L^p–MAPPING PROPERTIES FOR SCHRÖDINGER OPERATORS IN OPEN SETS OF $\mathbb{R}^d$

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Abstract. Let $H_V = -\Delta + V$ be a Schrödinger operator on an arbitrary open set $\Omega \subset \mathbb{R}^d$ ($d \geq 3$), where Δ is the Dirichlet Laplacian and the potential V belongs to the Kato class on Ω. The purpose of this paper is to show L^p–boundedness of an operator $\varphi(H_V)$ for any rapidly decreasing function φ on $\mathbb{R}$. $\varphi(H_V)$ is defined by the spectral resolution theorem. As a by-product, L^p–L^q–estimates for $\varphi(H_V)$ are also obtained.

1. Introduction and main result

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 3$) be an arbitrary open set. We consider the Schrödinger operator $H_V = H + V(x)$, where

$$H := -\Delta = -\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$$

is the Dirichlet Laplacian with domain

$$\mathcal{D}(H) = \left\{ u \in H_0^1(\Omega) \,\middle|\, \Delta u \in L^2(\Omega) \right\}$$

and $V(x)$ is a real-valued measurable function on Ω. If we impose an appropriate assumption on $V(x)$, H_V will be a self-adjoint operator on $L^2(\Omega)$. Let $\{E_{H_V}(\lambda)\}_{\lambda\in\mathbb{R}}$ be the spectral resolution of the identity for H_V. Then H_V is written as

$$H_V = \int_{-\infty}^{\infty} \lambda \, dE_{H_V}(\lambda).$$

Hence we can define $\varphi(H_V)$ by

$$\varphi(H_V) = \int_{-\infty}^{\infty} \varphi(\lambda) \, dE_{H_V}(\lambda)$$

for a Borel measurable function $\varphi(\lambda)$ on $\mathbb{R}$. These operators are initially defined on $L^2(\Omega)$. This paper is devoted to investigation of functional calculus for Schrödinger operators on Ω. More precisely, our purpose is to prove that $\varphi(H_V)$ is extended uniquely to a bounded linear operator on $L^p(\Omega)$ for $1 \leq p \leq \infty$ and that L^p–boundedness of $\varphi(\theta H_V)$ is uniform with respect to a parameter $\theta > 0$.

When $\Omega = \mathbb{R}^d$, Simon considered the Kato class K_d of potentials to reveal L^p–mapping properties of the Schrödinger operators H_V and e^{-tH_V} for $t > 0$ (see [9, Section A.2]). We now define a Kato class $K_d(\Omega)$ on an *open set* Ω as follows: We

Key words and phrases. Schrödinger operators, functional calculus, Kato class.

say that a real-valued measurable function V on Ω belongs to the class $K_d(\Omega)$ if and only if

$$\lim_{r\to 0}\sup_{x\in\Omega}\int_{\Omega\cap\{|x-y|<r\}}\frac{|V(y)|}{|x-y|^{d-2}}\,dy=0.$$

Throughout this paper, defining the "Kato norm":

$$\|V\|_{K_d(\Omega)}:=\sup_{x\in\Omega}\int_{\Omega}\frac{|V(y)|}{|x-y|^{d-2}}\,dy,$$

we impose an assumption on V as follows:

Assumption A. *Let $d\ge 3$. A real-valued measurable function $V(x)$ on Ω is decomposed into $V=V_+-V_-$, $V_\pm\ge 0$, belongs to $K_d(\Omega)$ and satisfies*

$$\|V_-\|_{K_d(\Omega)}<\gamma_d, \tag{1.1}$$

where γ_d is the constant given by

$$\gamma_d=\frac{\pi^{d/2}}{\Gamma(d/2-1)}.$$

Here $\Gamma(s)$ is the Gamma function for $s>0$.

If the potential V is satisfied with assumption A, it will be proved in Proposition 2.1 that H_V is the non-negative and self-adjoint operator on $L^2(\Omega)$ (see §2). For a Borel measurable function φ on $\mathbb{R}$, we define the operator $\varphi(H_V)$ on $L^2(\Omega)$ as follows:

$$\mathcal{D}(\varphi(H_V))=\Big\{f\in L^2(\Omega)\ \Big|\ \int_0^\infty|\varphi(\lambda)|^2\,d\langle E_{H_V}(\lambda)f,f\rangle_{L^2(\Omega)}<\infty\Big\},$$

$$\big\langle\varphi(H_V)f,g\big\rangle_{L^2(\Omega)}=\int_0^\infty\varphi(\lambda)\,d\langle E_{H_V}(\lambda)f,g\rangle_{L^2(\Omega)},\quad\forall f\in\mathcal{D}(\varphi(H_V)),\ \forall g\in L^2(\Omega),$$

where $\langle\cdot,\cdot\rangle_{L^2(\Omega)}$ stands for the inner product in $L^2(\Omega)$. Formally we write

$$\varphi(H_V)=\int_0^\infty\varphi(\lambda)dE_{H_V}(\lambda). \tag{1.2}$$

Denoting by $\mathscr{S}(\mathbb{R})$ the space of rapidly decreasing functions on $\mathbb{R}$, we shall prove here the following:

Theorem 1.1. *Let $d\ge 3$, $\varphi\in\mathscr{S}(\mathbb{R})$ and $1\le p\le\infty$. Assume that the measurable potential V satisfies assumption A. Then there exists a constant $C=C(d,\varphi,p)>0$ such that*

$$\|\varphi(\theta H_V)\|_{\mathscr{B}(L^p(\Omega))}\le C \tag{1.3}$$

for any $\theta>0$.

Let us give a few remarks on Theorem 1.1. We have restricted the result in this theorem to high space dimensions. So, one would expect the result to hold also for low space dimensions, i.e., $d=1,2$. But in the present paper, we will use the pointwise estimates for kernel of e^{-tH_V} on $\mathbb{R}^d$ that D'Ancona and Pierfelice proved for $d\ge 3$ (see [2]). Hence low dimensional cases will be a future problem. When

$V = 0$, Theorem 1.1 also holds in the cases $d = 1, 2$ by using the pointwise estimates for classical heat kernel of $e^{t\Delta}$.

One can easily see that $\varphi(H_V)$ is bounded on $L^2(\Omega)$ via direct application of the spectral resolution (1.2). From the point of view of harmonic analysis, it would be important to obtain L^p–boundedness ($p \neq 2$). For instance, Theorem 1.1 provides a generalization of L^p–boundedness for the Fourier multiplier in $\mathbb{R}^d$:

$$\big\|\mathscr{F}^{-1}\big[\hat{\varphi}(\theta|\cdot|^2)\hat{f}\,\big]\big\|_{L^p(\mathbb{R}^d)} \le C_p\|f\|_{L^p(\mathbb{R}^d)}, \quad \forall \theta > 0,$$

where $\varphi \in \mathscr{S}(\mathbb{R}^d)$, $\hat{}$ denotes the Fourier transform, and $\mathscr{F}^{-1}$ is the Fourier inverse transform. L^p–boundedness of $\varphi(\theta H_V)$ also plays a fundamental role in defining the Besov spaces associated with spectral resolution of H_V (see, e.g., [2, 4, 6]). Thus Theorem 1.1 would be a starting point of the study of spectral multiplier and Besov spaces on open sets.

When $\Omega = \mathbb{R}^d$, there are some known results on uniform L^p-estimates for $\varphi(\theta H_V)$ with respect to θ. For $0 < \theta \le 1$, Jensen and Nakamura proved the uniform estimates for $d \ge 1$, under the assumption that the potential $V = V_+ - V_-$, $V_\pm \ge 0$, satisfies $V_+ \in K_d^{\mathrm{loc}}$ and $V_- \in K_d$ (see [6, 7]). Here K_d^{loc} is the local Kato class, which is the space of all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that f belongs to the Kato class on any compact set in $\mathbb{R}^d$. For $\theta > 0$, Georgiev and Visciglia proved the uniform estimates under the assumption that the potential V satisfies $0 \le V(x) \le \frac{C}{|x|^2(|x|^\varepsilon + |x|^{-\varepsilon})}$ ($C > 0, \varepsilon > 0$) in dimension $d = 3$ (see [4]). D'Ancona and Pierfelice proved the uniform estimates for $d \ge 3$, under the assumption that the potential $V = V_+ - V_-$, $V_\pm \ge 0$, satisfies $V_\pm \in K_d$ and $\|V_-\|_{K_d} < \gamma_d$ (see [2]). As far as we know, Theorem 1.1 is new in the sense that there would not be no results on L^p–estimates for $\varphi(H_V)$ in open sets.

Let us overview the strategy of proof of Theorem 1.1. For the sake of simplicity, we consider the case $V = 0$, since the case $V \neq 0$ is similar. The original idea of proof of L^p–boundedness goes back to Jensen and Nakamura [7]. The method for the boundedness of $\varphi(-\Delta)$ is to use the amalgam spaces $\ell^p(L^q)$, pointwise estimates for the kernel of $e^{-t\Delta}$ and the commutator estimates for $-\Delta$ and polynomials. As to the uniformity of the boundedness of $\varphi(-\theta\Delta)$ with respect to θ (see [2, 4, 6]), the estimates on the operator $\varphi(-\theta\Delta)$ are reduced to those on $\varphi(-\Delta)$ via the following equality

$$(1.4) \qquad (\varphi(-\theta\Delta)f)(x) = \Big(\varphi(-\Delta)\left(f(\theta^{1/2}\cdot)\right)\Big)(\theta^{-1/2}x), \quad x \in \mathbb{R}^d, \quad \theta > 0.$$

There, scaling invariance of $\mathbb{R}^d$, i.e., $\mathbb{R}^d = \theta^{1/2}\mathbb{R}^d$, plays an essential role in the argument. On the other hand, when one tries to get (1.4) on open sets $\Omega \subsetneq \mathbb{R}^d$, the scaling invariance breaks down, i.e., $\Omega \neq \theta^{1/2}\Omega$. To avoid this problem, we shall introduce the scaled amalgam spaces $\ell^p(L^q)_\theta(\Omega)$ to estimate the operator norm of $\varphi(-\theta\Delta)$ directly. A scale exponent $1/2$ in $\theta^{1/2}$ of the spaces $\ell^p(L^q)_\theta(\Omega)$ is chosen to fit the scale exponent of the operator $\varphi(-\theta\Delta)$; thus we define the scaled amalgam spaces as follows:

Definition 1.2 (Scaled amalgam spaces $\ell^p(L^q)_\theta$)**.** Let $1 \le p, q \le \infty$ and $\theta > 0$. The space $\ell^p(L^q)_\theta$ is defined as

$$\ell^p(L^q)_\theta = \ell^p(L^q)_\theta(\Omega) := \Big\{ f \in L^q_{\mathrm{loc}}(\overline{\Omega}) \Big| \sum_{n\in\mathbb{Z}^d} \|f\|^p_{L^q(C_\theta(n))} < \infty \Big\}$$

with norm

$$\|f\|_{\ell^p(L^q)_\theta} = \Big(\sum_{n\in\mathbb{Z}^d} \|f\|^p_{L^q(C_\theta(n))} \Big)^{1/p},$$

where $C_\theta(n)$ is the cube centered at $\theta^{1/2}n \in \theta^{1/2}\mathbb{Z}^d$ with side length $\theta^{1/2}$;

$$C_\theta(n) = \Big\{ x \in \Omega \ \Big| \ \max_{i=1,\cdots,d} |x_i - \theta^{1/2} n_i| \le \frac{\theta^{1/2}}{2} \Big\}.$$

Here we adopt the Euclidean norm for $n = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d$;

$$|n| = \sqrt{n_1^2 + n_2^2 + \cdots + n_d^2}.$$

It can be checked that $\ell^p(L^q)_\theta$ is a Banach space with norm $\|\cdot\|_{\ell^p(L^q)_\theta}$ having the property that

$$\ell^p(L^q)_\theta \hookrightarrow L^p(\Omega) \cap L^q(\Omega)$$

for $1 \le p \le q \le \infty$.

