TOHOKU
UNIVERSITY

Lecture of FreeFEM++
2015.8.2

Update:2015/7/29

About this lecture

- The goal of this lecture is:
  - At the first, 3~4 participants are grouped,
  - They understand how to use throughout total 10 problems,
  - They should teach each other in the same group
  - They can search anything from internet.

Schedule

- 9:00-10:20
  - *Running Sample files and Mesh Generating*
- 10:30-11:50
  - *Poisson equations*
- 12:00-13:30 Lunch Time
- 13:30-14:50
  - *Convection-Diffusion equations*
- 15:00-16:20
  - *Navier-Stokes equations*
- 16:30-17:30
  - *Free Time*

One lecture is 80 minutes in which,
30 minutes is used for explanations from me,
50 minutes is used for exercise.

Grouping and notations

- Grouping
  - 3~4 participants are grouped.
- Notations
  - About evaluations from A. Prof. J. Masamune

Schedule

- ***9:00-10:20***
  - ***Running Sample files and Mesh Generating***
- 10:30-11:50
  - *Poisson equations*
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- 16:30-17:30
  - *Free Time*

Running Sample files

- Laplace.edp
- diffusion.edp
- convection.edp
- tunnel.edp
- LapComplexEigenValue.edp
- Mesh_square.edp
- Mesh_circle.edp
- Mesh_circle_in_square.edp

I show you how to generate finite element meshes

Mesh generation(Mesh_square.edp)

```
border a0(t=1,0){ x=0; y=t; label=1;}
border a1(t=0,1){ x=t; y=0; label=2;}
border a2(t=0,1){ x=1; y=t; label=3;}
border a3(t=1,0){ x=t; y=1; label=4;}
int n=5;
Mesh  Th
=buildmesh( a0(10*n)+a1(10*n)+a2(10*n)+a3(10*n));
```

Mesh generation(Mesh_square.edp)

```
border a0(t=1,0){ x=0; y=t; label=1;}
border a1(t=0,1){ x=t; y=0; label=2;}
border a2(t=0,1){ x=1; y=t; label=3;}
border a3(t=1,0){ x=t; y=1; label=4;}
int n=5;
Mesh   Th
=buildmesh( a0(10*n)+a1(10*n)+a2(10*n)+a3(10*n));
```

Command for definitions of borders

```
border a0(t=1,0){ x=0; y=t; label=1;}
```

border a0(t=1,0){ x=0; y=t; label=1;}

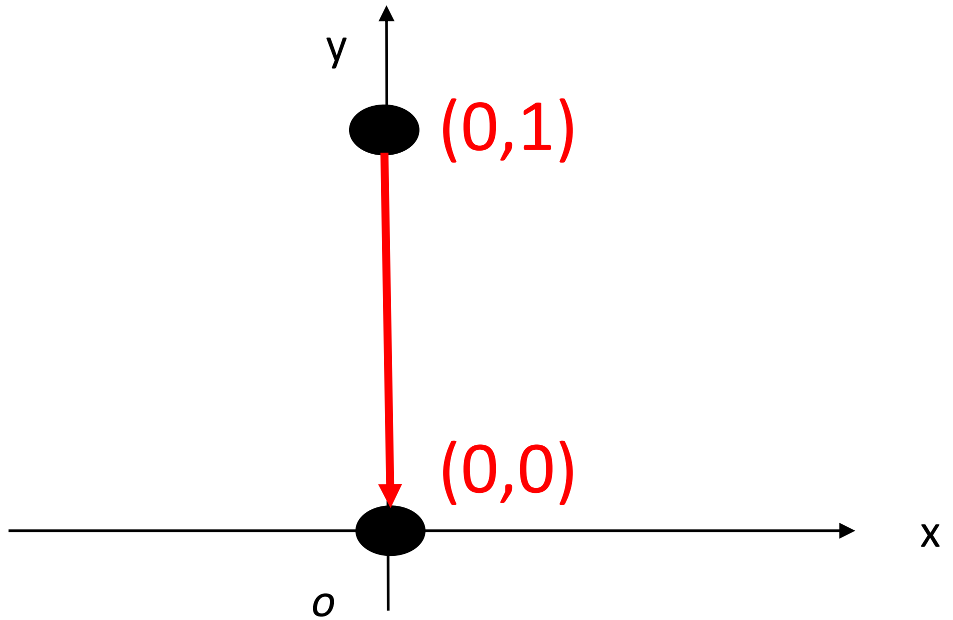
The name of border.
You can use any words which you want.

```
border a0(t=1,0){ x=0; y=t; label=1;}
```

A parameter range from 1 to 0.
You can use any number which you want.

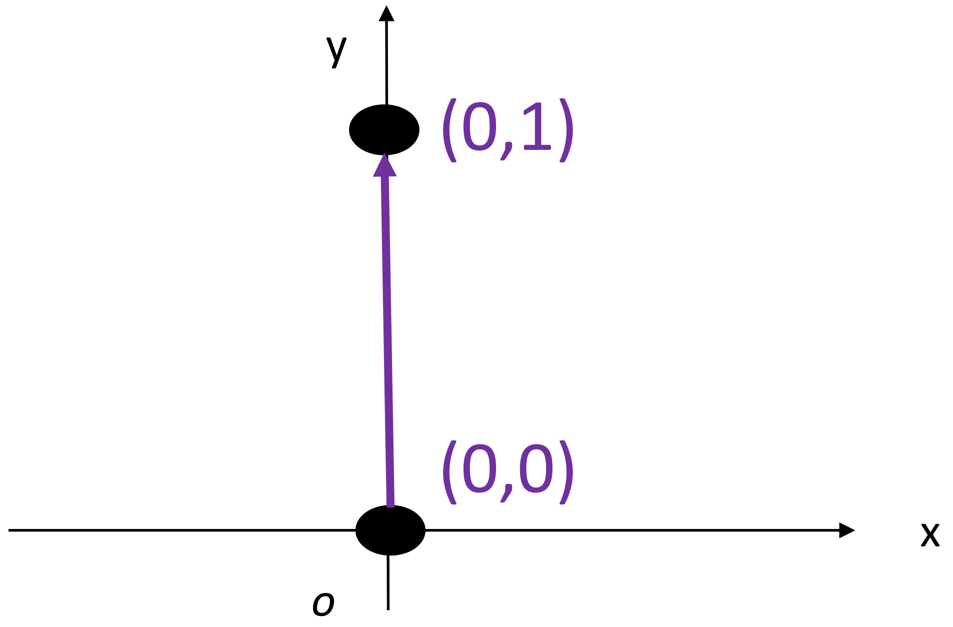
positions of borders
(Like position vector?)

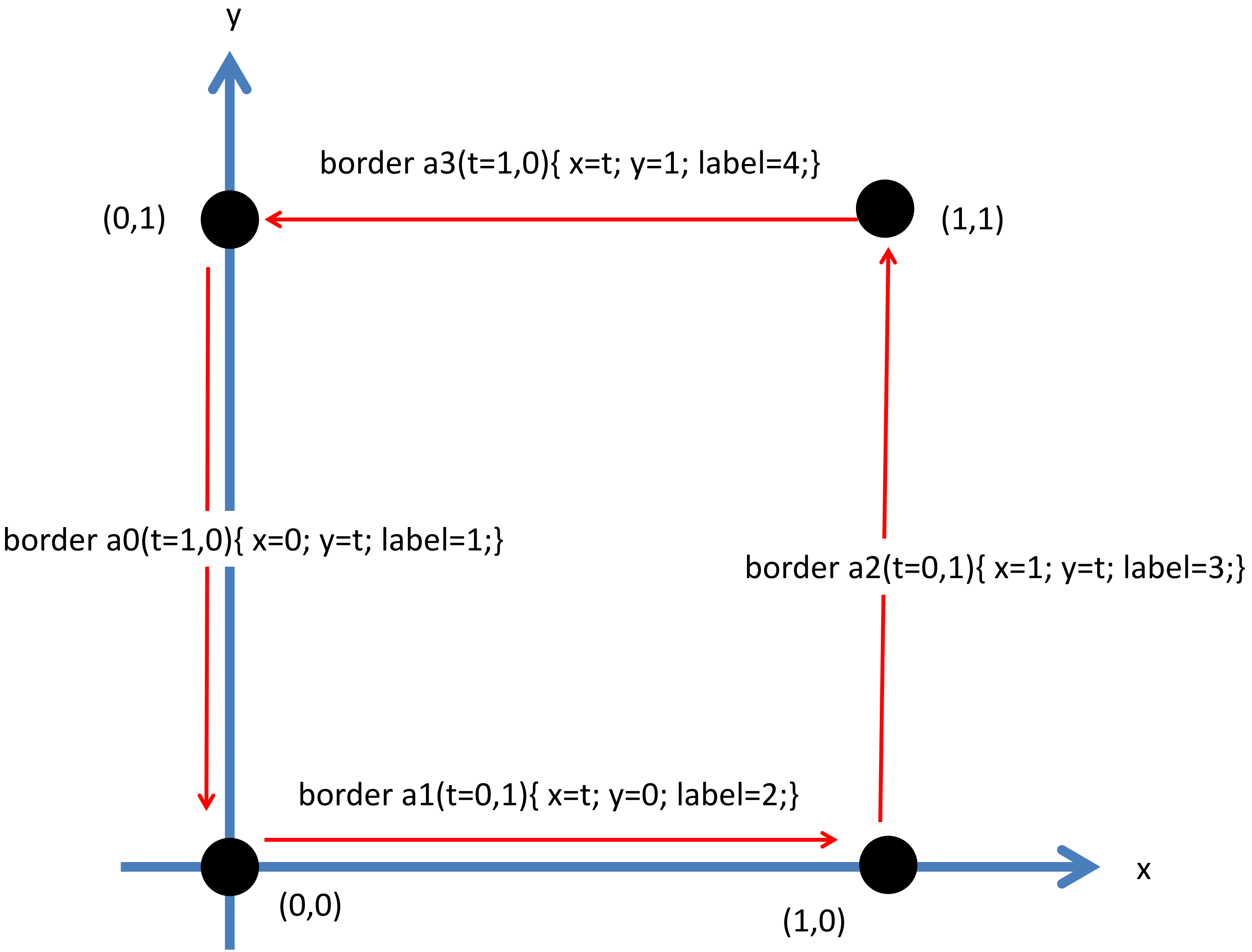
```
border a0(t=1,0){ x=0; y=t; label=1;}
```

