I_\omega^*(\psi_1) = 0$, we have

$$\lim_{\omega \to \infty} \|\tilde{\phi_{\omega}}\|_{H^1}^2 = \lim_{\omega \to \infty} \|\tilde{\phi_{\omega}}\|_{H^1}^2 = \|\psi_1\|_{H^1}^2,$$

which shows (iii).
Finally, from (ii) and $\tilde{I}_\omega(\tilde{\phi}_\omega) = 0$, we have
\[ \lim_{\omega \to \infty} \omega^{-1} \int_{\mathbb{R}^n} V\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\phi}_\omega(x)|^2 \, dx = 0, \]
which shows (iv).

We are now in a position to give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** As stated above, we have only to show (2.5). Recall that $V^*_l(x)$ is defined by (2.1), and we put
\[ V^*_l(x) := 3x \cdot \nabla V_l(x) + \sum_{j,k \leq 1} x_j x_k \partial_j \partial_k V_l(x), \quad (l = 1, 2). \]

By the assumption (V2) and Lemma 2.2, we have
\[ \omega^{-1} \int_{\mathbb{R}^n} |V^*_2\left(\frac{x}{\sqrt{\omega}}\right)||\tilde{\phi}_\omega(x)|^2 \, dx \leq C(\omega^{-1} + \omega^{-\theta(q)}) \|V^*_2\|_{L^q + L^\infty} \|	ilde{\phi}_\omega\|^2_{H^1}, \quad (2.9) \]
\[ \omega^{-1} \int_{\mathbb{R}^n} |V^*_2\left(\frac{x}{\sqrt{\omega}}\right)||\tilde{\phi}_\omega(x)|^2 \, dx \leq C(\omega^{-1} + \omega^{-\theta(q)}) \|V^*_2\|_{L^q + L^\infty} \|	ilde{\phi}_\omega\|^2_{H^1}. \quad (2.10) \]

From Lemma 2.1 (iii), (iv) and (2.10), we have
\[ \lim_{\omega \to \infty} \omega^{-1} \int_{\mathbb{R}^n} V^*_1\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\phi}_\omega(x)|^2 \, dx = 0. \quad (2.11) \]

Now, from the assumption (V1.2), we have
\[ -\int_{\mathbb{R}^n} \left|1 + V^*_1\left(\frac{x}{\sqrt{\omega}}\right)\right||\tilde{\phi}_\omega(x)|^2 \, dx \leq C \omega^{-1} \int_{\mathbb{R}^n} V^*_2\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\phi}_\omega(x)|^2 \, dx. \]

Thus, from (2.11) and Lemma 2.1 (iii), we have
\[ \lim_{\omega \to \infty} \omega^{-1} \int_{\mathbb{R}^n} V^*_1\left(\frac{x}{\sqrt{\omega}}\right) |\tilde{\phi}_\omega(x)|^2 \, dx = 0. \quad (2.12) \]

Since $V^*_2(x) = V^*_1(x) + V^*_2(x)$, it follows from (2.9) and (2.12) that
\[ \lim_{\omega \to \infty} \omega^{-1} \int_{\mathbb{R}^n} |V^*_2\left(\frac{x}{\sqrt{\omega}}\right)||\tilde{\phi}_\omega(x)|^2 \, dx = 0. \]
Hence, by Lemma 2.1 (i), we obtain (2.5).

**Remark 2.1.** Let $\phi_\omega(x) \in \mathcal{M}_{G,\omega}$. Without loss of generality, we may assume that $\phi_\omega(x)$ is positive in $\mathbb{R}^n$. By Lemma 2.1 and the concentration compactness principle, we see that for any sequence $\{\omega_j\}$ with $\omega_j \to \infty$, there exist a subsequence $\{\phi_{\omega_{j_k}}(x)\}$ of $\{\phi_{\omega_j}(x)\}$ and a sequence $\{y_k\} \subset \mathbb{R}^n$ such that
\[ \lim_{k \to \infty} \|\phi_{\omega_{j_k}}(\cdot + y_k)\|_{H^1} = 0 \quad (2.13) \]
(see Theorem III.1 in [18]). Although (2.13) may give some information on the asymptotic behavior of $\phi_\omega(x) \in \mathcal{M}_{G,\omega}$ as $\omega \to \infty$, we do not use (2.13) in the proof of Theorem 1.1 directly. We also note that Lemma 2 holds for any $p \in (1, 2^* - 1)$. Finally, we remark that, in the case $p = 1 + 4/n$, it follows from (2.13) that $\lim_{\omega \to \infty} \|\phi_\omega\|_2^2 = \|\psi_1\|_2^2$.