To prove Theorem 1.1, we also prepare the pointwise estimate for the kernel of $e^{t\Delta}$, and the commutator estimates for our problem in an open set Ω of $\mathbb{R}^d$. The pointwise estimate on $e^{t\Delta}$ is obtained by estimating the solution of linear heat equation in Ω from above by that in the whole space $\mathbb{R}^d$. For the commutator estimates, we utilize the explicit formula of the commutator to estimate optimally with respect to θ.

This paper is organized as follows. In §2 the self-adjointness of Schrödinger operator H_V will be shown. In §3 we will prove L^p–L^q–estimates for e^{-tH_V} and the pointwise estimates for integral kernel of e^{-tH_V}. In §4 L^p–$\ell^p(L^q)_\theta$–estimates for some power of resolvent of H_V will be proved. In §5 several commutator estimates for operators will be derived. In §6 the proof of Theorem 1.1 will be given. As a by-product of Theorem 1.1, L^p–L^q–boundedness for $\varphi(H_V)$ will be proved in §7.

2. Self-adjointness of Schrödinger operators

In this section we show that operator H_V is self-adjoint and non-negative under assumption A. When $\Omega = \mathbb{R}^d$, D'Ancona and Pierfelice had already proved these facts (see [2]). Hereafter we will often use the absolute constant γ_d in (1.1):

$$\gamma_d = \frac{\pi^{d/2}}{\Gamma(d/2-1)}.$$

Our purpose is to prove the following.

Proposition 2.1. *Let $d \ge 3$. Assume that the measurable potential V is a real-valued function on Ω and satisfies $V = V_+ - V_-$, $V_\pm \ge 0$ such that $V_\pm \in K_d(\Omega)$ and*

$$\|V_-\|_{K_d(\Omega)} < 4\gamma_d.$$

Let H_V be the operator with domain $\mathcal{D}(H_V) = \left\{u \in H_0^1(\Omega) \,\middle|\, H_V u \in L^2(\Omega)\right\}$, so that

$$\left\langle H_V u, v \right\rangle_{L^2(\Omega)} = \int_\Omega \nabla u(x) \cdot \overline{\nabla v(x)}\, dx + \int_\Omega V(x) u(x) \overline{v(x)}\, dx \tag{2.1}$$

for any $u \in \mathcal{D}(H_V)$ and $v \in H_0^1(\Omega)$. Then H_V is non-negative and self-adjoint on $L^2(\Omega)$.

We need a notion of quadratic forms on Hilbert spaces (see p.276 in Reed and Simon [8]).

Definition 2.2. Let $\mathscr{H}$ be a Hilbert space with the norm $\| \cdot \|$. A quadratic form is a map $q : \mathcal{Q}(q) \times \mathcal{Q}(q) \to \mathbb{C}$, where $\mathcal{Q}(q)$ is a dense linear subset in $\mathscr{H}$ called the form domain, such that $q(\cdot, v)$ is conjugate linear and $q(u, \cdot)$ is linear for $u, v \in \mathcal{Q}(q)$. We say that q is symmetric if $q(u, v) = \overline{q(v, u)}$. A symmetric quadratic form q is non-negative if $q(u, u) \geq 0$ for any $u \in \mathcal{Q}(q)$. A non-negative quadratic form q is closed if $\mathcal{Q}(q)$ is complete with respect to the norm:

$$\|u\|_{+1} := \sqrt{q(u, u) + \|u\|^2}. \tag{2.2}$$

The proof of Proposition 2.1 can be done by using the following two lemmas.

Lemma 2.3. *Let $\mathscr{H}$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and let $q : \mathcal{Q}(q) \times \mathcal{Q}(q) \to \mathbb{C}$ be a densely defined semibounded closed quadratic form. Then there exists a self-adjoint operator T on $\mathscr{H}$ uniquely such that*

$$\begin{cases} \mathcal{D}(T) = \left\{ u \in \mathcal{Q}(q) \,|\, \exists w_u \in \mathscr{H} \text{ such that } q(u, v) = \langle w_u, v \rangle, \ \forall v \in \mathcal{Q}(q) \right\}, \\ Tu = w_u, \quad u \in \mathcal{D}(T). \end{cases}$$

We note that $\mathcal{D}(T)$ can be simply written as

$$\mathcal{D}(T) = \left\{ u \in \mathcal{Q}(q) \,|\, Tu \in \mathscr{H} \right\}.$$

For the proof of Lemma 2.3, see [8, Theorem VIII.15].

The following lemma states that $V_\pm$ are relatively form bounded with respect to $-\Delta$.

Lemma 2.4. *Let V_+ and V_- be as in Proposition 2.1. Then for any $\varepsilon > 0$, there exists a constant $b_\varepsilon > 0$ such that the following estimates hold:*

$$\int_\Omega V_+(x) |u(x)|^2\, dx \leq \varepsilon \|\nabla u\|^2_{L^2(\Omega)} + b_\varepsilon \|u\|^2_{L^2(\Omega)}, \tag{2.3}$$

$$\int_\Omega V_-(x) |u(x)|^2\, dx \leq a_d \|\nabla u\|^2_{L^2(\Omega)}, \tag{2.4}$$

for any $u \in H_0^1(\Omega)$, where

$$a_d := \frac{\|V_-\|_{K_d(\Omega)}}{4\gamma_d} < 1.$$

Proof. The proof is similar to that of Lemma 3.1 from [2]. Let $u \in C_0^\infty(\Omega)$, and let $\tilde{u}$ and $\tilde{V}_\pm$ be the zero extensions of u and $V_\pm$ to $\mathbb{R}^d$, respectively. First, we prove that for any $\varepsilon > 0$, there exists a constant $b_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^d} \tilde{V}_+(x)|\tilde{u}(x)|^2\,dx \le \varepsilon\|\nabla\tilde{u}\|^2_{L^2(\mathbb{R}^d)} + b_\varepsilon\|\tilde{u}\|^2_{L^2(\mathbb{R}^d)}. \tag{2.5}$$

We divide the proof of (2.5) into two cases: $d = 3$ and $d > 3$. When $d = 3$, the inequality (2.5) is equivalent to

$$\begin{aligned}\int_{\mathbb{R}^3} \tilde{V}_+(x)|\tilde{u}(x)|^2\,dx \le&\varepsilon\langle\tilde{u}, -\Delta\tilde{u}\rangle_{L^2(\mathbb{R}^3)} + b_\varepsilon\|\tilde{u}\|^2_{L^2(\mathbb{R}^3)}\\ =&\varepsilon\left\|\left(H_0 + \frac{b_\varepsilon}{\varepsilon}\right)^{1/2}\tilde{u}\right\|^2_{L^2(\mathbb{R}^3)},\end{aligned}$$

where $H_0 = -\Delta$ is the self-adjoint operator with domain $H^2(\mathbb{R}^3)$. Put

$$v = \left(H_0 + \frac{b_\varepsilon}{\varepsilon}\right)^{1/2}\tilde{u}.$$

Then the estimate (2.5) takes the following form:

$$\left\|\tilde{V}_+^{1/2}\left(H_0 + \frac{b_\varepsilon}{\varepsilon}\right)^{-1/2} v\right\|^2_{L^2(\mathbb{R}^3)} \le \varepsilon\|v\|^2_{L^2(\mathbb{R}^3)}.$$

This estimate can be obtained if we show that

$$\|TT^*\|_{\mathscr{B}(L^2(\mathbb{R}^3))} \le \varepsilon, \tag{2.6}$$

where we set

$$T := \tilde{V}_+^{1/2}\left(H_0 + \frac{b_\varepsilon}{\varepsilon}\right)^{-1/2}.$$

Thus, it suffices to show that for any $\varepsilon > 0$, there exists a constant $b_\varepsilon > 0$ such that the estimate (2.6) holds. Let $\varepsilon > 0$ be fixed and $b > 0$. Using the formula:

$$\left(H_0 + \frac{b}{\varepsilon}\right)^{-1} v(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{-\sqrt{\frac{b}{\varepsilon}}|x-y|}}{|x-y|} v(y)\,dy$$

and Schwarz inequality, we can estimate

$$\begin{aligned}&\|TT^*v\|^2_{L^2(\mathbb{R}^3)}\\ =&\left\|\tilde{V}_+^{1/2}\left(H_0 + \frac{b}{\varepsilon}\right)^{-1}\tilde{V}_+^{1/2}v\right\|^2_{L^2(\mathbb{R}^3)}\\ =&\frac{1}{(4\pi)^2}\int_{\mathbb{R}^3}\tilde{V}_+(x)\left|\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\frac{b}{\varepsilon}}|x-y|}}{|x-y|}\tilde{V}_+^{1/2}(y)v(y)\,dy\right|^2 dx\\ \le&\frac{1}{(4\pi)^2}\int_{\mathbb{R}^3}\tilde{V}_+(x)\left(\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\frac{b}{\varepsilon}}|x-y|}}{|x-y|}\tilde{V}_+(y)\,dy\right)\left(\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\frac{b}{\varepsilon}}|x-y|}}{|x-y|}|v(y)|^2\,dy\right)dx.\end{aligned}$$

Now, we estimate the first integral on the right. We split the integral as follows:

$$\int_{\mathbb{R}^3} \frac{e^{-\sqrt{\frac{b}{\varepsilon}}|x-y|}}{|x-y|}\tilde{V}_+(y)\,dy = \int_{|x-y|<r} + \int_{|x-y|\geq r} =: I_1 + I_2$$

for any $r > 0$. Let $\delta > 0$ be fixed. Then, if we choose $r > 0$ small enough, we have $I_1 \leq \delta$, since $V_+ \in K^d(\Omega)$. Then, choosing $b = b_\delta > 0$ large enough, we have $I_2 \leq \delta$. Thus we obtain

$$\int_{\mathbb{R}^3} \frac{e^{-\sqrt{\frac{b}{\varepsilon}}|x-y|}}{|x-y|}\tilde{V}_+(y)\,dy \leq 2\delta. \tag{2.7}$$

Using this estimate, we can estimate

$$\|TT^*v\|^2_{L^2(\mathbb{R}^3)} \leq \frac{2\delta}{(4\pi)^2}\int_{\mathbb{R}^3}\tilde{V}_+(x)\left(\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\frac{b}{\varepsilon}}|x-y|}}{|x-y|}|v(y)|^2\,dy\right)dx.$$

Moreover, using Fubini–Tonnelli theorem and the estimate (2.7) once more, we can estimate

$$\begin{aligned}\|TT^*v\|^2_{L^2(\mathbb{R}^3)} \leq & \frac{2\delta}{(4\pi)^2}\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\frac{e^{-\sqrt{\frac{b}{\varepsilon}}|x-y|}}{|x-y|}\tilde{V}_+(x)\,dx\right)|v(y)|^2\,dy\\ \leq & \left(\frac{2\delta}{4\pi}\right)^2\|v\|^2_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Thus, by choosing $\delta = 2\pi\varepsilon$, we get (2.6), which implies (2.5) for $d = 3$.