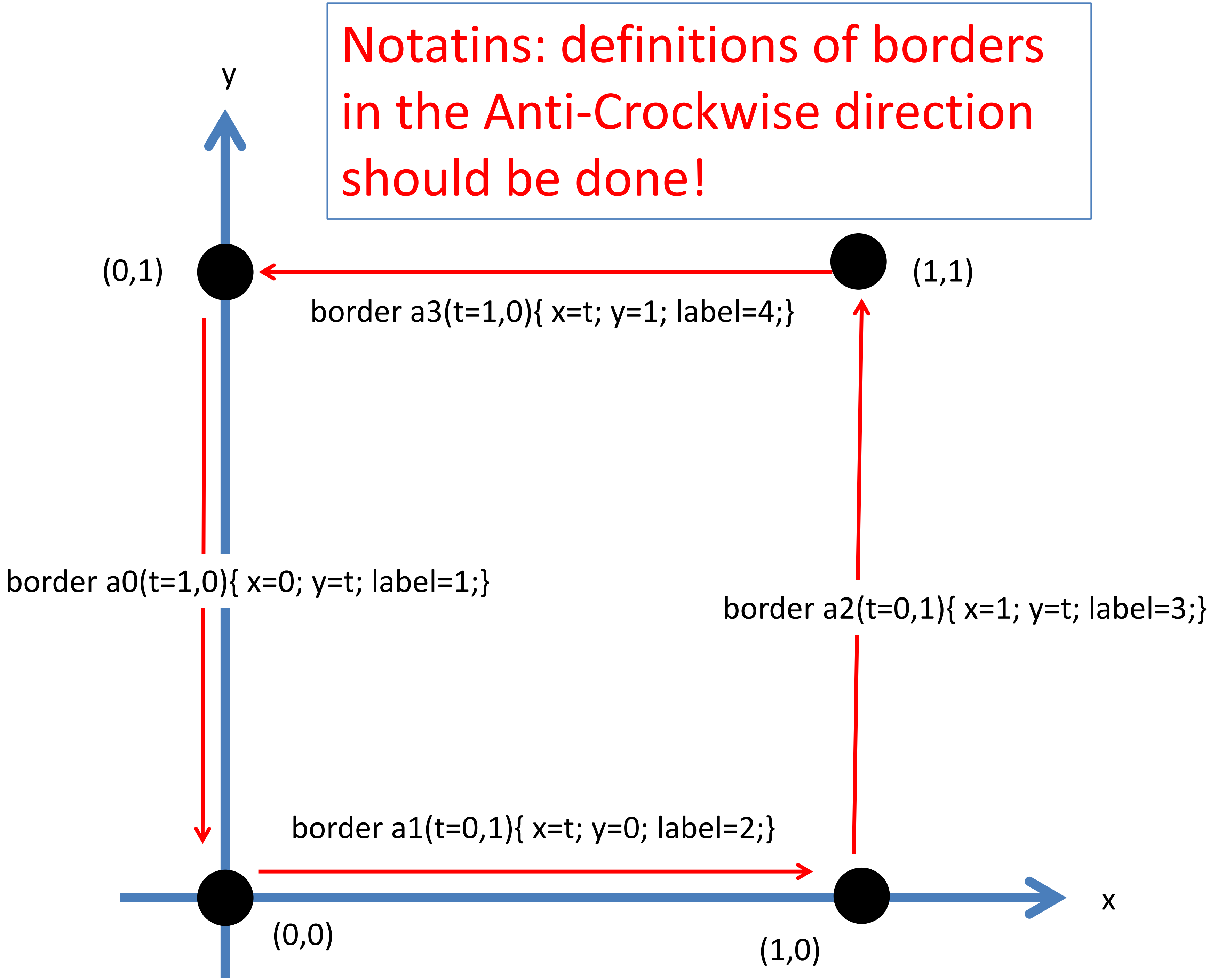


positions of borders
(Like position vector?)

```
border a0(t=0,1){ x=0; y=t; label=1;}
```




y
border a3(t=1,0){ x=t; y=1; label=4;}
(0,1)
(1,1)
border a0(t=1,0){ x=0; y=t; label=1;}
border a2(t=0,1){ x=1; y=t; label=3;}
border a1(t=0,1){ x=t; y=0; label=2;}
x
(0,0)
(1,0)


Notatins: definitions of borders in the Anti-Crockwise direction should be done!
y
(0,1)
(1,1)
border a3(t=1,0){ x=t; y=1; label=4;}
border a0(t=1,0){ x=0; y=t; label=1;}
border a2(t=0,1){ x=1; y=t; label=3;}
border a1(t=0,1){ x=t; y=0; label=2;}
x
(0,0)
(1,0)

Mesh generation(Mesh_square.edp)

```
border a0(t=1,0){ x=0; y=t; label=1;}
border a1(t=0,1){ x=t; y=0; label=2;}
border a2(t=0,1){ x=1; y=t; label=3;}
border a3(t=1,0){ x=t; y=1; label=4;}
```

```
int n=5;
Mesh   Th
=buildmesh( a0(10*n)+a1(10*n)+a2(10*n)+a3(10*n));
```

Commands for definitions of finite element meshes

```
Mesh   Th
=buildmesh( a0(10*n)+a1(10*n)+a2(10*n)+a3(10*n));
```

Commands for definitions of finite element meshes

The name of Mesh
You can use any words which you want.

Mesh Th
=buildmesh(a0(10*n)+a1(10*n)+a2(10*n)+a3(10*n));

Fineness of the mesh: should be defined by integer

Mesh Th
=buildmesh(a0(10*n)+a1(10*n)+a2(10*n)+a3(10*n));

Mesh generation(Mesh_square.edp)

```
border a0(t=1,0){ x=0; y=t; label=1;}
border a1(t=0,1){ x=t; y=0; label=2;}
border a2(t=0,1){ x=1; y=t; label=3;}
border a3(t=1,0){ x=t; y=1; label=4;}
```

Making boundaries

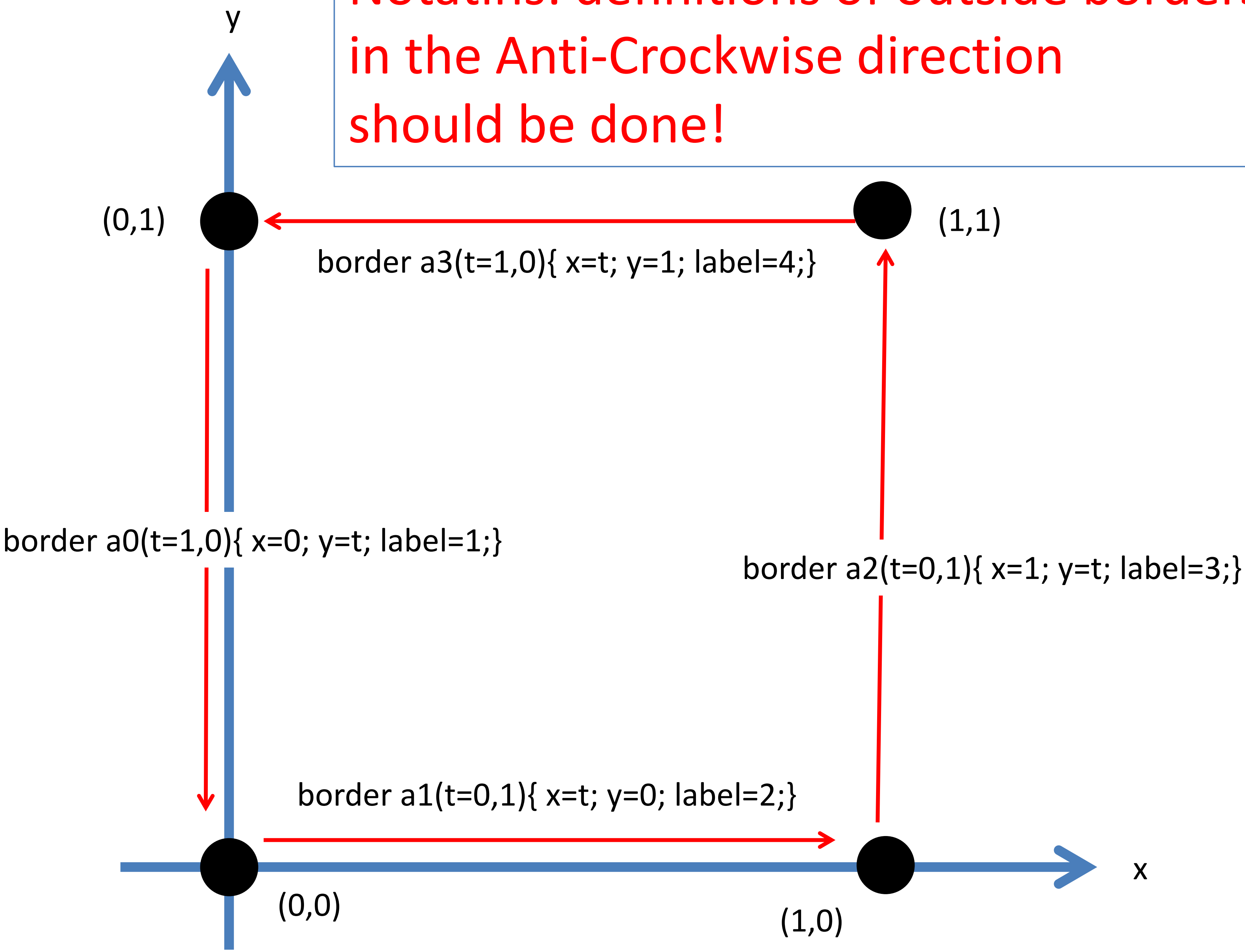
```
int n=5;
Mesh  Th
=buildmesh( a0(10*n)+a1(10*n)+a2(10*n)+a3(10*n));
```

Generating finite elements(triangles)