3. Proof of Proposition 1.1

In this section we give the proof of Proposition 1.1 following that of Theorem 3 in [20].

**Lemma 3.1.** Let $\phi_\omega \in \mathcal{M}_{G,\omega}$. Then, we have

(i) $\|\phi_\omega\|_{p+1} = \inf\{\|v\|_{p+1}: v \in X_G \setminus \{0\}, I_\omega(v) = 0\} = \inf\{\|v\|_{p+1}: v \in X_G \setminus \{0\}, I_\omega(v) \leq 0\}$,

(ii) $S_\omega(\phi_\omega) = \inf\{S_\omega(v): v \in X_G, \|v\|_{p+1} = \|\phi_\omega\|_{p+1}\}$.

**Proof.** (i) Since we have

$$S_\omega(v) = \frac{1}{2} I_\omega(v) + \frac{p-1}{2(p+1)} \|v\|_{p+1}^p, \quad v \in X_G,$$

we see that

$$d_1(\omega) := \inf\{S_\omega(v): v \in X_G \setminus \{0\}, I_\omega(v) = 0\} = \inf\left\{\frac{p-1}{2(p+1)} \|v\|_{p+1}: v \in X_G \setminus \{0\}, I_\omega(v) = 0\right\},$$

and $d_1(\omega) = S_\omega(\phi_\omega) = [(p-1)/(2(p+1))]\|\phi_\omega\|_{p+1}$. We put

$$d_2(\omega) := \inf\left\{\frac{p-1}{2(p+1)} \|v\|_{p+1}: v \in X_G \setminus \{0\}, I_\omega(v) \leq 0\right\}.$$

Since it is clear that $d_2(\omega) \leq d_1(\omega)$, we show $d_1(\omega) \leq d_2(\omega)$. For any $v \in X_G \setminus \{0\}$ satisfying $I_\omega(v) < 0$, there exists $\lambda_0 \in (0, 1)$ such that $I_\omega(\lambda_0 v) = 0$. Thus, we have

$$d_1(\omega) \leq \frac{p-1}{2(p+1)} \|\lambda_0 v\|_{p+1} = \frac{(p-1)}{2(p+1)} \lambda_0^{p+1} \|v\|_{p+1} < \frac{p-1}{2(p+1)} \|v\|_{p+1}.$$

Hence, we have $d_1(\omega) \leq d_2(\omega)$.

(ii) We put $d_3(\omega) := \inf\{S_\omega(v): v \in X_G, \|v\|_{p+1} = \|\phi_\omega\|_{p+1}\}$. Since $d_3(\omega) \leq S_\omega(\phi_\omega)$, it suffices to prove $S_\omega(\phi_\omega) \leq d_3(\omega)$. By (i), for any $v \in X_G$ satisfying $\|v\|_{p+1} = \|\phi_\omega\|_{p+1}$, we have $I_\omega(v) \geq 0$. Thus, we have

$$S_\omega(v) \geq \frac{p-1}{2(p+1)} \|v\|_{p+1} = \frac{p-1}{2(p+1)} \|\phi_\omega\|_{p+1} = S_\omega(\phi_\omega).$$
Therefore, we obtain $S_\omega(\phi_\omega) \leq d_3(\omega)$. \hfill \Box

**Lemma 3.2.** If $\partial^2_\lambda \bar{E}(\phi^\lambda_\omega)|_{\lambda=1} < 0$, then there exist positive constants $\varepsilon_1$ and $\delta_1$ with the following properties: for any $v \in N_{\delta_1}(\phi_\omega)$ satisfying $\|v\|^2 = \|\phi_\omega\|^2$, there exists $\lambda(v) \in (1 - \varepsilon_1, 1 + \varepsilon_1)$ such that $E(\phi_\omega) \leq E(v) + (\lambda(v) - 1)P(v)$, where $N_{\delta_1}(\phi_\omega)$ is the set defined in Definition 1.2.