When $d > 3$, we can also prove the estimate (2.5) in the same argument as in the case when $d = 3$, if we note that the kernel $K_M(x)$ of $(-\Delta + M)^{-1}$ for $M > 0$ satisfies

$$|K_M(x)| \leq \frac{1}{4\gamma_d|x|^{d-2}} \quad \text{and} \quad \lim_{M\to+\infty}\sup_{|x|>r} e^{|x|}K_M(x) = 0$$

for each $r > 0$ (see [9, p.454]). Indeed, we can perform the argument involving $H_0 + \frac{b_\varepsilon}{\varepsilon}$ by using the previous asymptotics, and as a result, we get also (2.5).

Based on (2.5), we can prove the required estimates (2.3). In fact, we can estimate, by using (2.5),

$$\begin{aligned}\int_\Omega V_+(x)|u(x)|^2\,dx &= \int_{\mathbb{R}^d}\tilde{V}_+(x)|\tilde{u}(x)|^2\,dx\\ &\leq \varepsilon\|\nabla\tilde{u}\|^2_{L^2(\mathbb{R}^d)} + b_\varepsilon\|\tilde{u}\|^2_{L^2(\mathbb{R}^d)}\\ &= \varepsilon\|\nabla u\|^2_{L^2(\Omega)} + b_\varepsilon\|u\|^2_{L^2(\Omega)}.\end{aligned}$$

As a consequence, by density argument, the inequality (2.3) is proved. The proof of (2.4) is almost identical to that of (2.3). The only difference is the estimate (2.7).

Instead of (2.7), we can apply the following estimate:

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{e^{-\sqrt{\frac{b}{\varepsilon}}|x-y|}}{|x-y|}\tilde{V}_-(y)\,dy \leq & \int_{\mathbb{R}^3} \frac{\tilde{V}_-(y)}{|x-y|}\,dy \\ \leq & \|\tilde{V}_-\|_{K_3(\mathbb{R}^3)} \\ = & \|V_-\|_{K_3(\Omega)},\end{aligned}$$

whence the argument in (2.3) works well in this case, and we get (2.4). The proof of Lemma 2.4 is complete. □

We are now in a position to prove Proposition 2.1.

Proof of Proposition 2.1. Let the quadratic form $q : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{C}$ be

$$q(u,v) = \int_\Omega (\nabla u \cdot \overline{\nabla v} + V u\bar{v})\,dx, \quad u,v \in H^1_0(\Omega).$$

It is clear that q is densely defined and semibounded. Hence, as a consequence of Lemma 2.3, it suffices to show that the quadratic form q is closed. Hence all we have to do is to show that the norm $\|\cdot\|_{+1}$ is equivalent to that of $H^1_0(\Omega)$, where $\|\cdot\|_{+1}$ is defined in (2.2), i.e.,

$$\|u\|_{+1} = \sqrt{q(u,u) + \|u\|^2_{L^2(\Omega)}}.$$

In fact, by Lemma 2.4 and $0 \leq a_d < 1$, we have

$$\begin{aligned}\|u\|^2_{+1} &\leq \|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega V(x)|u(x)|^2\,dx + \|u\|^2_{L^2(\Omega)} \\ &\leq C(\|\nabla u\|_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}),\end{aligned}$$

$$\begin{aligned}\|u\|^2_{+1} &\geq \|\nabla u\|^2_{L^2(\Omega)} - \int_\Omega V_-(x)|u(x)|^2\,dx + \|u\|^2_{L^2(\Omega)} \\ &\geq (1-a_d)\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\end{aligned}$$

for any $u \in H^1_0(\Omega)$, which implies that $\|\cdot\|_{+1}$ is equivalent to $\|\cdot\|_{H^1(\Omega)}$. The proof of Proposition 2.1 is complete. □

3. L^p–L^q–ESTIMATES AND POINTWISE ESTIMATES FOR e^{-tH_V}

In this section we shall prove L^p–L^q–estimates for e^{-tH_V} and pointwise estimates for the integral kernel of e^{-tH_V} on Ω.

More precisely, we have the following:

Proposition 3.1. *Assume that the measurable potential $V = V_+ - V_-$ satisfies $V_\pm \in K_d(\Omega)$. Let $1 \leq p \leq q \leq \infty$. If $\|V_-\|_{K_d(\Omega)} < 2\gamma_d$, then*

$$\|e^{-tH_V} f\|_{L^q(\Omega)} \leq \frac{(2\pi t)^{-d(1/p-1/q)/2}}{(1-\|V_-\|_{K_d(\Omega)}/2\gamma_d)^2}\|f\|_{L^p(\Omega)}, \quad \forall t > 0 \tag{3.1}$$

for any $f \in L^p(\Omega)$. In addition, if we further assume that V satisfies

$$\|V_-\|_{K_d(\Omega)} < \gamma_d,$$

then the kernel $K(t,x,y)$ of e^{-tH_V} enjoys with the proprerty that

$$0 \le K(t,x,y) \le \frac{(2\pi t)^{-d/2}}{1-\|V_-\|_{K_d(\Omega)}/\gamma_d} e^{-|x-y|^2/8t}, \quad \forall t>0 \tag{3.2}$$

for any $x, y \in \Omega$.

The following lemma is crucial in the proof of Proposition 3.1.

Lemma 3.2. *Let $d \ge 3$. Assume that the measurable potential $V = V_+ - V_-$ satisfies $V_\pm \in K_d(\Omega)$ and $\|V_-\|_{K_d(\Omega)} < 4\gamma_d$. Let $\tilde{V}$ be the zero extension of V to $\mathbb{R}^d$ and $\tilde{H}_{\tilde{V}}$ the self-adjoint operator $H_{\tilde{V}}$ on $\mathbb{R}^d$. Then for any non-negative function $f \in L^2(\Omega)$, the following estimates hold*:

$$\big(e^{-tH_V} f\big)(x) \ge 0, \tag{3.3}$$

$$\big(e^{-tH_V} f\big)(x) \le \big(e^{-t\tilde{H}_{\tilde{V}}} \tilde{f}\,\big)(x) \tag{3.4}$$

for $t>0$ and almost everywhere $x \in \Omega$, where $\tilde{f}$ is the zero extension of f to $\mathbb{R}^d$.

The proof of Lemma 3.2 is rather long, and will be postponed. Let us prove Proposition 3.1.

Proof of Proposition 3.1. Let $f \in C_0^\infty(\Omega)$. Applying (3.3) from Lemma 3.2 to non-negative functions $|f|-f$ and $|f|+f$, we obtain

$$-\big(e^{-tH_V}|f|\big)(x) \le \big(e^{-tH_V} f\big)(x) \le \big(e^{-tH_V}|f|\big)(x)$$

for any $t>0$ and almost everywhere $x \in \Omega$. Hence the above inequality and (3.4) from Lemma 3.2 imply that

$$\big|\big(e^{-tH_V} f\big)(x)\big| \le \big(e^{-t\tilde{H}_{\tilde{V}}}|\tilde{f}|\big)(x) \tag{3.5}$$

for any $t>0$ and almost everywhere $x \in \Omega$. Here we recall the result of L^p–L^q–estimates for $e^{-t\tilde{H}_{\tilde{V}}}$ on $\mathbb{R}^d$:

$$\|e^{-t\tilde{H}_{\tilde{V}}}|\tilde{f}|\|_{L^q(\mathbb{R}^d)} \le \frac{(2\pi t)^{-d(1/p-1/q)/2}}{(1-\|\tilde{V}_-\|_{K_d(\mathbb{R}^d)}/2\gamma_d)^2}\|\tilde{f}\|_{L^p(\mathbb{R}^d)}, \quad \forall t>0, \tag{3.6}$$

provided $1 \le p \le q \le \infty$. (see Proposition 5.1 from [2]). Combining (3.5)–(3.6), the estimate (3.1) can be obtained for $f \in C_0^\infty(\Omega)$. Thus, by density argument, we conclude the estimates (3.1) for any $f \in L^p(\Omega)$ if $p<\infty$. The case $p=\infty$ follows from the duality argument.

We now turn to prove (3.2). We adopt a sequence $\{j_\varepsilon(x)\}_{\varepsilon>0}$ of functions defined as the following:

$$j_\varepsilon(x) := \frac{1}{\varepsilon^d}\, j\left(\frac{x}{\varepsilon}\right), \quad x \in \mathbb{R}^d, \tag{3.7}$$

where

$$j(x) = \begin{cases} C_d\, e^{-1/(1-|x|^2)}, & |x|<1, \\ 0, & |x| \ge 1 \end{cases}$$

with

$$C_d := \left(\int_{\mathbb{R}^d} e^{-1/(1-|x|^2)}\, dx\right)^{-1}.$$

As is well-known, the sequence $\{j_\varepsilon(x)\}_\varepsilon$ enjoys with the following property:

$$j_\varepsilon(\cdot - y) \to \delta_y \quad \text{in } \mathscr{S}'(\mathbb{R}^d) \quad (\varepsilon \to 0), \tag{3.8}$$

where δ_y is the Dirac delta function at $y \in \Omega$. Let $y \in \Omega$ be fixed, and let $K(t,x,y)$ and $\tilde{K}(t,x,y)$ be kernels of e^{-tH_V} and $e^{-t\tilde{H}_{\tilde{V}}}$, respectively. Taking $\varepsilon > 0$ sufficiently small so that $\operatorname{supp} j_\varepsilon(\cdot - y) \subset \Omega$, and applying (3.3)–(3.4) from Lemma 3.2 to both f and $\tilde{f}$ replaced by $j_\varepsilon(\cdot - y)$, we get

$$0 \le \int_\Omega K(t,x,z) j_\varepsilon(z-y)\,dz \le \int_{\mathbb{R}^d} \tilde{K}(t,x,y) j_\varepsilon(z-y)\,dz$$

for any $x \in \Omega$. Noting (3.8) and taking the limit of the previous inequality as $\varepsilon \to 0$, we get

$$0 \le K(t,x,y) \le \tilde{K}(t,x,y)$$

for any $t > 0$ and $x \in \Omega$. Finally, by using the pointwise estimates:

$$\tilde{K}(t,x,y) \le \frac{(2\pi t)^{-d/2}}{1 - \|\tilde{V}_-\|_{K_d(\mathbb{R}^d)}/\gamma_d} e^{-|x-y|^2/8t} \left(= \frac{(2\pi t)^{-d/2}}{1 - \|V_-\|_{K_d(\Omega)}/\gamma_d} e^{-|x-y|^2/8t} \right)$$

(see Proposition 5.1 from [2]), we obtain the estimate (3.2), as desired. The proof of Proposition 3.1 is finished. □

In the rest of this section we shall prove Lemma 3.2. To prove Lemma 3.2 we need further the following two lemmas. The first one is concerned with the existence and uniqueness of solutions for evolution equations in abstract setting.