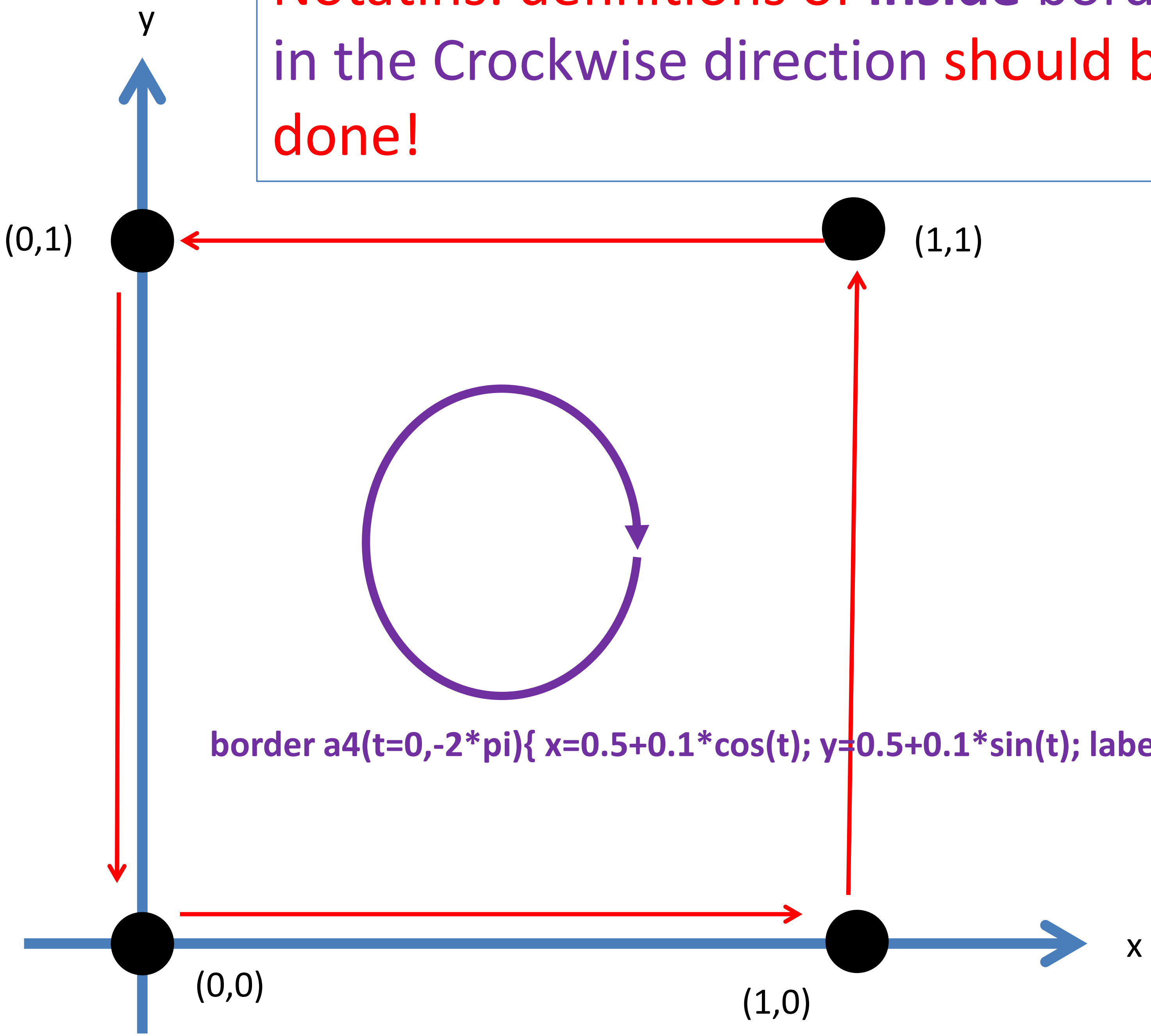
To define inside boundary Mesh_circle_in_square.edp

```
border a0(t=1,0){ x=0; y=t; label=1;}
border a1(t=0,1){ x=t; y=0; label=2;}
border a2(t=0,1){ x=1; y=t; label=3;}
border a3(t=1,0){ x=t; y=1; label=4;}
border a4(t=0,-2*pi){ x=0.5+0.1*cos(t); y=0.5+0.1*sin(t); label=1;}
int n=5;
mesh Th=
buildmesh( a0(10*n)+a1(10*n)+a2(10*n)+a3(10*n)+a4(10*n));
```

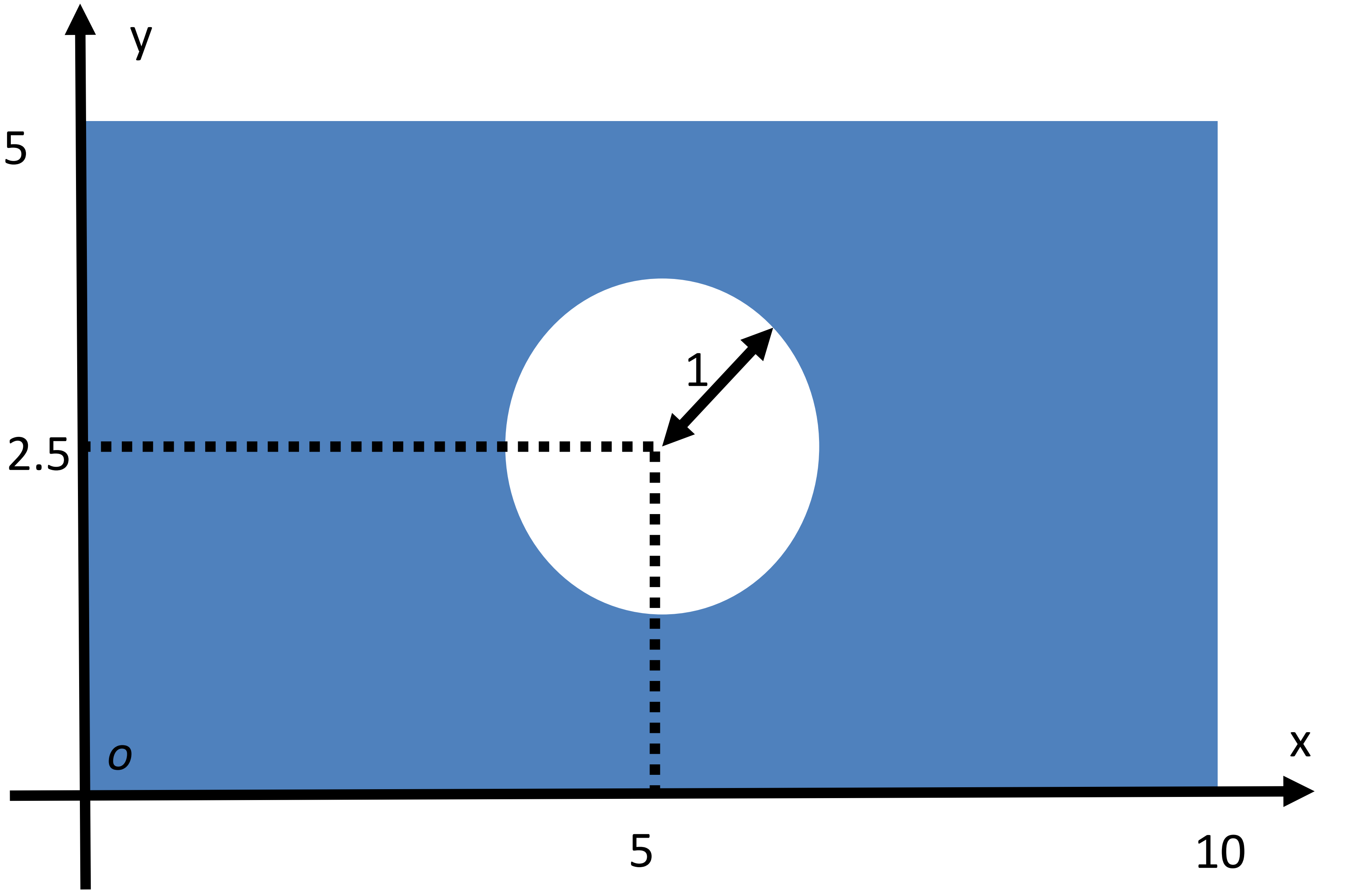
Notatins: definitions of outside borders in the Anti-Crockwise direction should be done!



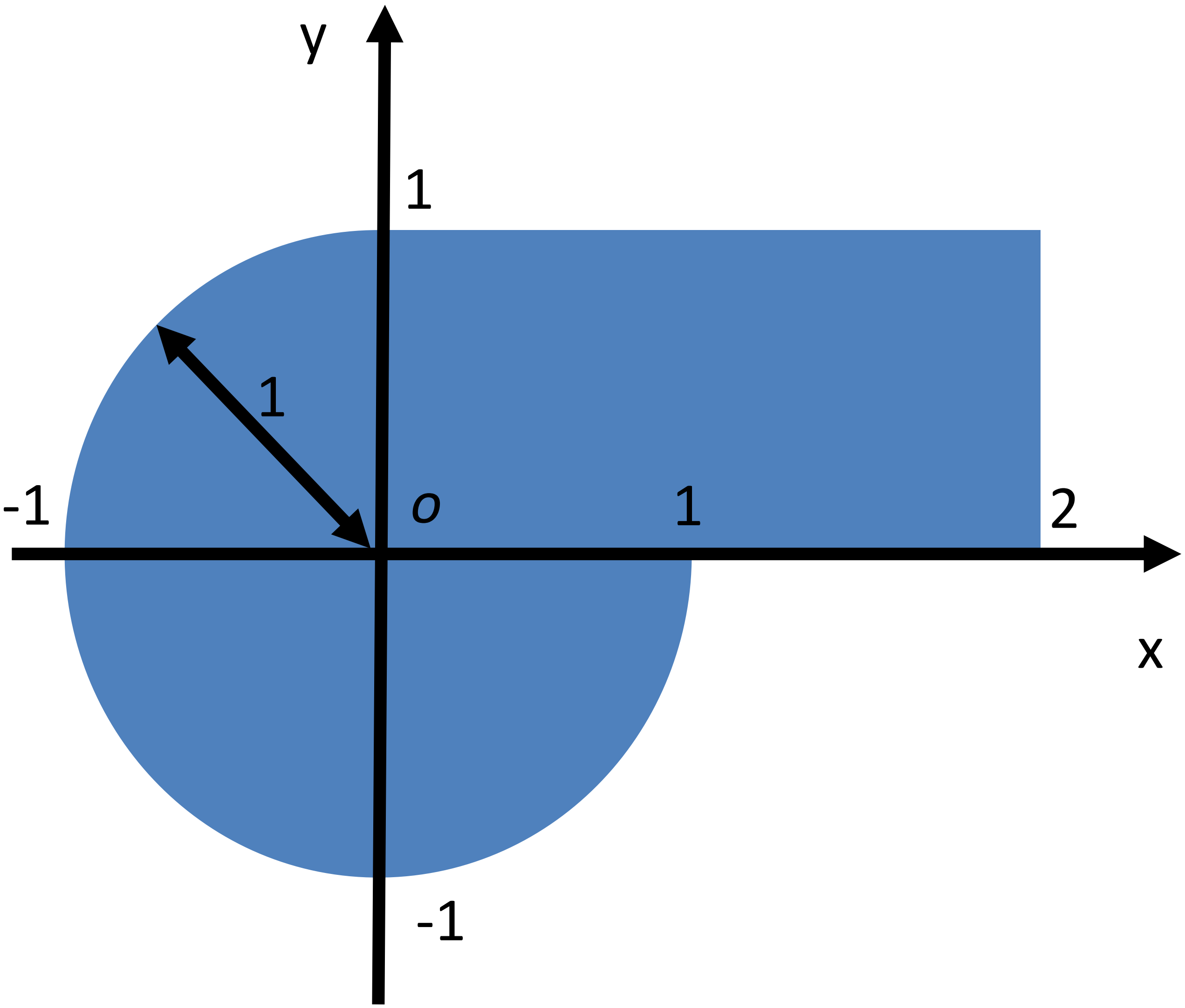
Notatins: definitions of **inside** borders in the Crockwise direction should be done!