**Proof.** From the assumption $\partial^2_\lambda \bar{E}(\phi^\lambda_\omega)|_{\lambda=1} < 0$ and the continuity of $\partial^2_\lambda \bar{E}(v^\lambda)$ in $\lambda$ and $v$, there exist $\varepsilon_1, \delta_1 > 0$ such that $\partial^2_\lambda \bar{E}(v^\lambda) < 0$ for any $\lambda \in (1 - \varepsilon_1, 1 + \varepsilon_1)$ and $v \in N_{\delta_1}(\phi_\omega)$. Since $\partial_\lambda E(v^\lambda)|_{\lambda=1} = P(v)$, the Taylor expansion at $\lambda = 1$ gives

$$E(v^\lambda) \leq E(v) + (\lambda - 1)P(v), \quad \lambda \in (1 - \varepsilon_1, 1 + \varepsilon_1), \quad v \in N_{\delta_1}(\phi_\omega). \quad (3.1)$$

For any $v \in N_{\delta_1}(\phi_\omega)$, we put $\lambda(v) := (\|\phi_\omega\|^2_1/\|v\|^2_1)^{2/n(p-1)}$. Then, we have $\|v^{\lambda(v)}\|^2_1 = \|\phi_\omega\|^2_1$, and we can take $\delta_1$ small enough to have $\lambda(v) \in (1 - \varepsilon_1, 1 + \varepsilon_1)$. Furthermore, from Lemma 3.1 (ii), if $\|v\|^2_2 = \|\phi_\omega\|^2_2$, we have

$$E(v^{\lambda(v)}) = S_\omega(v^{\lambda(v)}) - \frac{\omega}{2}\|v^{\lambda(v)}\|^2_2 \geq S_\omega(\phi_\omega) - \frac{\omega}{2}\|\phi_\omega\|^2_2 = E(\phi_\omega). \quad (3.2)$$

Therefore, from (3.1) and (3.2), we have $E(\phi_\omega) \leq E(v) + (\lambda(v) - 1)P(v)$ for any $v \in N_{\delta_1}(\phi_\omega)$ satisfying $\|v\|^2_2 = \|\phi_\omega\|^2_2$. \hfill \Box

**Definition 3.1.** Let $\delta_1$ be the positive constant in Lemma 3.2. We put

$$A := \{v \in N_{\delta_1}(\phi_\omega); \; E(v) < E(\phi_\omega), \; \|v\|^2_2 = \|\phi_\omega\|^2_2, \; P(v) < 0\},$$

and for any $u_0 \in N_{\delta_1}(\phi_\omega)$, we define the exit time from $N_{\delta_1}(\phi_\omega)$ by

$$T(u_0) = \sup\{T > 0 : u(t) \in N_{\varepsilon_1}(\phi_\omega), 0 \leq t \leq T\},$$

where $u(t)$ is a solution of (1.1) with $u(0) = u_0$.

**Lemma 3.3.** If $\partial^2_\lambda \bar{E}(\phi^\lambda_\omega)|_{\lambda=1} < 0$, then for any $u_0 \in A$, there exists $\varepsilon_0 = \varepsilon_0(u_0) > 0$ such that $P(u(t)) \leq -\varepsilon_0$ for $0 \leq t < T(u_0)$.

**Proof.** Take $u_0 \in A$ and put $\varepsilon_2 = E(\phi_\omega) - E(u_0) > 0$. From Lemma 3.2 and the conservation laws in the assumption (A1), we have

$$\varepsilon_2 \leq (\lambda(u(t)) - 1)P(u(t)), \quad 0 \leq t < T(u_0). \quad (3.3)$$

Thus, we see that $P(u(t)) \neq 0$ for $0 \leq t < T(u_0)$. Since the function $t \mapsto P(u(t))$ is continuous and $P(u_0) < 0$, we have $P(u(t)) < 0$ for $0 \leq t < T(u_0)$. Therefore, from Lemma 3.2 and (3.3), we have

$$-P(u(t)) \geq \frac{\varepsilon_2}{1 - \lambda(u(t))} \geq \frac{\varepsilon_2}{\varepsilon_1}, \quad 0 \leq t < T(u_0).$$

Hence, putting $\varepsilon_0 = \varepsilon_2/\varepsilon_1$, we have $P(u(t)) \leq -\varepsilon_0$ for $0 \leq t < T(u_0)$. \hfill \Box
Proof of Proposition 1.1. Since $\partial_\lambda E(\phi_\omega^\lambda)|_{\lambda=1} = 0$, $\partial^2_{\lambda} E(\phi_\omega^\lambda)|_{\lambda=1} < 0$ and $P(\phi_\omega^\lambda) = \lambda \partial_\lambda E(\phi_\omega^\lambda)$, we have $E(\phi_\omega^\lambda) < E(\phi_\omega)$ and $P(\phi_\omega^\lambda) < 0$ for $\lambda > 1$ sufficiently close to 1. Furthermore, since $\|\phi_\omega^\lambda\|_2^2 = \|\phi_\omega\|_2^2$ and $\lim_{\lambda \to 1} \|\phi_\omega^\lambda - \phi_\omega\|_X = 0$, we have $\phi_\omega^\lambda \in A$ for $\lambda > 1$ sufficiently close to 1. Since we assume $|x|\phi_\omega^\lambda(x) \in L^2(\mathbb{R}^n)$ in the assumption (A2), it follows from the virial identity (1.4) in the assumption (A1) that