Lemma 3.3. *Let $\mathscr{H}$ be a Hilbert space. Assume that A is a non-negative self-adjoint operator on $\mathscr{H}$. Let $\{T(t)\}_{t\ge 0}$ be the semigroup generated by A, and let $f \in \mathscr{H}$ and $u(t) = T(t)f$. Then u is the unique solution of the following problem:*

$$\begin{cases} u \in C([0,\infty);\mathscr{H}) \cap C((0,\infty);\mathcal{D}(A)) \cap C^1((0,\infty);\mathscr{H}), \\ u'(t) + Au(t) = 0, \quad t > 0, \\ u(0) = f. \end{cases}$$

For the proof of Lemma 3.3, see, e.g., Cazenave and Haraux [1, Theorem 3.2.1].

The second one is about the differentiability properties for composite functions of Lipschitz continuous functions and $W^{1,p}$–functions.

Lemma 3.4. *Let $d \ge 1$ and Ω be an open set in $\mathbb{R}^d$, and let $1 \le p \le \infty$. Consider the positive and negative parts of a real-valued function $u \in W^{1,p}(\Omega)$:*

$$u^+ = \chi_{\{u>0\}} u, \quad u^- = -\chi_{\{u<0\}} u.$$

Then $u^\pm \in W^{1,p}(\Omega)$ and

$$\partial_{x_i} u^+ = \chi_{\{u>0\}} \partial_{x_i} u, \quad \partial_{x_i} u^- = -\chi_{\{u<0\}} \partial_{x_i} u \quad (1 \le i \le d),$$

where $\partial_{x_i} = \partial/\partial x_i$.

For the proof of Lemma 3.4, see Gilbarg and Trudinger [5, Lemma 7.6].

To prove (3.3), we show that the negative part of $e^{-tH_V} f$ vanishes in Ω, provided $f \ge 0$. For this purpose, we prepare the following lemma.

Lemma 3.5. *For any non-negative function $f \in L^2(\Omega)$, let $u(t) = e^{-tH_V} f$. Then the negative part $u^-(t)$ of $u(t)$ belongs to $H_0^1(\Omega)$ for each $t > 0$.*

Proof. Obviously, $u(t)$ satisfies

$$\partial_t u(t,x) + H_V u(t,x) = 0, \quad t > 0, \quad x \in \Omega,$$
$$u(0,x) = f(x), \quad x \in \Omega.$$

Lemma 3.3 assures that

$$u \in C([0,\infty); L^2(\Omega)) \cap C^1((0,\infty); L^2(\Omega))$$

and

$$u(t) \in H_0^1(\Omega), \quad H_V u(t) \in L^2(\Omega) \quad \text{for each } t > 0.$$

Since $u(t) \in H_0^1(\Omega)$ for each $t > 0$, there exist $\varphi_n(t) \in C_0^\infty(\Omega)$ $(n = 1, 2, \dots)$ such that

$$\varphi_n(t) \to u(t) \quad \text{in } H^1(\Omega) \tag{3.9}$$

as $n \to \infty$ for each $t > 0$. Here $\{\varphi_n\}_n$ also depends on t. For the convenience of notation, we may omit the time variable t of φ_n without any confusion. Let us take a non-negative function $\psi \in C^\infty(\mathbb{R})$ as

$$\psi(x) \begin{cases} = -x, & x \le -1, \\ \le -x, & -1 < x < 0, \\ = 0, & x \ge 0, \end{cases}$$

and put

$$\psi_n(x) := \frac{\psi(nx)}{n}, \quad n = 1, 2, \cdots . \tag{3.10}$$

Then there exists a constant $C > 0$ such that

$$|\psi_n'(x)| \le C, \quad \forall x \in \mathbb{R}, \ \forall n \in \mathbb{N}. \tag{3.11}$$

Let us consider two kinds of composite functions $\psi_n \circ \varphi_n$ and $\psi_n \circ u$. We show that

$$\psi_n \circ \varphi_n - \psi_n \circ u \to 0 \quad \text{in } H^1(\Omega), \tag{3.12}$$

$$\psi_n \circ u - u^- \to 0 \quad \text{in } H^1(\Omega) \tag{3.13}$$

as $n \to \infty$. In fact, by the mean value theorem, we have

$$\begin{aligned} \|\psi_n \circ \varphi_n - \psi_n \circ u\|_{L^2(\Omega)} =& \left\| \int_0^1 \psi_n'\big(\theta\varphi_n + (1-\theta)u\big)(\varphi_n - u)\, d\theta \right\|_{L^2(\Omega)} \\ \le& C\|\varphi_n - u\|_{L^2(\Omega)}, \end{aligned} \tag{3.14}$$

and the derivative of $\psi_n \circ \varphi_n - \psi_n \circ u$ is written as

$$\begin{aligned} &\|\partial_{x_i}(\psi_n \circ \varphi_n - \psi_n \circ u)\|_{L^2(\Omega)} \\ =&\|\psi_n'(\varphi_n)\partial_{x_i}\varphi_n - \psi_n'(u)\partial_{x_i}u\|_{L^2(\Omega)} \\ \le&\|\psi_n'(\varphi_n)(\partial_{x_i}\varphi_n - \partial_{x_i}u)\|_{L^2(\Omega)} + \|\{\psi_n'(\varphi_n) - \psi_n'(u)\}\partial_{x_i}u\|_{L^2(\Omega)} \\ \le&C\|\partial_{x_i}\varphi_n - \partial_{x_i}u\|_{L^2(\Omega)} + \|\{\psi_n'(\varphi_n) - \psi_n'(u)\}\partial_{x_i}u\|_{L^2(\Omega)}, \end{aligned} \tag{3.15}$$

where we used (3.11) in the last step. Noting the pointwise convergence and uniform boundedness with respect to n:

$$\begin{aligned}&\big\{\psi_n'(\varphi_n)(x)-\psi_n'(u)(x)\big\}\partial_{x_i}u(x)\to 0 \quad \text{as } n\to\infty \text{ for a.e. } x\in\Omega,\\ &\big|\big\{\psi_n'(\varphi_n)(x)-\psi_n'(u)(x)\big\}\partial_{x_i}u(x)\big|\le 2C|\partial_{x_i}u(x)|\in L^2(\Omega),\end{aligned}$$

we can apply Lebesgue's dominated convergence theorem to obtain

$$\|\{\psi_n'(\varphi_n)-\psi_n'(u)\}\partial_{x_i}u\|_{L^2(\Omega)}\to 0 \tag{3.16}$$

as $n\to\infty$. Hence, summarizing (3.9) and (3.14)–(3.16), we obtain (3.12).

As to the latter convergence (3.13), since

$$|(\psi_n\circ u)(x)-u^-(x)|\le 2|u(x)|\in L^2(\Omega),$$

$$|\partial_{x_i}(\psi_n\circ u)(x)-\partial_{x_i}u^-(x)|\le (C+1)|\partial_{x_i}u(x)|\in L^2(\Omega),$$

and

$$(\psi_n\circ u)(x)-u^-(x)\to 0,$$

$$\partial_{x_i}(\psi_n\circ u)(x)-\partial_{x_i}u^-(x)=\{\psi_n'(u)+\chi_{\{u<0\}}\}\partial_{x_i}u(x)\to 0$$

as $n\to\infty$ for almost everywhere $x\in\Omega$, Lebesgue's dominated convergence theorem allows as to obtain (3.13). Thus (3.12)–(3.13) imply that

$$\psi_n\circ\varphi_n-u^-\to 0 \quad \text{in } H^1(\Omega) \quad (n\to\infty).$$

Since $\psi_n\circ\varphi_n\in C_0^\infty(\Omega)$, we conclude that $u^-\in H_0^1(\Omega)$. The proof of Lemma 3.5 is finished. □

We are now in a position to prove Lemma 3.2.

Proof of Lemma 3.2. Let $f\in L^2(\Omega)$ and $f\ge 0$ almost everywhere on Ω. Put

$$u(t)=e^{-tH_V}f \quad \text{for } t\ge 0.$$

If we show that $\|u^-(t)\|_{L^2(\Omega)}^2$ is monotonically decreasing with respect to $t\ge 0$, then we can obtain

$$u^-(t,x)=0$$

for each $t>0$ and almost everywhere $x\in\Omega$, since $u^-(0,x)=f^-(x)=0$ for almost everywhere $x\in\Omega$. This means that

$$u(t,x)\ge 0$$

for each $t>0$ and almost everywhere $x\in\Omega$; thus we conclude (3.3). Hence it is sufficient to show that

$$\frac{d}{dt}\int_\Omega (u^-)^2\,dx\le 0. \tag{3.17}$$

We compute

$$\begin{aligned}\frac{d}{dt}\int_\Omega (u^-)^2\,dx =& \frac{d}{dt}\int_{\{u<0\}} u^2\,dx \\ =& 2\int_{\{u<0\}} u_t u\,dx \\ =& -2\int_\Omega u_t u^-\,dx \\ =& 2\int_\Omega (H_V u)u^-\,dx\end{aligned} \tag{3.18}$$

where we use the equation $u_t + H_V u = 0$ in the last step. Since $u^- \in H_0^1(\Omega)$ by Lemma 3.5, we have, by going back to Definition 2.1 of H_V,

$$\int_\Omega (H_V u)u^-\,dx = \int_\Omega \nabla u\cdot\nabla u^-\,dx + \int_\Omega Vuu^-\,dx. \tag{3.19}$$

Here, by using Lemma 3.4, we get

$$\nabla u^- = -\chi_{\{u<0\}}\nabla u,$$

and hence, the first term on the right of (3.19) can be estimated as

$$\int_\Omega \nabla u\cdot\nabla u^-\,dx = -\int_\Omega |\nabla u^-|^2\,dx.$$

As to the second, by (2.4) from Lemma 2.4, we have

$$\begin{aligned}\int_\Omega Vuu^-\,dx =& -\int_\Omega V|u^-|^2\,dx \\ \le& \int_\Omega V_-|u^-|^2\,dx \\ \le& a_d\int_\Omega |\nabla u^-|^2\,dx;\end{aligned}$$

thus we find from $a_d < 1$ that

$$\begin{aligned}\int_\Omega (H_V u)u^-\,dx \le& -(1-a_d)\int_\Omega |\nabla u^-|^2\,dx \\ \le& 0,\end{aligned}$$

and hence, combining this inequality and (3.18), we conclude (3.17).