Problem1



Problem2



Schedule

- 9:00-10:20
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  - ***Poisson equations***
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- 16:30-17:30
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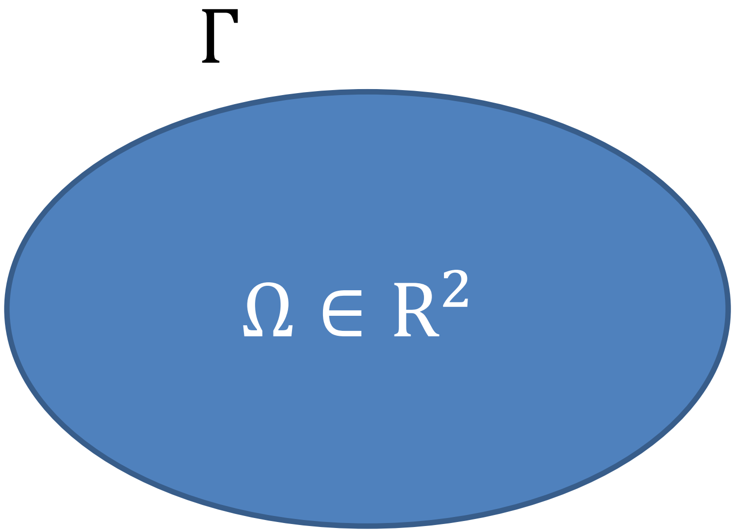
Strong form of Poisson eq.



$$-\Delta u - f = 0 \text{ in } \Omega$$

$$u = 0 \text{ on } \Gamma$$

Weak form of Poisson eq.

$$-\int_{\Omega}(\Delta u - f)v\, d\Omega = 0$$

$$lhs = -\int_{\Omega}\nabla\cdot(\nabla u)v d\Omega + \int_{\Omega} f v d\Omega$$

$$= -\int_{\Omega}\left(\frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}\right)v\, d\Omega + \int_{\Omega} f v d\Omega$$

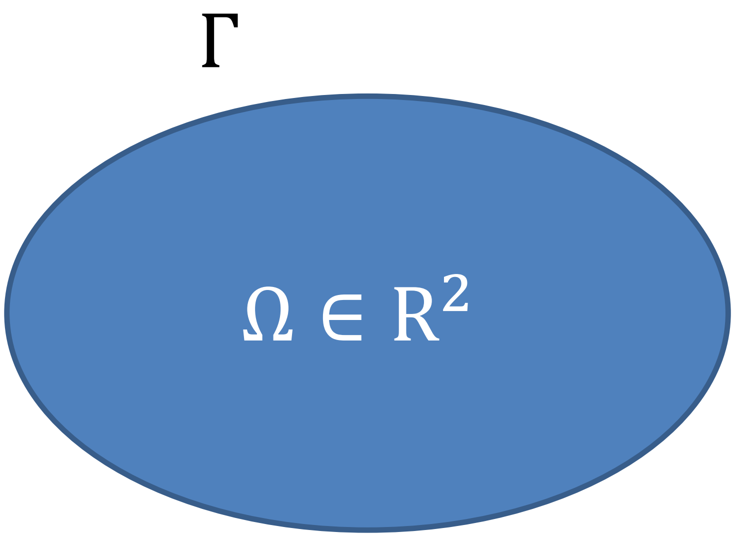
$$= -\int_{\Omega}\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}v\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}v\right)\right)\, d\Omega + \int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\, d\Omega + \int_{\Omega} f v d\Omega$$

$$= -\int_{\Omega}\nabla\cdot\left(\frac{\partial u}{\partial x}v + \frac{\partial u}{\partial y}v\right)\, d\Omega + \int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\, d\Omega + \int_{\Omega} f v d\Omega$$

$$= -\int_{\Gamma}\left(\frac{\partial u}{\partial x}v + \frac{\partial u}{\partial y}v\right)\cdot \boldsymbol{n} d\gamma + \int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\, d\Omega + \int_{\Omega} f v d\Omega$$

$$= -\int_{\Gamma}\left(\frac{\partial u}{\partial n}\right)v d\gamma + \int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\, d\Omega + \int_{\Omega} f v d\Omega$$

Strong form and Weak form of Poisson eq.



$$-\Delta u - f = 0 \text{ in } \Omega$$
$$u = 0 \text{ on } \Gamma$$

$$\int_{\Omega} \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega - \int_{\Gamma} \left(\frac{\partial u}{\partial n} \right) v d\gamma + \int_{\Omega} f v d\Omega = 0$$

Boundary Conditions

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) d\Omega - \int_\Gamma \left(\frac{\partial u}{\partial n}\right) v d\gamma + \int_\Omega f v d\Omega = 0$$

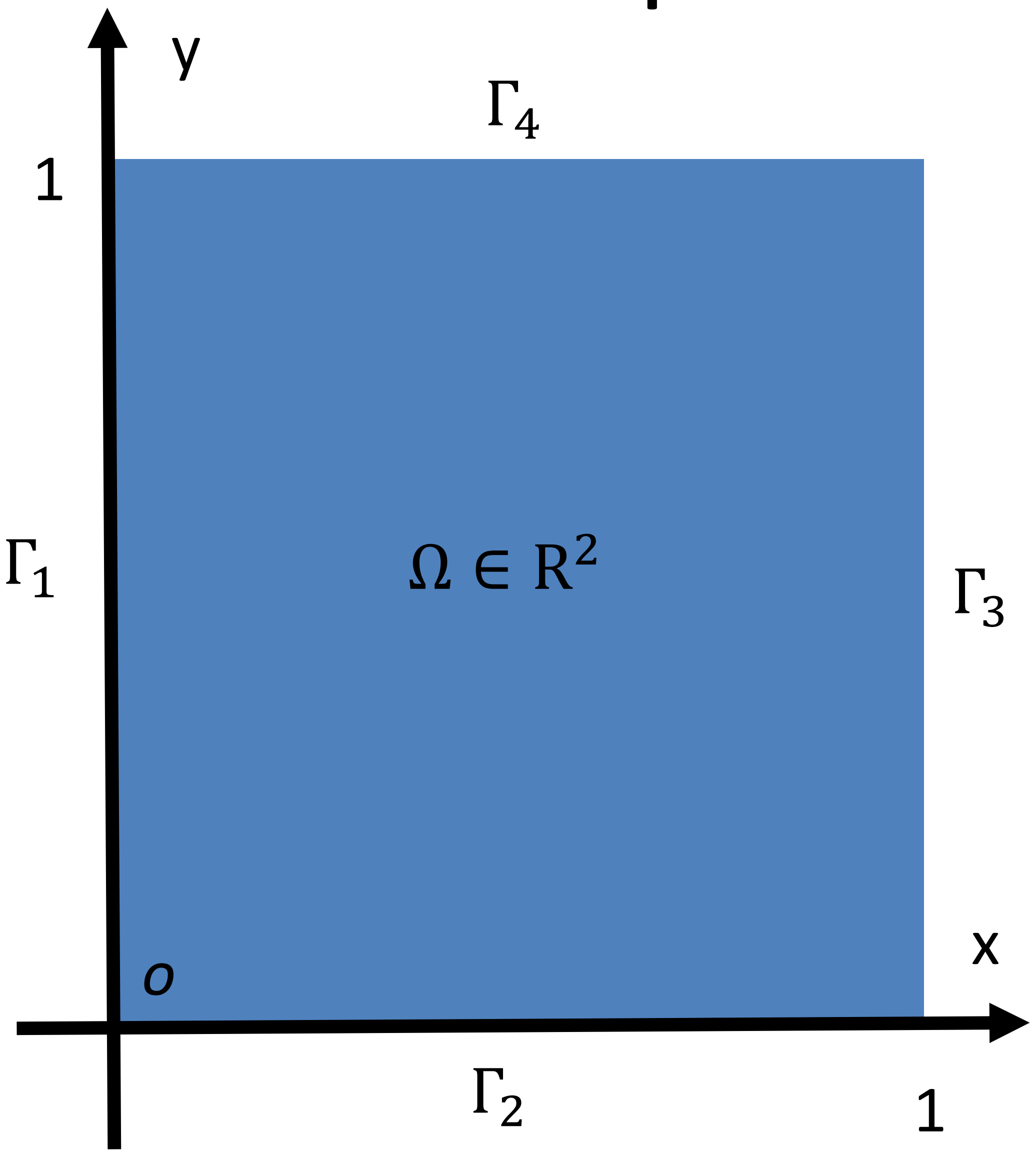
- In the case of Dirichlet condition
  - Let the trial function v be 0 on the boundary, we have

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) d\Omega + \int_\Omega f v d\Omega = 0$$

  - But commands for the Dirichlet condition should be written in the source code in FreeFEM++.
- In the case of Neumann condition
  - Substitute $\frac{\partial u}{\partial n} = g$ into boundary integration term, we have

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) d\Omega - \int_\Gamma g\, v d\gamma + \int_\Omega f v d\Omega = 0$$

Poisson equation: Laplace.edp



$$-\Delta u = 0 \text{ in } \Omega$$

$$u = 1 \text{ on } \Gamma_1$$

$$u = 0 \text{ on } \Gamma_2$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_3$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_4$$

Poisson equation: Laplace.edp

```
fespace Vh(Th,P2);        <- Definition of finite element space
Vh uh,vh;                 <- Definition of variable

problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)              <- Weak forms and numerical scheme
+ on(2,uh=0) ;
laplace;                  <- Running Numerical calculations
plot(uh);                 <- Visualizations
```

Poisson equation: Laplace.edp

Mesh

Th=buildmesh(a0(10)+a1(10)+a2(10)+a3(10));

Commands for definitions of finite element space

fespace Vh(Th,P2);

Vh uh,vh;

Poisson equation: Laplace.edp

Mesh

Th=buildmesh(a0(10)+a1(10)+a2(10)+a3(10));

The name of finite element space
You can use any words which you want.

fespace Vh(Th,P2);

Vh uh,vh;

Poisson equation: Laplace.edp

Mesh

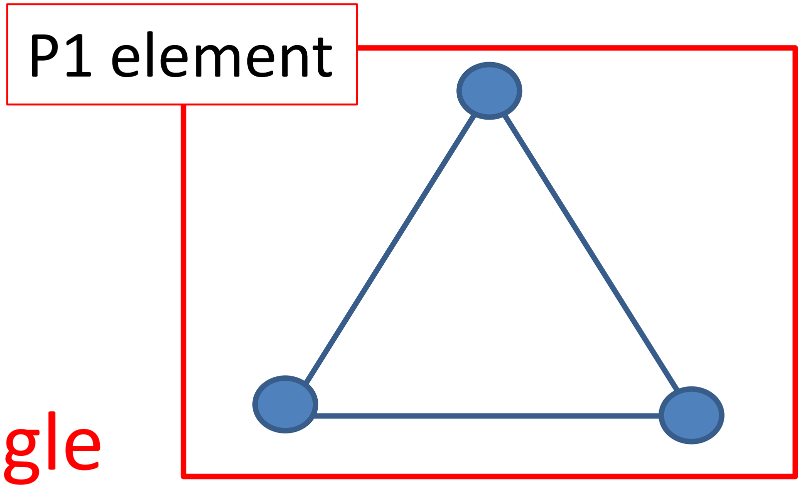
Th=buildmesh(a0(10)+a1(10)+a2(10)+a3(10));

fespace Vh(Th,P2);

Vh uh,vh;

Poisson equation: Laplace.edp

Mesh
Th=buildmesh(a0(10)+a1(10)+a2(10)+a3(10));



fespace Vh(Th,P1);