$$\frac{d^2}{dt^2} \|xu_\lambda(t)\|_2^2 = 8P(u_\lambda(t)), \quad 0 \leq t < T(\phi_\omega^\lambda),$$  

(3.4)

where $u_\lambda(t)$ is the solution of (1.1) with $u_\lambda(0) = \phi_\omega^\lambda$. From Lemma 3.3, there exists $\varepsilon_\lambda > 0$ such that

$$P(u_\lambda(t)) \leq -\varepsilon_\lambda, \quad 0 \leq t < T(\phi_\omega^\lambda).$$  

(3.5)

Hence, from (3.4) and (3.5), we can conclude that $T(\phi_\omega^\lambda) < \infty$. Since $\lim_{\lambda \to 1} \|\phi_\omega^\lambda - \phi_\omega\|_X = 0$, the proof is completed. \qed

4. NLS WITH A CONSTANT MAGNETIC FIELD

In this section, we consider the nonlinear Schrödinger equation with a constant magnetic field $B = (0, 0, b)$:

$$i\partial_t u = -(\nabla + iA(x))^2 u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R}^{1+3},$$  

(4.1)

where $1 < p < 5$ and

$$A(x_1, x_2, x_3) = \frac{b}{2}(-x_2, x_1, 0), \quad b \in \mathbb{R} \setminus \{0\}.$$  

Here, we note that $B = \text{rot } A(x) = (0, 0, b)$, $\text{div } A(x) = 0$ and

$$-(\nabla + iA(x))^2 u = -\Delta u - 2iA(x) \cdot \nabla u + |A(x)|^2 u = -\Delta u - bi\frac{\partial u}{\partial \theta} + \frac{b^2}{4}\rho^2 u,$$

where we used the cylindrical coordinates $(\rho, \theta, z)$ in $\mathbb{R}^3$:

$$x_1 = \rho \cos \theta, \quad x_2 = \rho \sin \theta, \quad x_3 = z.$$  

As in [11], we consider (4.1) in the closed subspace $H^1_{A, 0}(\mathbb{R}^3) = \{v \in H^1(\mathbb{R}^3) : \rho v \in L^2(\mathbb{R}^3), \quad v = v(\rho, z) \text{ does not depend on } \theta\}$ of the energy space $H^1_A(\mathbb{R}^3) = \{v \in L^2(\mathbb{R}^3) : (\nabla + iA(x))v \in L^2(\mathbb{R}^3)\}$. We note that in $H^1_{A, 0}(\mathbb{R}^3)$, equation (4.1) is equivalent to

$$i\partial_t u = -\Delta u + \frac{b^2}{4}\rho^2 u - |u|^{p-1}u, \quad (t, x) \in \mathbb{R}^{1+3}.$$  

(4.2)

Let $V_1(x) = (b^2/4)(x_1^2 + x_2^2) = (b^2/4)\rho^2$, $V_2(x) \equiv 0$, and let $G$ be the group of rotations around the $x_3$-axis in $\mathbb{R}^3$. Then, $V(x) = V_1(x) + V_2(x) = \frac{b^2}{4}\rho^2$.\]
the standing wave solution 4.1, Gonçalves Ribeiro [11] proved that if $1 + 4^p < 3$, and $E$, $S_\omega$ and $L_\omega$ on $H^1_{A,0}(\mathbb{R}^3)$ are defined as in Section 1. The assumption (A1) is verified by [5, 12]. For the assumption (A2), the existence of minimal action solution $\phi_\omega(\rho, z)$ of the stationary problem:

$$-\Delta \phi + \omega \phi + \frac{b^2}{4} \rho^2 \phi - |\phi|^{p-1} \phi = 0, \quad x \in \mathbb{R}^3$$

in $H^1_{A,0}(\mathbb{R}^3)$ was proved by Esteban and Lions [8] for $\omega \in (-|b|, \infty)$. More precisely, we have