Next, we prove (3.4). Let us define two functions $v^{(1)}(t)$ and $v^{(2)}(t)$ as follows:

$$v^{(1)}(t) := e^{-t\tilde{H}_{\tilde{V}}}\tilde{f} \quad\text{and}\quad v^{(2)}(t) := e^{-tH_V}f$$

for $t \ge 0$. Then it follows from Lemma 3.3 that $v^{(1)}$ and $v^{(2)}$ satisfy

$$\begin{cases} v^{(1)} \in C([0,\infty), L^2(\mathbb{R}^d)) \cap C^1((0,\infty), L^2(\mathbb{R}^d)), \\ v^{(1)}(t) \in H^1(\mathbb{R}^d), \quad \tilde{H}_{\tilde{V}} v^{(1)}(t) \in L^2(\mathbb{R}^d), \\ v_t^{(1)}(t) + \tilde{H}_{\tilde{V}} v^{(1)}(t) = 0, \\ v^{(1)}(0) = \tilde{f} \end{cases} \tag{3.20}$$

and

$$
(3.21) \qquad \begin{cases} v^{(2)} \in C([0,\infty), L^2(\Omega)) \cap C^1((0,\infty), L^2(\Omega)), \\ v^{(2)}(t) \in H_0^1(\Omega), \quad H_V v^{(2)}(t) \in L^2(\Omega), \\ v_t^{(2)}(t) + H_V v^{(2)}(t) = 0, \\ v^{(2)}(0) = f \end{cases}
$$

for each $t > 0$, respectively. We define a new function v as

$$
v(t) := v^{(1)}(t)|_\Omega - v^{(2)}(t)
$$

for $t \geq 0$, where $v^{(1)}(t)|_\Omega$ is the restriction of $v^{(1)}(t)$ to Ω. Let us consider the negative part of v:

$$
v^- = -\chi_{\{v<0\}} v.
$$

Then, resorting to (3.20)–(3.21), we have

$$
v^- \in C([0,\infty), L^2(\Omega)) \cap C^1((0,\infty), L^2(\Omega)).
$$

Moreover, by using Lemma 3.4, we have $v^- \in H^1(\Omega)$, since $v \in H^1(\Omega)$. Once we prove that

$$
(3.22) \qquad v^- \in H_0^1(\Omega),
$$

we can get, by the previous argument,

$$
(3.23) \qquad \frac{d}{dt} \int_\Omega \left(v^-\right)^2 dx \leq 0.
$$

In fact, by the definition of v^-, we have

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega \left(v^-\right)^2 dx =& -2 \int_{\{v<0\}} v_t^{(1)} v^- \, dx + 2 \int_{\{v<0\}} v_t^{(2)} v^- \, dx \\
=& 2 \int_{\mathbb{R}^d} (\tilde{H}_{\tilde{V}} v^{(1)}) \tilde{v}^- \, dx - 2 \int_\Omega (H_V v^{(2)}) v^- \, dx
\end{aligned}
$$

where $\tilde{v}^-$ is the zero extension of v^- to $\mathbb{R}^d$, and we use equations $v_t^{(1)} + \tilde{H}_{\tilde{V}} v^{(1)} = 0$ and $v_t^{(2)} + H_V v^{(2)} = 0$ in the last step. Since $v^- \in H_0^1(\Omega)$ by (3.22), we have, by definitions of $\tilde{H}_{\tilde{V}}$ and H_V

$$
\begin{aligned}
& \int_{\mathbb{R}^d} (\tilde{H}_{\tilde{V}} v^{(1)}) \tilde{v}^- \, dx - \int_\Omega (H_V v^{(2)}) v^- \, dx \\
=& \int_{\mathbb{R}^d} \nabla v^{(1)} \cdot \nabla \tilde{v}^- \, dx + \int_{\mathbb{R}^d} \tilde{V} v^{(1)} \tilde{v}^- \, dx - \int_\Omega \nabla v^{(2)} \cdot \nabla v^- \, dx - \int_\Omega V v^{(2)} v^- \, dx \\
=& \int_\Omega \nabla v \cdot \nabla v^- \, dx + \int_\Omega V v v^- \, dx \\
\leq& -(1-a_d) \int_\Omega |\nabla v^-|^2 \, dx \\
\leq& \; 0.
\end{aligned}
$$

Hence we obtain (3.23), which implies the required inequality (3.4).

It remains to prove (3.22). The proof is similar to that of Lemma 3.5. Since $v^{(2)}(t) \in H_0^1(\Omega)$ for each $t > 0$ by (3.21), there exist $\varphi_n = \varphi_n(t) \in C_0^\infty(\Omega)$ such that

$$\varphi_n \to v^{(2)} \quad \text{in } H^1(\Omega)$$

as $n \to \infty$. Put

$$v_n(t) := v^{(1)}(t)|_\Omega - \varphi_n(t), \quad n = 1, 2, \cdots,$$

for each $t > 0$. Let ψ_n be as in (3.10). As in the proof of Lemma 3.5, we can show that

$$\psi_n \circ v_n^- - v^- \to 0 \quad \text{in } H^1(\Omega)$$

as $n \to \infty$. Since v_n^- have compact supports in $\underbrace{\text{supp}\,\varphi_n}$ by $v^{(1)} \geq 0$ on Ω, the functions $\psi_n \circ v_n^-$ also have compact supports in Ω. Let $\widetilde{\psi_n \circ v_n^-}$ be the zero extension of $\psi_n \circ v_n^-$ to $\mathbb{R}^d$, and let J_ε be Friedrichs' mollifier: For $u \in L^1_{\text{loc}}(\mathbb{R}^d)$,

$$(J_\varepsilon u)(x) := (j_\varepsilon * u)(x) = \int_{\mathbb{R}^d} j_\varepsilon(x-y)u(y)\,dy, \quad x \in \mathbb{R}^d,$$

where $\{j_\varepsilon(x)\}_\varepsilon$ are functions defined in (3.7). Taking $\varepsilon = \varepsilon_n$ sufficiently small so that $\varepsilon_n \to 0$ $(n \to \infty)$ and supp $J_{\varepsilon_n}\left(\widetilde{\psi_n \circ v_n^-}\right)$ is contained in Ω, we have

$$J_{\varepsilon_n}(\widetilde{\psi_n \circ v_n^-})|_\Omega \in C_0^\infty(\Omega).$$

Since

$$J_{\varepsilon_n}\left(\widetilde{\psi_n \circ v_n^-}\right)\Big|_\Omega - v^- \to 0 \quad \text{in } H^1(\Omega)$$

as $n \to \infty$, we conclude (3.22). The proof of Lemma 3.2 is complete. □

4. L^p–$\ell^p(L^q)_\theta$–BOUNDEDNESS FOR $(\theta H_V - z)^{-\beta}$

In this section we shall prove the boundedness of resolvent $(\theta H_V - z)^{-\beta}$ $(\beta > 0)$ in scaled amalgam spaces. The result in this section will play an important role in the proof of Theorem 1.1.

More precisely, we have:

Theorem 4.1. *Let $1 \leq p \leq q \leq \infty$ and $\beta > d(1/p - 1/q)/2$, and let $z \in \mathbb{C}$ with* $\mathrm{Re}(z) < 0$. *Then there exists a constant $C = C(d, p, q, \beta, z) > 0$ such that*

$$\|(\theta H_V - z)^{-\beta}\|_{\mathscr{B}(L^p(\Omega), L^q(\Omega))} \leq C\theta^{-d(1/p-1/q)/2}, \tag{4.1}$$

$$\|(\theta H_V - z)^{-\beta}\|_{\mathscr{B}(L^p(\Omega), \ell^p(L^q)_\theta)} \leq C\theta^{-d(1/p-1/q)/2} \tag{4.2}$$

for any $\theta > 0$.

Proof. Let us first prove (4.1). We use the following well-known formula: For $z \in \mathbb{C}$ with $\mathrm{Re}(z) < 0$ and $\beta > 0$,

$$(H_V - z)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{zt} e^{-tH_V}\,dt. \tag{4.3}$$

Resorting to the formula (4.3) and L^p–L^q–estimates (3.1) for $e^{-t\theta H_V}$ in Proposition 3.1, we can estimate

$$\begin{aligned}\|(\theta H_V - z)^{-\beta} f\|_{L^q(\Omega)} &\le \frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1} e^{\mathrm{Re}(z)t}\|e^{-t\theta H_V} f\|_{L^q(\Omega)}\,dt\\ &\le C\theta^{-d(1/p-1/q)/2}\int_0^\infty t^{\beta-1}e^{\mathrm{Re}(z)t}t^{-d(1/p-1/q)/2}\,dt\cdot\|f\|_{L^p(\Omega)}.\end{aligned}$$

Since $\beta > d(1/p-1/q)/2$ and $\mathrm{Re}(z) < 0$, the integral on the right is absolutely convergent. Hence we obtain

$$\|(\theta H_V - z)^{-\beta} f\|_{L^q(\Omega)} \le C\theta^{-d(1/p-1/q)/2}\|f\|_{L^p(\Omega)}.$$

This proves (4.1).

Let us turn to the proof of (4.2). If we can prove that

$$\|e^{-t\theta H_V} f\|_{\ell^p(L^q)_\theta} \le C\theta^{-d(1/p-1/q)/2}(t^{-d(1/p-1/q)/2}+1)\|f\|_{L^p(\Omega)}, \quad \forall t > 0 \tag{4.4}$$

for any $f \in L^p(\Omega)$ provided $1 \le p \le q \le \infty$, then the estimate (4.2) will be obtained by combining (4.3)–(4.4). In fact, we can estimate

$$\begin{aligned}&\|(\theta H_V - z)^{-\beta} f\|_{\ell^p(L^q)_\theta}\\ \le&\frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}e^{\mathrm{Re}(z)t}\|e^{-t\theta H_V} f\|_{\ell^p(L^q)_\theta}\,dt\\ \le&C\theta^{-d(1/p-1/q)/2}\int_0^\infty t^{\beta-1}e^{\mathrm{Re}(z)t}(t^{-d(1/p-1/q)/2}+1)\,dt\cdot\|f\|_{L^p(\Omega)}.\end{aligned}$$

Since $\beta > d(1/p-1/q)/2$ and $\mathrm{Re}(z) < 0$, the integral on the right is absolutely convergent. Hence we conclude that

$$\|(\theta H_V - z)^{-\beta}\|_{\ell^p(L^q)_\theta} \le C\theta^{-d(1/p-1/q)/2}\|f\|_{L^p(\Omega)}.$$

This proves (4.2). Therefore, all we have to do is to prove the estimate (4.4). To this end, we prove the following estimate: For $1 \le q \le \infty$ and any $\theta > 0$,

$$\|K_0(\theta t,\cdot)\|_{l^1(L^q)_\theta} \le C\theta^{-d(1-1/q)/2}(t^{-d(1-1/q)/2}+1), \quad \forall t > 0, \tag{4.5}$$

where $K_0(t,x)$ is defined by

$$K_0(t,x) = \frac{(2\pi t)^{-d/2}}{1-\|V_-\|_{K_d(\Omega)}/\gamma_d}e^{-|x|^2/8t} =: C_1 t^{-d/2}e^{-|x|^2/8t}$$

for any $t > 0$ and $x \in \mathbb{R}^d$. In fact, we compute $\|K_0(\theta t,\cdot)\|_{L^q(C_\theta(n))}$ for the case $n = 0$ and $n \ne 0$, separately:

The case $n = 0$: We can estimate

$$\begin{aligned}
(4.6)\qquad \|K_0(\theta t,\cdot)\|_{L^q(C_\theta(0))} &\le C_1(\theta t)^{-d/2}\left(\int_{|x|<\frac{\theta^{1/2}}{2}} e^{-\frac{q|x|^2}{8\theta t}}\,dx\right)^{1/q}\\
&= C_1(\theta t)^{-d/2}\left(\int_{|x|<\frac{1}{2t^{1/2}}} e^{-\frac{q|x|^2}{8}}(\theta t)^{d/2}\,dx\right)^{1/q}\\
&\le C(\theta t)^{-d(1-1/q)/2}\left(\int_{\mathbb{R}^d} e^{-\frac{q|x|^2}{8}}\,dx\right)^{1/q}\\
&\le C(\theta t)^{-d(1-1/q)/2}.
\end{aligned}$$

The case $n \neq 0$: We can estimate

$$\begin{aligned}
(4.7)\qquad \sum_{n\neq 0}\|K_0(\theta t,\cdot)\|_{L^q(C_\theta(n))} &\le C_1(\theta t)^{-d/2}\sum_{n\neq 0}\left(\int_{C_\theta(n)} e^{-\frac{q|x|^2}{8\theta t}}\,dx\right)^{1/q}\\
&\le C_1(\theta t)^{-d/2}\sum_{n\neq 0}\left(\sup_{x\in C_\theta(n)} e^{-\frac{|x|^2}{8\theta t}}\right)\cdot\left(\int_{C_\theta(n)} dx\right)^{1/q}.
\end{aligned}$$

Here observing that

$$\frac{|\theta^{1/2}n|}{2} \le |x| \le 2|\theta^{1/2}n|, \quad x\in C_\theta(n),$$

we can estimate the right member of (4.7) as

$$C_1(\theta t)^{-d/2}\left(\sum_{n\neq 0} e^{-\frac{|n|^2}{16t}}\right)(\theta^{d/2})^{1/q},$$

and hence, we get

$$\sum_{n\neq 0}\|K_0(\theta t,\cdot)\|_{L^q(C_\theta(n))} \le C_1(\theta t)^{-d/2}\left(\sum_{n\neq 0} e^{-\frac{|n|^2}{16t}}\right)(\theta^{d/2})^{1/q}.$$

Here, by explicit calculations, we see that

$$\begin{aligned}
\sum_{n\neq 0} e^{-\frac{|n|^2}{16t}} = \sum_{n\neq 0} e^{-\frac{n_1^2+n_2^2+\cdots+n_d^2}{16t}} &= 2^d\left(\sum_{j=1}^{\infty} e^{-\frac{j^2}{16t}}\right)^d\\
&\le 2^d\left(\int_0^\infty e^{-\frac{x^2}{16t}}\,dx\right)^d\\
&= 4^d t^{d/2}\pi^{d/2}.
\end{aligned}$$

Summarizing the estimates obtained now, we conclude that

$$\begin{aligned}
(4.8)\qquad \sum_{n\neq 0}\|K_0(\theta t,\cdot)\|_{L^q(C_\theta(n))} &\le 4^d t^{d/2}\pi^{d/2}\cdot C_1(\theta t)^{-d/2}(\theta^{d/2})^{1/q}\\
&= 4^d\pi^{d/2}C_1\theta^{-d(1-1/q)/2}.
\end{aligned}$$

Combining the estimates (4.6)–(4.8), we obtain (4.5), as desired.

We are now in a position to prove the key estimate (4.4). Let $f \in L^p(\Omega)$ and $\tilde{f}$ be a zero extension of f to $\mathbb{R}^d$. Resorting to the estimate (3.2) in Proposition 3.1, we have

$$\begin{aligned}
\|e^{-t\theta H_V} f\|_{\ell^p(L^q)_\theta} &= \left\| \int_\Omega K(\theta t, x, y) f(y)\, dy \right\|_{\ell^p(L^q)_\theta} \\
&\le \left\| \int_\Omega K(\theta t, x, y) |f(y)|\, dy \right\|_{\ell^p(L^q)_\theta} \\
&\le \left\| \int_{\mathbb{R}^d} K_0(\theta t, x, y) |\tilde{f}(y)|\, dy \right\|_{\ell^p(L^q)_\theta} \\
&= \left\| K_0(\theta t, \cdot) * |\tilde{f}| \right\|_{\ell^p(L^q)_\theta(\mathbb{R}^d)} .
\end{aligned}$$

Applying the Young inequality (A.1) (see appendix A) to the right member, and using the estimate (4.5), we can estimate

$$\begin{aligned}
\|e^{-t\theta H_V} f\|_{\ell^p(L^q)_\theta} &\le 3^d \|K_0(\theta t, \cdot)\|_{l^1(L^r)_\theta(\mathbb{R}^d)} \|\tilde{f}\|_{\ell^p(\ell^p)_\theta(\mathbb{R}^d)} \\
&\le C\theta^{-d(1-1/r)/2}(t^{-d(1-1/r)/2} + 1)\|\tilde{f}\|_{L^p(\mathbb{R}^d)} \\
&= C\theta^{-d(1/p-1/q)/2}(t^{-d(1/p-1/q)/2} + 1)\|f\|_{L^p(\Omega)},
\end{aligned}$$

provided that p, q, r satisfy $1 \le p, q, r \le \infty$ and $1/p + 1/r - 1 = 1/q$. This proves (4.4). The proof of Theorem 4.1 is finished. □

5. Commutator estimates

In this section we shall prepare commutator estimates. These estimates will be also an important tool in the proof of Theorem 1.1. Among other things, we introduce an operator Ad as follows:

Definition. *Let X and Y be topological vector spaces, and let A and B be continuous linear operators from X and Y into themselves, respectively. For a continuous linear operator L from X into Y, the operator $\mathrm{Ad}^k(L)$ from X into Y, $k = 0, 1, \cdots$, is successively defined by*

$$\mathrm{Ad}^0(L) = L, \quad \mathrm{Ad}^k(L) = \mathrm{Ad}^{k-1}(BL - LA), \quad k \ge 1.$$

The result in this section is concerned with L^2-boundedness for $\mathrm{Ad}^k(e^{-itR_\theta})$, where R_θ is the resolvent operator defined by

$$R_\theta := (\theta H_V + M)^{-1}, \quad \theta > 0$$

for a fixed $M > 0$. Hereafter, operators A and B will be taken as

$$A = B = x_i - \theta^{1/2} n_i, \quad \text{for some } i \in \{1, \cdots, d\}. \tag{5.1}$$

Then we shall prove here the following.

Proposition 5.1. *Let $d \ge 3$. Assume that the measurable potential V is a real-valued function on Ω and satisfies $V = V_+ - V_-$, $V_\pm \ge 0$ such that $V_\pm \in K_d(\Omega)$ and*

$$\|V_-\|_{K_d(\Omega)} < 4\gamma_d.$$

Let A and B be the operators as in (5.1)*, and let $L = e^{-itR_\theta}$. For any non-negative integer k, there exists a constant $C = C(d, M, k) > 0$ such that*

$$\|\mathrm{Ad}^k(e^{-itR_\theta})\|_{\mathscr{B}(L^2(\Omega))} \le C\theta^{k/2}\langle t\rangle^k \tag{5.2}$$

for any $t > 0$ and $\theta > 0$, where we put

$$\langle t\rangle = \sqrt{1+t^2}.$$

First, we prepare L^2-boundedness for R_θ and $\partial_{x_i} R_\theta$ to prove Proposition 5.1.

Lemma 5.2. *Let $d \ge 3$ and V be as in Proposition 5.1. Then the following estimates hold*:

$$\|R_\theta\|_{\mathscr{B}(L^2(\Omega))} \le M^{-1}, \tag{5.3}$$

$$\|\partial_{x_i} R_\theta\|_{\mathscr{B}(L^2(\Omega))} \le M^{-1/2}(1-a_d)^{-1/2}\theta^{-1/2} \tag{5.4}$$

for any $\theta > 0$, where

$$a_d = \frac{\|V_-\|_{K_d(\Omega)}}{4\gamma_d} < 1.$$

Proof. Since H_V is the self-adjoint operator with domain

$$\mathcal{D}(H_V) = \{u \in H_0^1(\Omega) \mid H_V u \in L^2(\Omega)\},$$

we can obtain (5.3)–(5.4) by the spectral resolution. In fact, we have

$$\begin{aligned}\|R_\theta f\|^2_{L^2(\Omega)} &= \int_0^\infty \frac{1}{(\theta\lambda+M)^2}\, d\|E_{H_V}(\lambda)f\|^2_{L^2(\Omega)}\\ &\le M^{-2}\int_0^\infty d\,\|E_{H_V}(\lambda)f\|^2_{L^2(\Omega)}\\ &\le M^{-2}\|f\|^2_{L^2(\Omega)}\end{aligned}$$

for any $f \in L^2(\Omega)$. This proves (5.3).

Since $R_\theta f \in \mathcal{D}(H_V)$ for any $f \in L^2(\Omega)$, we can write

$$\begin{aligned}\|\nabla R_\theta f\|^2_{L^2(\Omega)} &= \int_\Omega \Big(\nabla R_\theta f\cdot\nabla R_\theta f + V|R_\theta f|^2 - V|R_\theta f|^2\Big)\,dx\\ &= \langle H_V R_\theta f, R_\theta f\rangle_{L^2(\Omega)} - \int_\Omega V|R_\theta f|^2\,dx\\ &= I + II.\end{aligned}$$

Then we can estimate

$$\begin{aligned} I &\le \int_0^\infty \frac{\lambda}{(\theta\lambda+M)^2}\, d\|E_{H_V}(\lambda)f\|^2_{L^2(\Omega)}\\ &= \int_0^\infty \theta^{-1}\cdot\frac{\theta\lambda}{\theta\lambda+M}\cdot\frac{1}{\theta\lambda+M}\, d\|E_{H_V}(\lambda)f\|^2_{L^2(\Omega)}\\ &\le \theta^{-1}M^{-1}\int_0^\infty d\,\|E_{H_V}(\lambda)f\|^2_{L^2(\Omega)}\\ &\le \theta^{-1}M^{-1}\|f\|^2_{L^2(\Omega)},\end{aligned}$$

and by Lemma 2.4,

$$\begin{aligned} II &\le \int_\Omega V_-|R_\theta f|^2\,dx \\ &\le a_d \int_\Omega |\nabla R_\theta f|^2\,dx. \end{aligned}$$

Combining the previous estimates, we conclude

$$\|\nabla R_\theta f\|^2_{L^2(\Omega)} \le \theta^{-1}(1-a_d)^{-1}M^{-1}\|f\|^2_{L^2(\Omega)}$$

for any $f \in L^2(\Omega)$. The proof of Lemma 5.2 is complete. □

We are now in a position to prove Proposition 5.1.