Variables are calculated

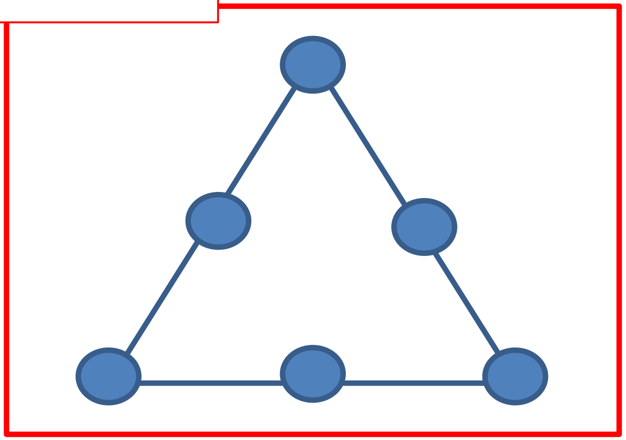
on 3 points on one triangle

Vh uh,vh;

Poisson equation: Laplace.edp

Mesh
Th=buildmesh(a0(10)+a1(10)+a2(10)+a3(10));

P2 element



fespace Vh(Th,P2);

Variables are calculated on 6 points on one triangle

Vh uh,vh;

Poisson equation: Laplace.edp

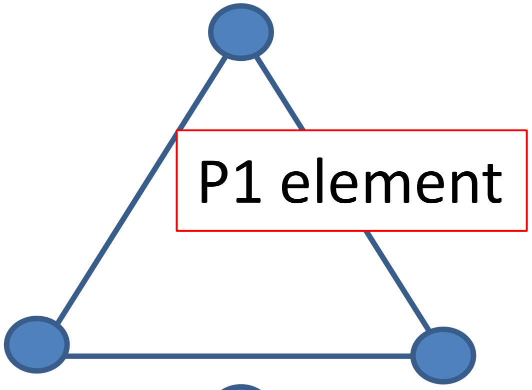
Mesh

Th=buildmesh(a0(10)+a1(10)+a2(10)+a3(10));

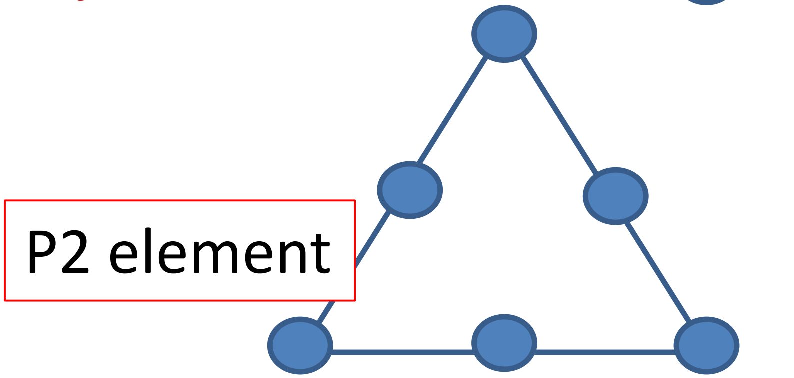
fespace Vh(Th,P1 or P2);

P1: low cost and low accuracy

P2: high cost and high accuracy



Vh uh,vh;



Poisson equation: Laplace.edp

Commands for definition of numerical calculations

```
problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0) ;
laplace;
```

Poisson equation: Laplace.edp

The name of the problem
You can use any words which you want.

```
problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0) ;
laplace;
```

Poisson equation: Laplace.edp

Variables that you need to calculate

```
problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0) ;
laplace;
```

Poisson equation: Laplace.edp

Numerical Scheme
LU decompositions

```
problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0) ;
laplace;
```

Poisson equation: Laplace.edp

Integral on finite elements Th
If you want to Integrate in 2 dimension, you should int2d.

```
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
```

Weak form of Poisson equation
dx() and dy() mean derivatives with respect to x and y.

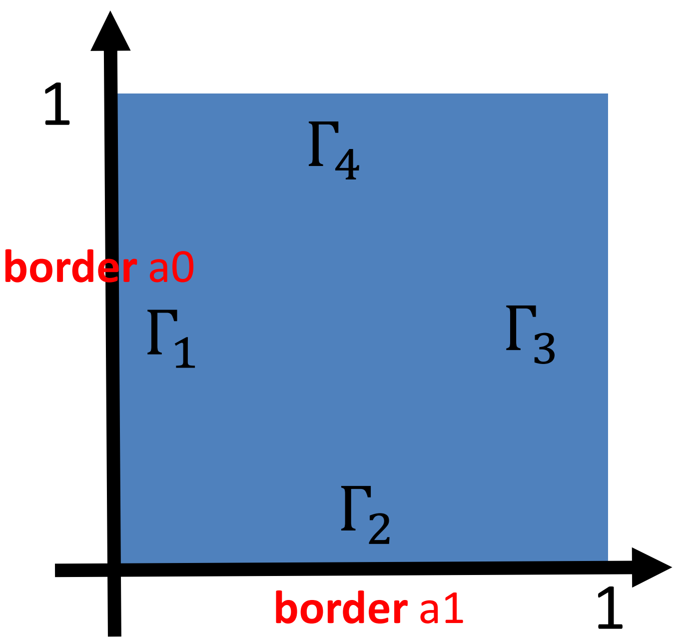
$$\int_{\Omega} \left(\frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \right) \, d\Omega$$

Poisson equation: Laplace.edp (Dirichlet conditions)

```
problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0) ;
```

Dirichlet condition

$$u = 1 \text{ on } \Gamma_1$$
$$u = 0 \text{ on } \Gamma_2$$




```
border a0(t=1,0){ x=0; y=t; label=1;}
border a1(t=0,1){ x=t; y=0; label=2;}
```

Poisson equation: Laplace.edp Neumann conditions

```
problem laplace(uh,vh,solver=LU) =
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
+ on(1,uh=1)
+ on(2,uh=0) ;
```

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right) d\Omega - \int_\Gamma \left(\frac{\partial u}{\partial n}\right) v d\gamma + \int_\Omega f v d\Omega = 0$$

$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_3$

$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_4$

Substitute $\frac{\partial u}{\partial n} = 0$ into the second term,

You should not type anything.

If you want to use Neumann conditions…

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) d\Omega - \int_\Gamma g\, v d\gamma + \int_\Omega f\, v d\Omega = 0$$

- Please type boundary integration as follows;

  +int1d(Th, (*label number*))(- g *vh)

ex)

on the boundary border a1, for g =2, you should type

+int1d(Th, 2)(-2*vh)

border a1(t=0,1){ x=t; y=0; label=2;}

Poisson equation: Laplace.edp

Integral on finite elements Th
If you want to Integrate in 2 dimension, you should int2d.

```
int2d(Th)( dx(uh)*dx(vh) + dy(uh)*dy(vh) )
```

Weak form of Poisson equation
dx() and dy() mean derivatives with respect to x and y.

$$\int_{\Omega} \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} \right) \, d\Omega$$

If you want to use Neumann conditions…

$$\int_{\Omega}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)\,d\Omega-\int_{\Gamma}g\,v d\gamma+\int_{\Omega}f\,v d\Omega=0$$

- Please type boundary integration as follows;

  +int1d(Th, *label number*)(- g *vh)

ex)

on the boundary border a1, for g =2,

you should type

+int1d(Th, 2)(-2*vh)

```
border a1(t=0,1){ x=t; y=0; label=2;}
```

Poisson equation: Laplace.edp

problem laplace(uh,vh,solver=LU) =
int2d(Th)(dx(uh)*dx(vh) + dy(uh)*dy(vh))
+ on(1,uh=1)
+ on(2,uh=0);

Dirichlet condition

laplace;

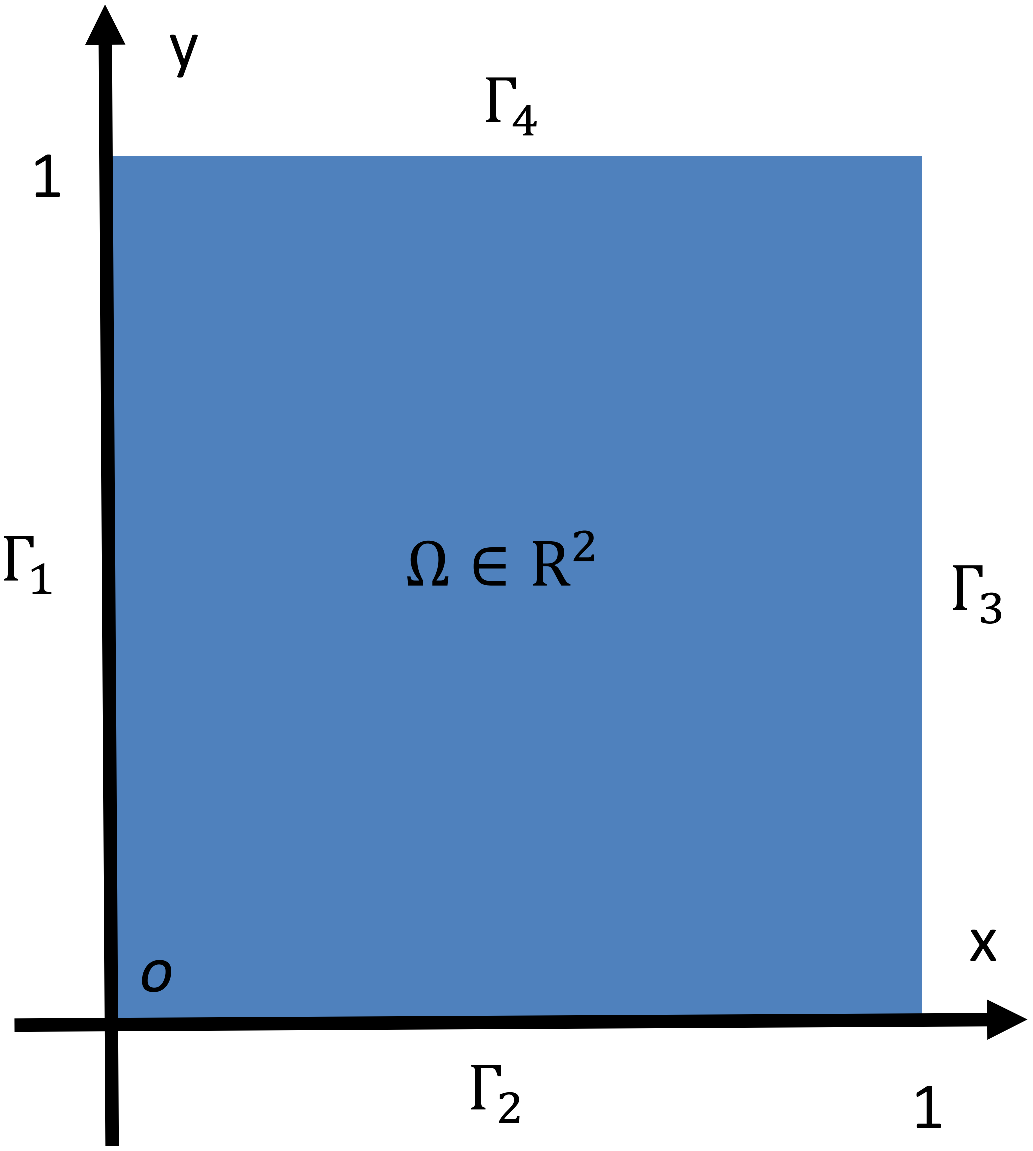
Numerical Calculation

Output with Paraview
Laplace_output.edp

Please add the following commands in the line where you want output data.

```
load "iovtk";
string
vtkOutputFile="./output.vtk";savevtk(vtkOutputF
ile, Th, uh, dataname="poisson");
```