**Lemma 4.1.** Let $1 < p < 5$ and $\omega \in (-|b|, \infty)$. Then, the set $M_{G,\omega}$ is not empty, i.e., there exists a minimizer $\phi_\omega(\rho, z)$ of

$$\inf \{ S_\omega(v) : v \in H^1_{A,0}(\mathbb{R}^3) \setminus \{0\}, \quad I_\omega(v) = 0 \}.$$

**Proof.** Esteban and Lions [8] proved that for any $\omega \in (-|b|, \infty)$, there exists a minimizer $\varphi_\omega(x)$ for

$$\alpha_\omega := \inf \{ W_\omega(v) : v \in H^1_{A,0}(\mathbb{R}^3), \quad \|v\|_{p+1} = 1 \},$$

where

$$W_\omega(v) = I_\omega(v) + \|v\|^{p+1}_{p+1} = \|\nabla v\|^2_2 + \omega \|v\|^2_2 + \frac{b^2}{4} \int_{\mathbb{R}^3} \rho^2 |v(x)|^2 dx.$$

Here, we put $\phi_\omega(x) = \alpha_\omega^{1/(p-1)} \varphi_\omega(x)$. Then, we have $\phi_\omega \in H^1_{A,0}(\mathbb{R}^3) \setminus \{0\}$ and $I_\omega(\phi_\omega) = 0$. Moreover, for any $v \in H^1_{A,0}(\mathbb{R}^3) \setminus \{0\}$ satisfying $I_\omega(v) = 0$, we have

$$S_\omega(\phi_\omega) = \frac{p-1}{2(p+1)} \alpha_\omega^{(p+1)/(p-1)} \leq \frac{p-1}{2(p+1)} W_\omega \left( \frac{v}{\|v\|_{p+1}} \right)^{(p+1)/(p-1)} = \frac{p-1}{2(p+1)} \|v\|^{p+1}_{p+1} = S_\omega(v).$$

Hence, we conclude that $\phi_\omega \in M_{G,\omega}$. \hfill \Box

The stability of standing wave solutions of (4.1) was studied by Cazenave and Esteban [5] for the case $1 < p < 1 + 4/3$. For $\phi_\omega(\rho, z) \in M_{G,\omega}$ in Lemma 4.1, Gonçalves Ribeiro [11] proved that if $1 + 4/3 + (4\sqrt{10} - 8)/9 \leq p < 5$, the standing wave solution $e^{i\omega t} \phi_\omega(\rho, z)$ of (4.1) is unstable in $H^1_{A,0}(\mathbb{R}^3)$ for any $\omega > 0$. Here, we remark that $\phi_\omega(\rho, z)$ exists for $\omega \in (-|b|, \infty)$. Applying Theorem 1.1 to (4.2), we obtain the following theorem, which covers the case $1 + 4/3 < p < 1 + 4/3 + (4\sqrt{10} - 8)/9$ and gives an improvement of the above result by Gonçalves Ribeiro [11].
Theorem 4.1. Let \( 1 + 4/3 < p < 5 \) and \( \phi_\omega(\rho, z) \in M_{G,\omega} \). Then there exists \( \omega_* = \omega_*(p, b) \) such that the standing wave solution \( e^{i\omega t} \phi_\omega(\rho, z) \) of (4.1) is unstable in \( H^1_{A,0}(\mathbb{R}^3) \) for any \( \omega \in (\omega_*, \infty) \).

Proof. We apply Theorem 1.1 to (4.2). As stated above, \( V(x) = V_1(x) + V_2(x) = (b^2/4)\rho^2 \) satisfies (V0)–(V2) and (A1). For (A2), by Lemma 4.1, the set \( M_{G,\omega} \) is not empty for \( \omega \in (-|b|, \infty) \). Thus, we have only to show that
\[
M_{G,\omega} \subset \{ v \in H^1_{A,0}(\mathbb{R}^3) : |x|v(x) \in L^2(\mathbb{R}^3) \}, \quad \omega \in (0, \infty).
\] (4.4)
For any \( \omega > 0 \), it follows from [21, Theorem 2.5] that the operator \(-\Delta + (b^2/4)\rho^2 + \omega\) is m-accretive in \( L^r(\mathbb{R}^n) \) for \( 1 < r < \infty \). By following the argument of Cazenave [4, Theorem 8.1.1], we see that all \( v \in M_{G,\omega} \) decay exponentially. Therefore, we have (4.4). This completes the proof. \( \square \)

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