Proof of Proposition 5.1. Let us denote by $\mathscr{D}(\Omega)$ the totality of the test functions on Ω, and by $\mathscr{D}'(\Omega)$ its dual space. We regard X as $\mathscr{D}(\Omega)$ and Y as $\mathscr{D}'(\Omega)$ in the definition of operator Ad. Then we have, by Lemma B.2 in appendix B

$$\mathrm{Ad}^0(R_\theta) = R_\theta, \quad \mathrm{Ad}^1(R_\theta) = -2\theta R_\theta \partial_{x_i} R_\theta, \tag{5.5}$$

$$\mathrm{Ad}^k(R_\theta) = \theta\left\{-2k\mathrm{Ad}^{k-1}(R_\theta)\partial_{x_i}R_\theta + k(k-1)\mathrm{Ad}^{k-2}(R_\theta)R_\theta\right\}, \quad k \ge 2. \tag{5.6}$$

Since R_θ and $\partial_{x_i}R_\theta$ are bounded on $L^2(\Omega)$ by Lemma 5.2, $\mathrm{Ad}^k(R_\theta)$ is also bounded on $L^2(\Omega)$ for $k \ge 0$. Before going to prove (5.2), we prepare the following estimates for $\mathrm{Ad}^k(R_\theta)$: For any non-negative integer k, there exists a constant $C_k > 0$ such that

$$\|\mathrm{Ad}^k(R_\theta)\|_{\mathscr{B}(L^2(\Omega))} \le C_k\theta^{k/2} \tag{5.7}$$

for any $\theta > 0$. We can prove (5.7) by induction. For $k = 0, 1$, we have, by using (5.5) and Lemma 5.2,

$$\|\mathrm{Ad}^0(R_\theta)\|_{\mathscr{B}(L^2(\Omega))} = \|R_\theta\|_{\mathscr{B}(L^2(\Omega))} \le C_0,$$

$$\begin{aligned} \|\mathrm{Ad}^1(R_\theta)\|_{\mathscr{B}(L^2(\Omega))} &= 2\theta\|R_\theta\partial_{x_i}R_\theta\|_{\mathscr{B}(L^2(\Omega))} \\ &\le 2\theta M^{-1}\cdot M^{-1/2}(1-a_d)^{-1/2}\theta^{-1/2} \\ &= C_1\theta^{1/2}. \end{aligned}$$

Let us suppose that (5.7) is true for $k = 0, 1, \dots, \ell$. Combining (5.6) and Lemma 5.2, we get (5.7) for $k = \ell + 1$:

$$\begin{aligned} &\left\|\mathrm{Ad}^{\ell+1}(R_\theta)\right\|_{\mathscr{B}(L^2(\Omega))} \\ =&\left\|\theta\left\{-2(\ell+1)\mathrm{Ad}^\ell(R_\theta)\partial_{x_i}R_\theta + \ell(\ell+1)\mathrm{Ad}^{\ell-1}(R_\theta)R_\theta\right\}\right\|_{\mathscr{B}(L^2(\Omega))} \\ \le&2\ell(\ell+1)\theta\left\{\|\mathrm{Ad}^\ell(R_\theta)\|_{\mathscr{B}(L^2(\Omega))}\|\partial_{x_i}R_\theta\|_{\mathscr{B}(L^2(\Omega))} + \|\mathrm{Ad}^{\ell-1}(R_\theta)\|_{\mathscr{B}(L^2(\Omega))}\|R_\theta\|_{\mathscr{B}(L^2(\Omega))}\right\} \\ \le&C_{\ell+1}\theta\left\{\theta^{\ell/2}\cdot\theta^{-1/2} + \theta^{(\ell-1)/2}\right\} \\ \le&C_{\ell+1}\theta^{(\ell+1)/2}. \end{aligned}$$

Thus (5.7) is true for $k \ge 0$.

We prove (5.2) also by induction. For $k = 1$, by using the estimate (5.7) and the formula (see Lemma B.3 in appendix B)

$$\mathrm{Ad}^1(e^{-itR_\theta}) = -i\int_0^t e^{-isR_\theta}\mathrm{Ad}^1(R_\theta)e^{-i(t-s)R_\theta}\,ds, \tag{5.8}$$

we have

$$\begin{aligned}\|\mathrm{Ad}^1(e^{-itR_\theta})\|_{\mathscr{B}(L^2(\Omega))} \le & \int_0^t \|e^{-isR_\theta}\|_{\mathscr{B}(L^2(\Omega))}\|\mathrm{Ad}^1(R_\theta)\|_{\mathscr{B}(L^2(\Omega))}\|e^{-i(t-s)R_\theta}\|_{\mathscr{B}(L^2(\Omega))}\,ds \\ \le & C_1\int_0^t \theta^{1/2}\,ds \\ \le & C_1\theta^{1/2}\langle t\rangle.\end{aligned}$$

Let us suppose that (5.2) holds for $k = 1, \ldots, \ell$. Then, by using the estimate (5.7) and the formula (see Lemma B.3)

$$\begin{aligned}&\mathrm{Ad}^{\ell+1}(e^{-itR_\theta}) \\ =& -i\int_0^t \sum_{\ell_1+\ell_2+\ell_3=\ell}\Gamma(\ell_1,\ell_2,\ell_3)\mathrm{Ad}^{\ell_1}(e^{-isR_\theta})\mathrm{Ad}^{\ell_2+1}(R_\theta)\mathrm{Ad}^{\ell_3}(e^{-i(t-s)R_\theta})\,ds,\end{aligned}$$

where constants $\Gamma(\ell_1,\ell_2,\ell_3)$ are trinomial coefficients:

$$\Gamma(\ell_1,\ell_2,\ell_3) = \frac{\ell!}{\ell_1!\ell_2!\ell_3!},$$

we can estimate

$$\begin{aligned}&\|\mathrm{Ad}^{\ell+1}(e^{-itR_\theta})\|_{\mathscr{B}(L^2(\Omega))} \\ \le & C_{\ell+1}\int_0^t \sum_{\ell_1+\ell_2+\ell_3=\ell}\|\mathrm{Ad}^{\ell_1}(e^{-isR_\theta})\|_{\mathscr{B}(L^2(\Omega))}\|\mathrm{Ad}^{\ell_2+1}(R_\theta)\|_{\mathscr{B}(L^2(\Omega))}\times \\ & \qquad\qquad \times\|\mathrm{Ad}^{\ell_3}(e^{-i(t-s)R_\theta})\|_{\mathscr{B}(L^2(\Omega))}\,ds \\ \le & C_{\ell+1}\int_0^t \sum_{\ell_1+\ell_2+\ell_3=\ell}\theta^{\ell_1/2}\langle s\rangle^{\ell_1}\cdot\theta^{(\ell_2+1)/2}\cdot\theta^{\ell_3/2}\langle t-s\rangle^{\ell_3}\,ds \\ \le & C_{\ell+1}\theta^{(\ell+1)/2}\langle t\rangle^{\ell+1}.\end{aligned}$$

Thus we conclude (5.2). The proof of Proposition 5.1 is complete. □

6. Proof of Theorem 1.1.

In this section we shall prove Theorem 1.1. To begin with, let us introduce a family of operators which is useful to prove the theorem. For any non-negative integer N, we define a family $\mathscr{A}_N$ of operators as follows: We say that $A \in \mathscr{A}_N$ if and only if $A \in \mathscr{B}(L^2(\Omega))$ and

$$|||A|||_N := \sup_{n\in\mathbb{Z}^d}\left\||\cdot - \theta^{1/2}n|^N A\chi_{C_\theta(n)}\right\|_{\mathscr{B}(L^2(\Omega))} < \infty,$$

where $\chi_{C_\theta(n)}$ is the characteristic function of the set $C_\theta(n)$.

First, we prepare two lemmas.

Lemma 6.1. *For any positive integer N, there exists a constant $C = C(d,N) > 0$ such that*

$$\begin{aligned}&\sum_{m\in\mathbb{Z}^d} \|\chi_{C_\theta(m)} A \chi_{C_\theta(n)} f\|_{L^2(\Omega)}\\ \le& C\left(\|A\|_{\mathscr{B}(L^2(\Omega))} + \theta^{-d/4}\|\!|A|\!\|_N^{d/2N}\|A\|_{\mathscr{B}(L^2(\Omega))}^{1-d/2N}\right)\|\chi_{C_\theta(n)} f\|_{L^2(\Omega)}\end{aligned}$$

for all $n \in \mathbb{Z}^d$, $A \in \mathscr{A}_N$ and $f \in L^2(\Omega)$.

Proof. For the convenience of notation, we set

$$a_{mn} = \left\|\chi_{C_\theta(m)} A \chi_{C_\theta(n)} f\right\|_{L^2(\Omega)}.$$

Let $n \in \mathbb{Z}^d$ be fixed. For $\omega > 0$ and $N \in \mathbb{N}$ with $N > d/2$, we have, by Schwarz inequality,

$$\begin{aligned}\sum_{m\in\mathbb{Z}^d} a_{mn} =& \sum_{|m-n|>\omega} |\theta^{1/2}m - \theta^{1/2}n|^{-N}|\theta^{1/2}m - \theta^{1/2}n|^{N} a_{mn} + \sum_{|m-n|\le\omega} a_{mn}\\ \le& \theta^{-N/2}\Big(\sum_{|m-n|>\omega} |m-n|^{-2N}\Big)^{1/2}\Big(\sum_{|m-n|>\omega} |\theta^{1/2}m - \theta^{1/2}n|^{2N} a_{mn}^2\Big)^{1/2}\\ &+ \Big(\sum_{|m-n|\le\omega} 1\Big)^{1/2}\Big(\sum_{|m-n|\le\omega} a_{mn}^2\Big)^{1/2}\\ =:& I(n) + II(n).\end{aligned}$$

As to the first factor of $I(n)$, since $N > d/2$, we can estimate

$$\sum_{|m-n|>\omega} |m-n|^{-2N} = \sum_{|m|>\omega} |m|^{-2N} \le C_N \omega^{-2N+d}.$$

As to the second factor of $I(n)$, there exists a constant $C > 0$, independent of m, n and θ, such that

$$\begin{aligned}\sum_{|m-n|>\omega} |\theta^{1/2}m - \theta^{1/2}n|^{2N} a_{mn}^2 \le& \sum_{|m-n|>\omega} |\theta^{1/2}m - \theta^{1/2}n|^{2N}\left\|\chi_{C_\theta(m)} A \chi_{C_\theta(n)} f\right\|_{L^2(\Omega)}^2\\ \le& \sum_{|m-n|>\omega} |\theta^{1/2}m - \theta^{1/2}n|^{2N}\int_{C_\theta(m)} |A\chi_{C_\theta(n)} f|^2\, dx\\ \le& \sum_{|m-n|>\omega} \int_{C_\theta(m)} ||x - \theta^{1/2}n|^N A\chi_{C_\theta(n)} f|^2\, dx\\ \le& C\int_\Omega ||x - \theta^{1/2}n|^N A\chi_{C_\theta(n)} f|^2\, dx\\ \le& C\left\||\cdot - \theta^{1/2}n|^N A\chi_{C_\theta(n)} f\right\|_{L^2(\Omega)}^2\\ \le& C\|\!|A|\!\|_N^2\left\|\chi_{C_\theta(n)} f\right\|_{L^2(\Omega)}^2;\end{aligned}$$

thus we find from the estimates obtained now that

$$I(n) \le C\theta^{-N/2}\omega^{-(N-d/2)}\|\!|A|\!\|_N\left\|\chi_{C_\theta(n)} f\right\|_{L^2(\Omega)}. \tag{6.1}$$