Problem3



$$-\Delta u = u \text{ in } \Omega$$

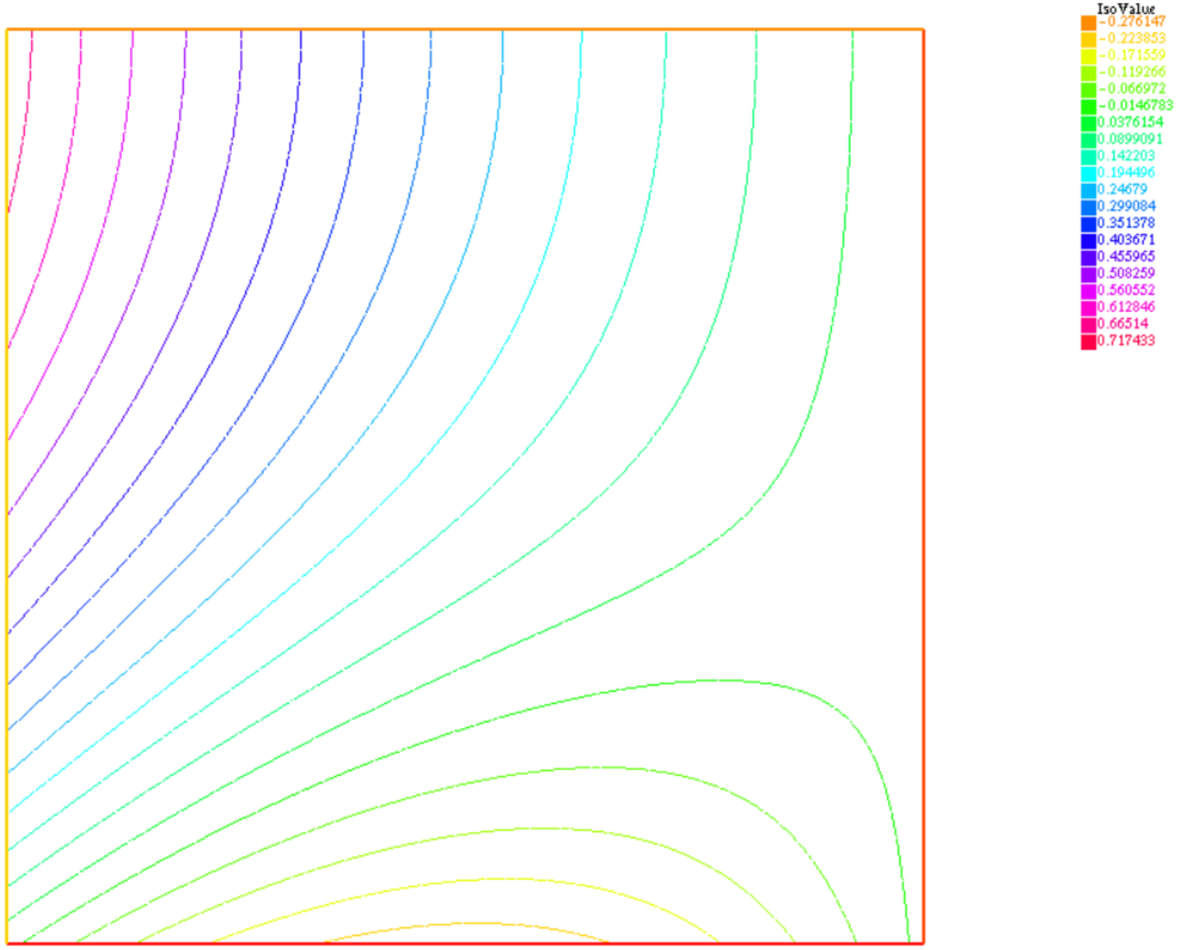
$$u = x * x - x \text{ on } \Gamma_1$$

$$u = 0 \text{ on } \Gamma_2$$

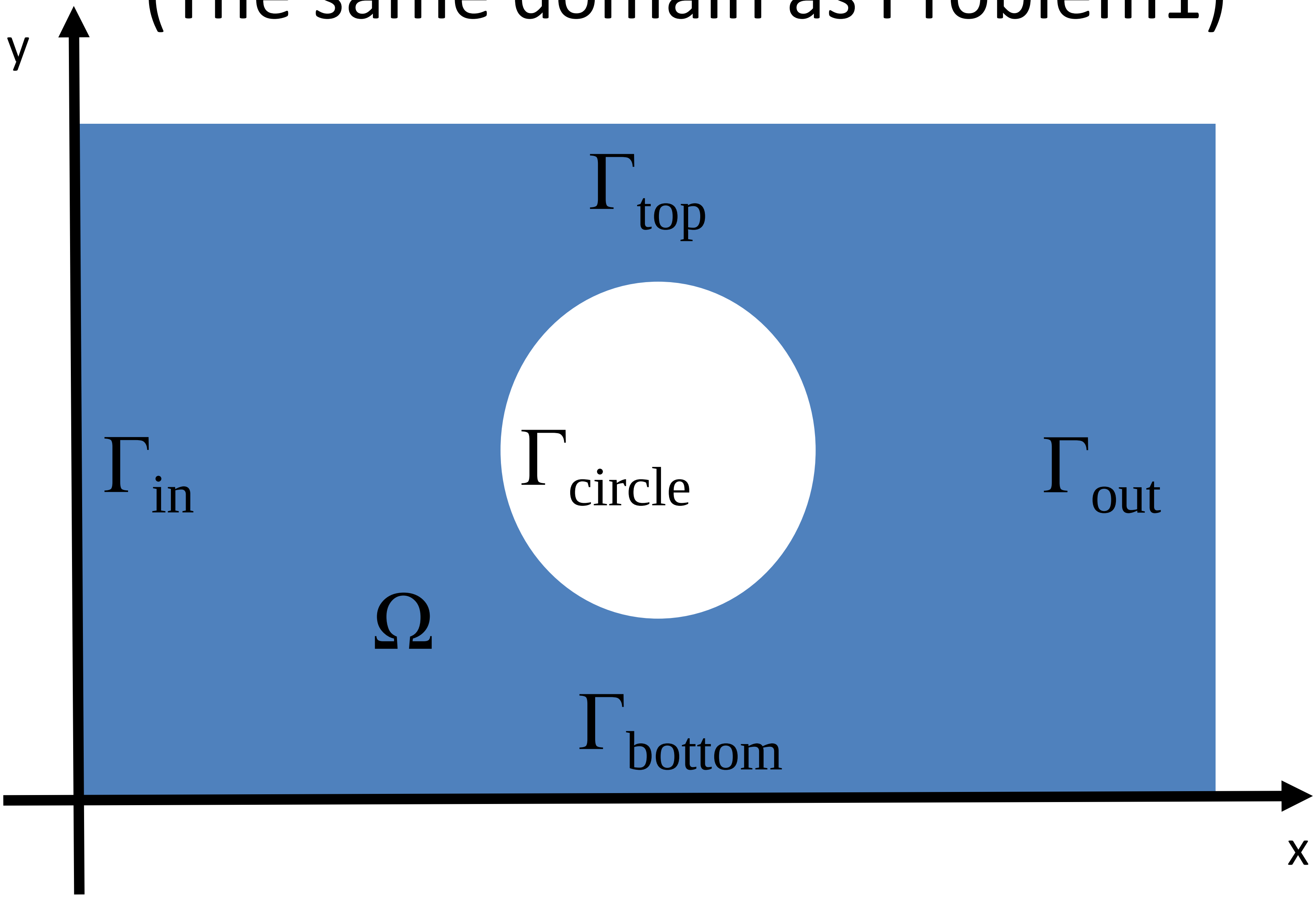
$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_3$$

$$\frac{\partial u}{\partial n} = 1 \text{ on } \Gamma_4$$

Answer of Problem3



Problem4
(The same domain as Problem1)



Problem4
(The same domain as Problem1)

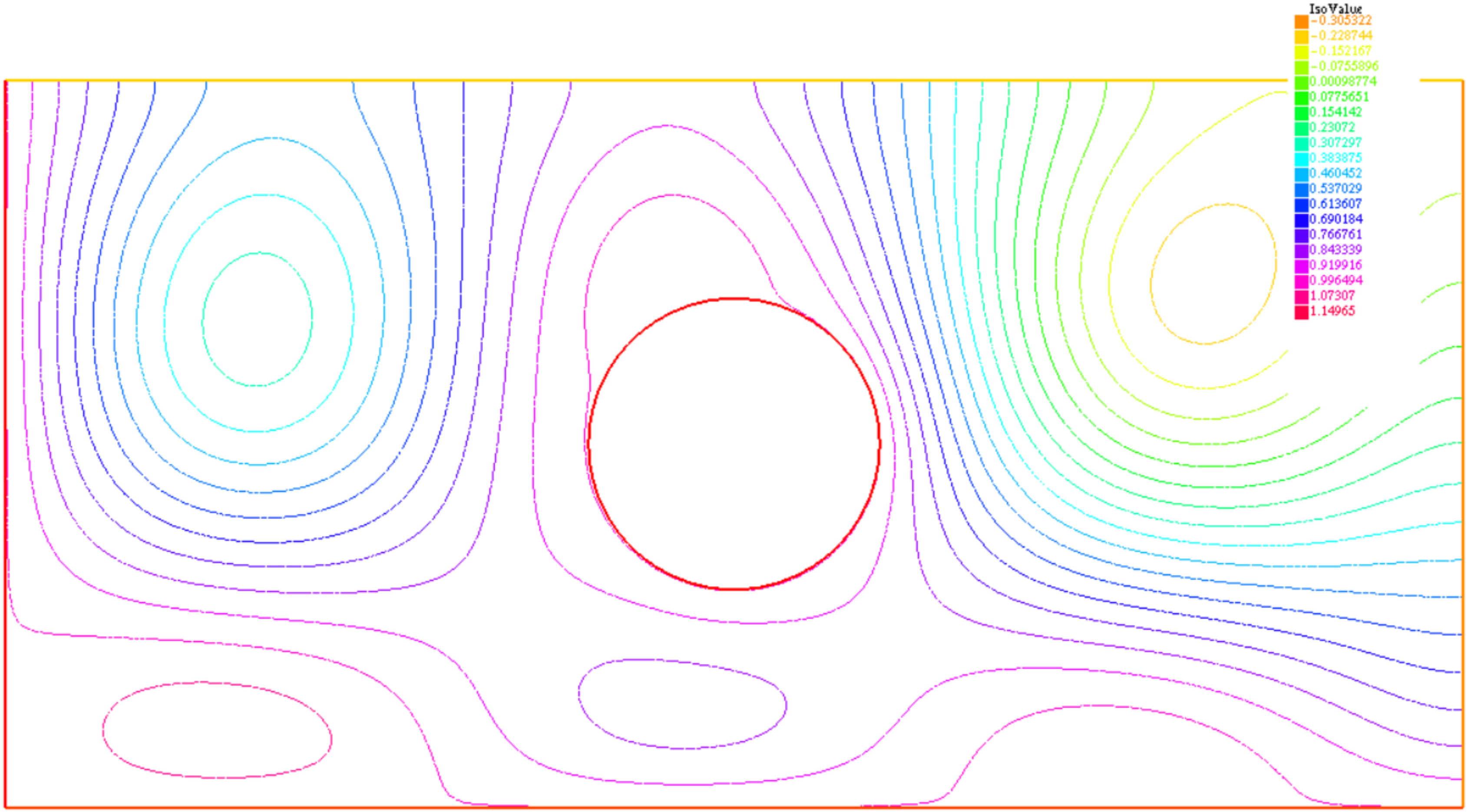
$$\Delta u = -\sin(x) * \cos(y) \text{ in } \Omega$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{\mathrm{top}} \cup \Gamma_{\mathrm{bottom}}$$

$$u = 1 \text{ on } \Gamma_{\mathrm{in}} \cup \Gamma_{\mathrm{out}}$$

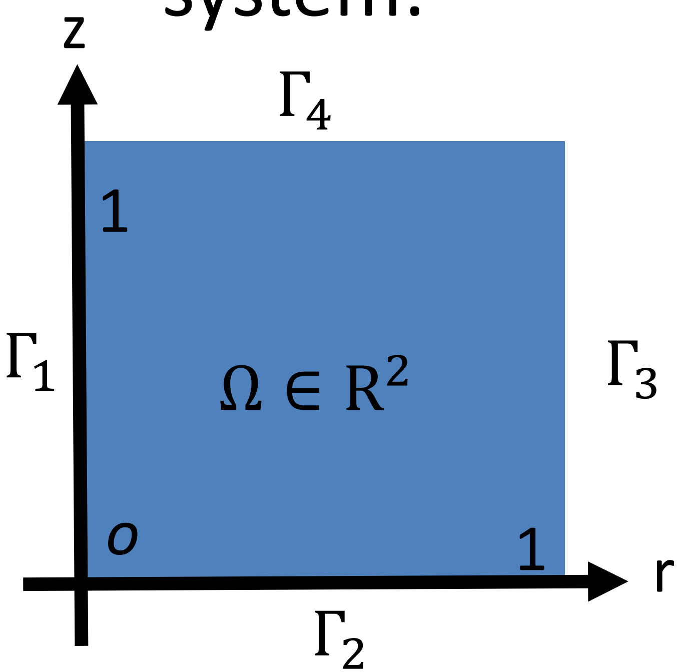
$$\frac{\partial u}{\partial n} = 1 \text{ on } \Gamma_{\mathrm{circle}}$$

Answer of Problem4



Problem 5

- In the axi-symmetric Cylindrical coordinate system:



$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \text{ in } \Omega$$

$$u = 1 \text{ on } \Gamma_2$$

$$u = 0 \text{ on } \Gamma_3$$

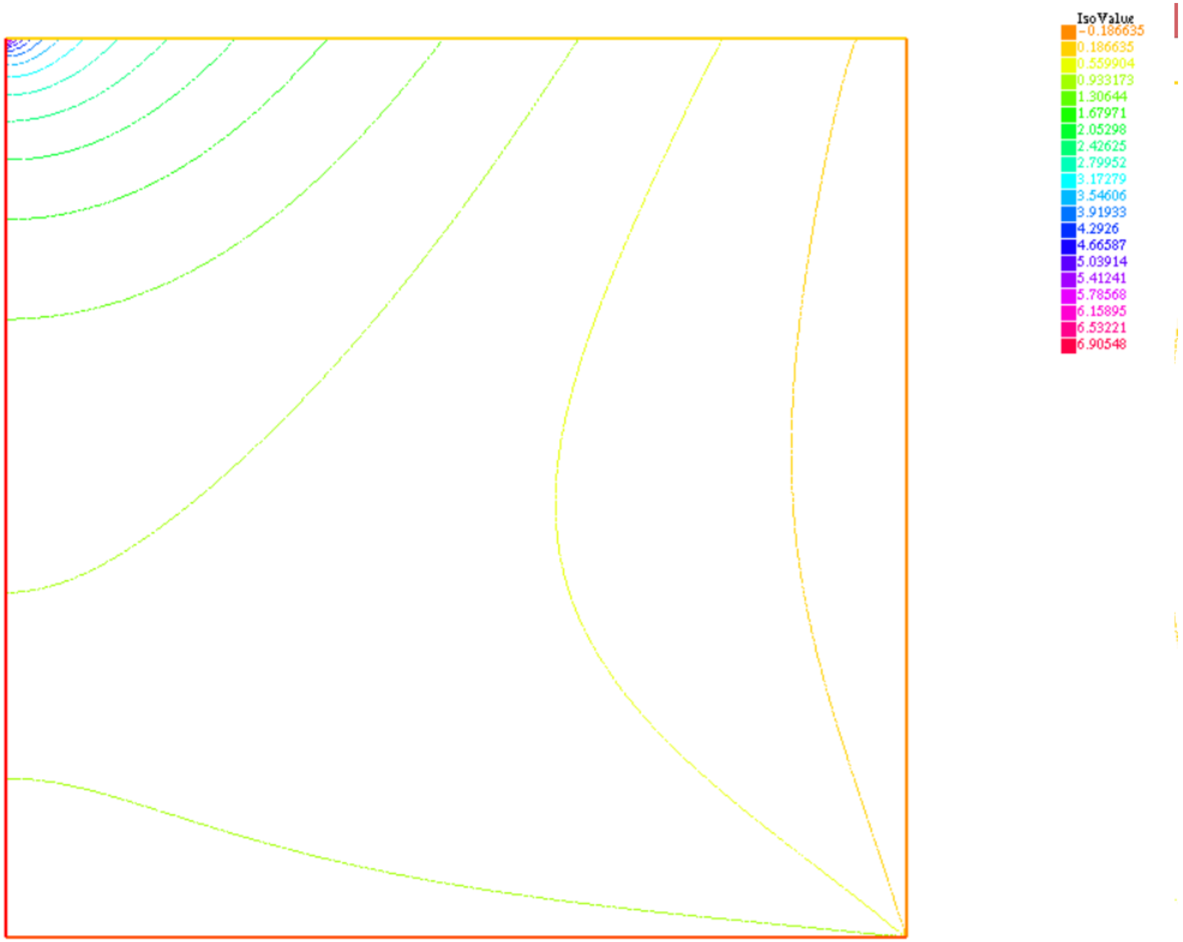
$$\frac{\partial u}{\partial n} = 1 \text{ on } \Gamma_4$$

$$\Delta u = \frac{1}{r}\frac{\partial u}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}$$

- Space integration In this system:

$$\int_\Omega (\Delta u) v r d\Omega = 0$$

Answer of Problem 5



Schedule

- 9:00-10:20
  - *Running Sample files and Mesh Generating*
- 10:30-11:50
  - *Poisson equations*
- 12:00-13:30   Lunch Time
- ***13:30-14:50***
  - ***Convection-Diffusion equations***
- 15:00-16:20
  - *Navier-Stokes equations*
- 16:30-17:30
  - *Free Time*

Diffusion equation: diffusion.edp

$$\frac{\partial u}{\partial t} - \frac{1}{\mathrm{Re}} \Delta u = 0$$

$$\Longrightarrow \frac{u^{n+1} - u^n}{\Delta t} - \frac{1}{\mathrm{Re}} \Delta u^{n+1} = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{u^{n+1} - u^n}{\Delta t} - \frac{1}{\mathrm{Re}} \Delta u^* \right) w d\Omega = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{u^{n+1}}{\Delta t} w - \frac{u^n}{\Delta t} w + \frac{1}{\mathrm{Re}} \nabla u^{n+1} \cdot \nabla w \right) d\Omega$$

$$- \int_\Gamma \frac{1}{\mathrm{Re}} \frac{\partial u^{n+1}}{\partial n} w = 0$$

Convection equation:convection.edp

$$\frac{\partial u}{\partial t} + (\boldsymbol{u} \cdot \nabla)u = 0$$

$$\Longrightarrow \frac{u^{n+1} - u^n}{\Delta t} + (\boldsymbol{u} \cdot \nabla) = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{u^{n+1} - u^n}{\Delta t} + (\boldsymbol{u} \cdot \nabla)u^n \right) w d\Omega = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{u^{n+1}}{\Delta t} w - \frac{u^n}{\Delta t} w + \{(\boldsymbol{u} \cdot \nabla)u^n\} w \right) d\Omega = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{u^{n+1}}{\Delta t} w - \frac{w}{\Delta t} convect(\boldsymbol{u}, -\Delta t, u^n) \right) d\Omega = 0$$

$$\Longrightarrow u^{n+1} = convect(\boldsymbol{u}, -\Delta t, u^n) = 0$$

Convection diffusion equation: convection_diffusion.edp

$$\frac{\partial u}{\partial t} + (\boldsymbol{u} \cdot \nabla)u - \frac{1}{\mathrm{Re}}\Delta u = 0$$

$$\Longrightarrow \frac{u^{n+1} - u^n}{\Delta t} + (\boldsymbol{u} \cdot \nabla)u^n - \frac{1}{\mathrm{Re}}\Delta u^{n+1} = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{u^{n+1} - u^n}{\Delta t} + (\boldsymbol{u} \cdot \nabla)u^n - \frac{1}{\mathrm{Re}}\Delta u^{n+1}\right) w d\Omega = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{u^{n+1}}{\Delta t} w - \frac{u^n}{\Delta t} w + \{(\boldsymbol{u^n} \cdot \nabla)u^n\}w + \frac{1}{\mathrm{Re}}\nabla u^{n+1} \cdot \nabla w\right) d\Omega - \int_\Gamma \frac{1}{\mathrm{Re}}\frac{\partial u^{n+1}}{\partial n} w = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{u^{n+1}}{\Delta t} w - \frac{w}{\Delta t} convect(\boldsymbol{u}, -\Delta t, u^n) + \frac{1}{\mathrm{Re}}\nabla u^{n+1} \cdot \nabla w\right) d\Omega - \int_\Gamma \frac{1}{\mathrm{Re}}\frac{\partial u^{n+1}}{\partial n} w = 0$$

Problem6
(The same domain as Problem1)

$$\frac{\partial u}{\partial t} - \mathit{\Delta} u = 0 \text{ in } \Omega$$

Initial condition:

$$u = \exp\{-10(x-1)^2 + (y-1.5)^2\}$$

Boundary conditions:

$$u = 0 \text{ on } \Gamma_{\text{top}} \times [0, t]$$
$$u = 0 \text{ on } \Gamma_{\text{bottom}} \times [0, t]$$
$$\frac{\partial u}{\partial n} = 0 \text{ on} \Gamma_{\text{in}} \times [0, t]$$
$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{\text{out}} \times [0, t]$$
$$u = 0 \text{ on } \Gamma_{\text{circle}} \times [0, t]$$

Problem7
(The same domain as Problem1)

$$\frac{\partial u}{\partial t} + (\boldsymbol{f} \cdot \nabla)u - \Delta u = 0 \text{ in } \Omega$$

where $\boldsymbol{f} = (-y(y-5), 0)$

Initial condition:

$$u = \exp\{-10(x-1)^2 + (y-1.5)^2\}$$

Boundary conditions:

$$u = 0 \text{ on } \Gamma_{\text{top}} \times [0, t]$$

$$u = 0 \text{ on } \Gamma_{\text{bottom}} \times [0, t]$$

$$\frac{\partial u}{\partial n} = 0 \text{ on} \Gamma_{\text{in}} \times [0, t]$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{\text{out}} \times [0, t]$$

$$u = 0 \text{ on } \Gamma_{\text{circle}} \times [0, t]$$

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Navier-Stokes equation

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \dfrac{1}{\mathrm{Re}} \Delta \mathbf{u} \\ \nabla \cdot \mathbf{u} = 0 \end{cases}$$

$$\mathbf{u} = (u, v)$$

- Smac method

1st step $$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} + (\mathbf{u}^n \cdot \nabla)\mathbf{u}^n = -\nabla p^n + \frac{1}{\mathrm{Re}} \Delta \mathbf{u}^*$$

2nd step $$\Delta q = -\frac{\nabla \cdot \mathbf{u}^*}{\Delta t}$$

3rd step $$p^{n+1} = p^n - q$$

4th step $$\mathbf{u}^{n+1} = \mathbf{u}^* + \nabla q \Delta t$$

1st step on Smac method: Only x-direction component

$$\frac{\partial u}{\partial t} + (\boldsymbol{u} \cdot \nabla) u + \frac{\partial p}{\partial x} - \frac{1}{\mathrm{Re}} \Delta u = 0$$

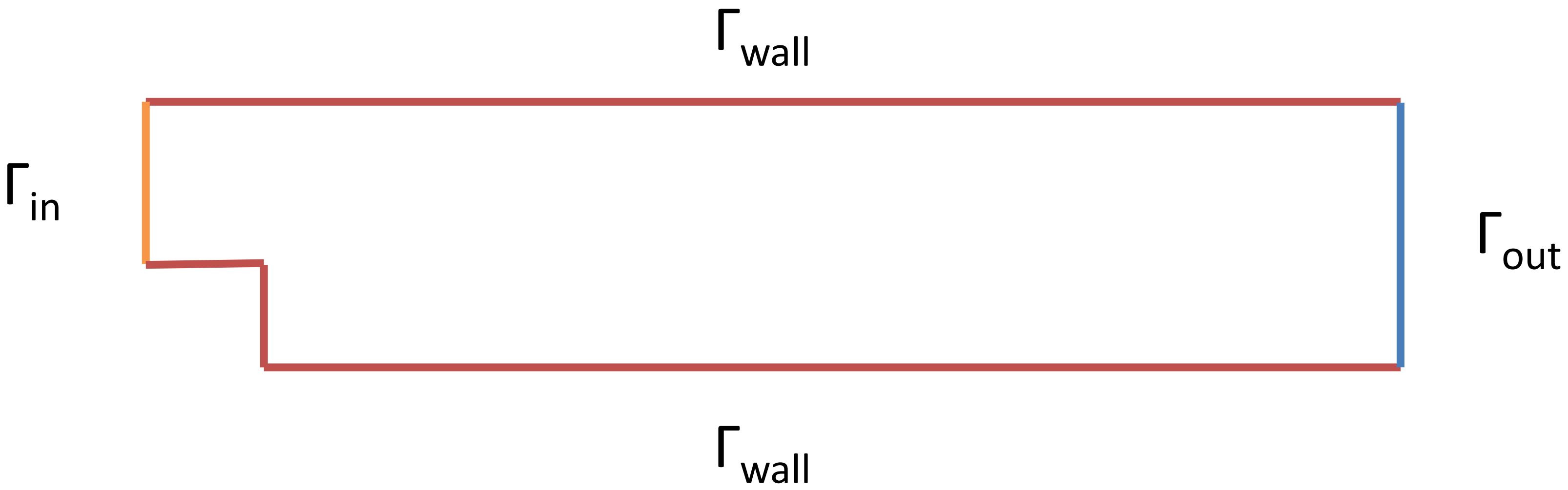
$$\Longrightarrow \frac{u^* - u^n}{\Delta t} + (\boldsymbol{u^n} \cdot \nabla) u^n + \frac{\partial p^n}{\partial x} - \frac{1}{\mathrm{Re}} \Delta u^* = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{u^* - u^n}{\Delta t} + (\boldsymbol{u^n} \cdot \nabla) u^n + \frac{\partial p^n}{\partial x} - \frac{1}{\mathrm{Re}} \Delta u^* \right) w d\Omega = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{u^*}{\Delta t} w - \frac{u^n}{\Delta t} w + \{(\boldsymbol{u^n} \cdot \nabla) u^n\} w + \frac{\partial p^n}{\partial x} w + \frac{1}{\mathrm{Re}} \nabla u^* \cdot \nabla w \right) d\Omega - \int_\Gamma \frac{1}{\mathrm{Re}} \frac{\partial u}{\partial n} w = 0$$

$$\Longrightarrow \int_\Omega \left(\frac{u^*}{\Delta t} w - \frac{w}{\Delta t} convect(\boldsymbol{u^n}, -\Delta t, u^n) + \frac{\partial p^n}{\partial x} w + \frac{1}{\mathrm{Re}} \nabla u^* \cdot \nabla w \right) d\Omega - \int_\Gamma \frac{1}{\mathrm{Re}} \frac{\partial u}{\partial n} w = 0$$

Boundary condition



$$u = y(1-y), v = 0, \frac{\partial q}{\partial n} = 0 \quad \text{on } \Gamma_{in}$$

$$\frac{\partial u}{\partial x} = 0, v = 0, q = 0 \quad \text{on } \Gamma_{out}$$

$$u = 0, v = 0, \frac{\partial q}{\partial n} = 0 \quad \text{on } \Gamma_{wall}$$

Problem8
(The same domain as Problem1)

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}} \Delta u \text{ in } \Omega$$

$$\nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega$$

Parameters: $\mathrm{Re} = 10, \Delta t = 0.05$

Initial condition: $u = -y(y-5), v = 0$

Boundary conditions:

$$u = 0, v = 0, \frac{\partial q}{\partial n} = 0 \text{ on } \Gamma_{\mathrm{top}} \times [0, t]$$

$$u = 0, v = 0, \frac{\partial q}{\partial n} = 0 \text{on } \Gamma_{\mathrm{bottom}} \times [0, t]$$

$$u = -y(y-5), v = 0, \frac{\partial q}{\partial n} = 0 \text{on} \Gamma_{\mathrm{in}} \times [0, t]$$

$$\frac{\partial u}{\partial n} = 0, v = 0, p = 0 \text{ on } \Gamma_{\mathrm{out}} \times [0, t]$$

$$u = 0, v = 0, \frac{\partial q}{\partial n} = 0 \text{on } \Gamma_{\mathrm{circle}} \times [0, t]$$

Problem9
(The same domain as Problem1)

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}} \Delta u \text{ in } \Omega$$

$$\nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega$$

$$\frac{\partial f}{\partial t} + (\boldsymbol{u} \cdot \nabla) f = \Delta f \text{ in } \Omega$$

Parameters: $\mathrm{Re} = 10, \Delta t = 0.05$

Initial condition: $u = -y(y-5), v = 0, f = \exp\{-10(x-1)^2 + (y-1.5)^2\}$

Boundary conditions:

$$u = 0, v = 0, \frac{\partial q}{\partial n} = 0, f = 0 \text{ on } \Gamma_{\mathrm{top}} \times [0, t]$$

$$u = 0, v = 0, \frac{\partial q}{\partial n} = 0, f = 0 \text{ on } \Gamma_{\mathrm{bottom}} \times [0, t]$$

$$u = -y(y-5), v = 0, \frac{\partial q}{\partial n} = 0, \frac{\partial f}{\partial n} = 0 \text{ on} \Gamma_{\mathrm{in}} \times [0, t]$$

$$\frac{\partial u}{\partial n} = 0, v = 0, p = 0, \frac{\partial f}{\partial n} = 0 \text{ on } \Gamma_{\mathrm{out}} \times [0, t]$$

$$u = 0, v = 0, \frac{\partial q}{\partial n} = 0, f = 0 \text{ on } \Gamma_{\mathrm{circle}} \times [0, t]$$

Problem10
(The same domain as Problem1)

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{\mathrm{Re}} \Delta u \text{ in } \Omega$$

$$\nabla \cdot \boldsymbol{u} = 0 \text{ in } \Omega$$

Parameters: $\mathrm{Re} = 10^5, \Delta t = 0.005$

Initial condition: $u = 1, v = 0.4$

Boundary conditions:

$$\frac{\partial u}{\partial n} = 0, \frac{\partial v}{\partial n} = 0, q = 0 \text{ on } \Gamma_{\mathrm{top}} \times [0, t]$$

$$u = 1, v = 0.4, \frac{\partial q}{\partial n} = 0 \text{ on } \Gamma_{\mathrm{bottom}} \times [0, t]$$

$$u = 1, v = 0.4, \frac{\partial q}{\partial n} = 0 \text{ on} \Gamma_{\mathrm{in}} \times [0, t]$$

$$\frac{\partial u}{\partial n} = 0, \frac{\partial v}{\partial n} = 0, q = 0 \text{ on } \Gamma_{\mathrm{out}} \times [0, t]$$

$$u = 0, v = 0, \frac{\partial q}{\partial n} = 0 \text{ on } \Gamma_{\mathrm{wing}} \times [0, t]$$

Borders for Problem 10

```
border a41(t=0.5*pi,1.5*pi){ x=1+2*cos(t); y=2*sin(t); label=1;}
border a42(t=-0.5*pi,0.5*pi){ x=1+2*cos(t); y=2*sin(t); label=2;}
border Splus(t=0,1){  x = t; y = 0.17735*sqrt(t)-0.075597*t -
0.212836*(t^2)+0.17363*(t^3)-0.06254*(t^4); label=5;}
border Sminus(t=1,0){  x =t; y= -(0.17735*sqrt(t)-0.075597*t-
0.212836*(t^2)+0.17363*(t^3)-0.06254*(t^4)); label=5;}
int n=5;
mesh Th=
buildmesh( a41(30*n)+a42(30*n)+Splus(60*n)+Sminus(60*n));plot(Th);
```

Output with Paraview

Please add the following commands at the top of code

load "iovtk";

Please add the following commands in the line where you want output data.

String vtkOutputFile="./Problem10_out"+n+".vtk";

savevtk(vtkOutputFile, Th, [u,v,0],p,dataname="velocity pressure");