We now turn to estimate $II(n)$. Since

$$\sum_{|m-n|\le\omega} 1 \le 1+\omega^d,$$

we can estimate, by the same argument as in $I(n)$,

$$\begin{aligned} II(n) &\le (1+\omega^{d/2})\Big(\sum_{|m-n|\le\omega}\big\|\chi_{C_\theta(m)}A\chi_{C_\theta(n)}f\big\|^2_{L^2(\Omega)}\Big)^{1/2} \\ &\le (1+\omega^{d/2})\big\|A\chi_{C_\theta(n)}f\big\|_{L^2(\Omega)} \\ &\le (1+\omega^{d/2})\|A\|_{\mathscr{B}(L^2(\Omega))}\big\|\chi_{C_\theta(n)}f\big\|_{L^2(\Omega)}. \end{aligned} \tag{6.2}$$

Combining the estimates (6.1)–(6.2), we get

$$\sum_{m\in\mathbb{Z}^d} a_{mn} \le C\Big\{\theta^{-N/2}\omega^{-(N-d/2)}|||A|||_N + (1+\omega^{d/2})\|A\|_{\mathscr{B}(L^2(\Omega))}\Big\}\big\|\chi_{C_\theta(n)}f\big\|_{L^2(\Omega)}.$$

Finally, taking $\omega = (|||A|||_N/\|A\|_{\mathscr{B}(L^2(\Omega))})^{1/N}\cdot\theta^{-1/2}$, we obtain

$$\begin{aligned} &\sum_{m\in\mathbb{Z}^d}\|\chi_{C_\theta(m)}A\chi_{C_\theta(n)}f\|_{L^2(\Omega)} \\ \le& C\Big(\|A\|_{\mathscr{B}(L^2(\Omega))} + \theta^{-d/4}|||A|||_N^{d/2N}\|A\|^{1-d/2N}_{\mathscr{B}(L^2(\Omega))}\Big)\|\chi_{C_\theta(n)}f\|_{L^2(\Omega)}, \end{aligned}$$

as desired. The proof of Lemma 6.1 is complete. □

Lemma 6.2. *Let N be a positive integer, and let $\psi\in\mathscr{S}(\mathbb{R})$. Then $\psi(R_\theta)\in\mathscr{A}_N$. Furthermore, there exists a constant $C_\psi>0$ such that*

$$\|\psi(R_\theta)\|_{\mathscr{B}(L^2(\Omega))}\le C_\psi,\quad \forall\theta>0, \tag{6.3}$$

and there exists a constant $C_N>0$ such that

$$|||\psi(R_\theta)|||_N \le C_N\theta^{N/2}\int_{-\infty}^{\infty}\langle t\rangle^N|\hat{\psi}(t)|\,dt,\quad \forall\theta>0. \tag{6.4}$$

Proof. The proof is based on the well-known formula;

$$\psi(R_\theta) = (2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{-itR_\theta}\hat{\psi}(t)\,dt, \tag{6.5}$$

where $\hat{\psi}$ is the Fourier transform of ψ on $\mathbb{R}$. The estimate (6.3) is an immediate consequence of the unitarity of e^{-itR_θ}, the formula (6.5) and $\psi\in\mathscr{S}(\mathbb{R})$.

As to the estimate (6.4), applying the formula (6.5), we obtain

$$\begin{aligned} &|||\psi(R_\theta)|||_N \\ =&\sup_{n\in\mathbb{Z}^d}\big\||\cdot-\theta^{1/2}n|^N\psi(R_\theta)\chi_{C_\theta(n)}\big\|_{\mathscr{B}(L^2(\Omega))} \\ \le&(2\pi)^{-1/2}\sup_{n\in\mathbb{Z}^d}\int_{-\infty}^{\infty}\big\||\cdot-\theta^{1/2}n|^N e^{-itR_\theta}\chi_{C_\theta(n)}\big\|_{\mathscr{B}(L^2(\Omega))}|\hat{\psi}(t)|\,dt. \end{aligned}$$

Resorting to Lemma B.1 with taking $A = B = x_i - \theta^{1/2} n_i$ and $L = e^{-itR_\theta}$, we can find from Proposition 5.1 that

$$\begin{aligned}
&\big\| |\cdot - \theta^{1/2} n|^N e^{-itR_\theta} \chi_{C_\theta(n)} \big\|_{\mathscr{B}(L^2(\Omega))} \\
\le & \sum_{k=0}^{N} C(N,k) \big\| \mathrm{Ad}^k (e^{-itR_\theta}) \big\|_{\mathscr{B}(L^2(\Omega))} \big\| |\cdot - \theta^{1/2} n|^{N-k} \chi_{C_\theta(n)} \big\|_{\mathscr{B}(L^2(\Omega))} \\
\le & \sum_{k=0}^{N} C(N,k) \theta^{k/2} \langle t \rangle^k \theta^{(N-k)/2};
\end{aligned}$$

thus we conclude that

$$|||\psi(R_\theta)|||_N \le \theta^{N/2} \sum_{k=0}^{N} C(N,k) \int_{-\infty}^{\infty} \langle t \rangle^k |\hat{\psi}(t)| \, dt,$$

which proves (6.4). The proof of Lemma 6.2 is finished. □

We are now in a position to prove the main theorem.

Proof of Theorem 1.1. It suffices to show L^1–boundedness of $\varphi(\theta H_V)$. Let $\beta > d/4$ and $M > 0$. Let us define $\psi \in \mathscr{S}(\mathbb{R})$ as

$$\psi(\mu) := \mu^{-\beta} \varphi(\mu^{-1} - M), \quad \mu \in (0, 1/M].$$

Then we can write

$$\psi\big((\lambda + M)^{-1}\big) = \varphi(\lambda)(\lambda + M)^{\beta}, \quad \lambda \ge 0. \tag{6.6}$$

Now we can estimate, by Hölder's inequality and the definition of amalgam spaces $\ell^p(L^q)_\theta$,

$$\begin{aligned}
\|\varphi(\theta H_V) f\|_{L^1(\Omega)} =& \sum_{n \in \mathbb{Z}^d} \|\varphi(\theta H_V) f\|_{L^1(C_\theta(n))} \\
\le & \sum_{n \in \mathbb{Z}^d} |C_\theta(n)|^{1/2} \|\varphi(\theta H_V) f\|_{L^2(C_\theta(n))} \\
\le & \theta^{d/4} \|\varphi(\theta H_V) f\|_{l^1(L^2)_\theta},
\end{aligned}$$

where we used $|C_\theta(n)|^{1/2} = \theta^{d/4}$. The right member in the above inequality can be estimated as

$$\begin{aligned}
\|\varphi(\theta H_V) f\|_{l^1(L^2)_\theta} =& \|\varphi(\theta H_V)(\theta H_V + M)^{\beta} R_\theta^{\beta} f\|_{l^1(L^2)_\theta} \\
=& \|\psi(R_\theta) R_\theta^{\beta} f\|_{l^1(L^2)_\theta} \\
\le & \sum_{n \in \mathbb{Z}^d} \sum_{m \in \mathbb{Z}^d} \big\| \chi_{C_\theta(m)} \psi(R_\theta) \chi_{C_\theta(n)} R_\theta^{\beta} f \big\|_{L^2(\Omega)},
\end{aligned}$$

where we used (6.6) in the second step. Resorting to Lemma 6.1 for A and f replaced by $\psi(R_\theta)$ and $R_\theta^{\beta} f$, respectively, we can estimate

$$\begin{aligned}
& \sum_{m \in \mathbb{Z}^d} \|\chi_{C_\theta(m)} \psi(R_\theta) \chi_{C_\theta(n)} R_\theta^{\beta} f\|_{L^2(\Omega)} \\
\le & C\Big(\|\psi(R_\theta)\|_{\mathscr{B}(L^2(\Omega))} + \theta^{-d/4} |||\psi(R_\theta)|||_N^{d/2N} \|\psi(R_\theta)\|_{\mathscr{B}(L^2(\Omega))}^{1-d/2N} \Big) \|\chi_{C_\theta(n)} R_\theta^{\beta} f\|_{L^2(\Omega)}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}&\|\varphi(\theta H_V)f\|_{L^1(\Omega)}\\ \le & C\theta^{d/4}\Big(\|\psi(R_\theta)\|_{\mathscr{B}(L^2(\Omega))}+\theta^{-d/4}|\!|\!|\psi(R_\theta)|\!|\!|_N^{d/2N}\|\psi(R_\theta)\|_{\mathscr{B}(L^2(\Omega))}^{1-d/2N}\Big)\|R_\theta^\beta f\|_{l^1(L^2)_\theta}.\end{aligned}$$

Applying Theorem 4.1 and Lemma 6.2 to the above estimate, we conclude that

$$\begin{aligned}\|\varphi(\theta H_V)f\|_{L^1(\Omega)} &\le C\theta^{d/4}\left\{1+\theta^{-d/4}\cdot(\theta^{N/2})^{d/2N}\right\}\theta^{-d/4}\|f\|_{L^1(\Omega)}\\ &\le C\|f\|_{L^1(\Omega)},\end{aligned}$$

where the constant C is independent of θ. The proof of Theorem 1.1 is complete. □

7. A final remark

As a consequence of Theorems 1.1 and 4.1, we have L^p–L^q–boundedness of $\varphi(\theta H_V)$. L^p–L^q–boundedness of $\varphi(\theta H_V)$ is useful to prove the embedding theorem for Besov spaces.

Proposition 7.1. *Let φ and V be as in Theorem 1.1. Then there exists a constant $C=C(d,\varphi)>0$ such that for $1\le p\le q\le\infty$,*

$$\|\varphi(\theta H_V)\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))}\le C\theta^{-d(1/p-1/q)/2},\quad \forall\theta>0.$$

Proof. Let us define $\tilde\varphi\in\mathscr{S}(\mathbb{R})$ as

$$\tilde\varphi(\lambda)=(\lambda+M)^\beta\varphi(\lambda),\quad \lambda\ge 0.$$

By Theorems 1.1 and 4.1, for $1\le p\le q\le\infty$ and $\beta>d(1/p-1/q)/2$, we can estimate

$$\begin{aligned}\|\varphi(\theta H_V)\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))}&=\|\varphi(\theta H_V)(\theta H_V+M)^\beta R_\theta^\beta\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))}\\ &\le\|\tilde\varphi(\theta H_V)\|_{\mathscr{B}(L^q(\Omega))}\|R_\theta^\beta\|_{\mathscr{B}(L^p(\Omega),L^q(\Omega))}\\ &\le C\theta^{-d(1/p-1/q)/2}.\end{aligned}$$

This proves Proposition 7.1. □

Appendix A. (The Young inequality)

In this appendix we introduce the Young inequality for scaled amalgam spaces. The proof of this result is given in Fournier and Stewart [3].

Lemma A.1. *Let $\Omega=\mathbb{R}^d$, $\ell^p(L^q)_\theta=\ell^p(L^q)_\theta(\mathbb{R}^d)$, and let $1\le p,p_1,p_2,q,q_1,q_2\le\infty$ with $1/p_1+1/p_2-1=1/p$ and $1/q_1+1/q_2-1=1/q$. If $f\in l^{p_1}(L^{q_1})_\theta$ and $g\in l^{p_2}(L^{q_2})_\theta$, then $f*g\in l^p(L^q)_\theta$ and*

$$\|f*g\|_{\ell^p(L^q)_\theta}\le 3^d\|f\|_{l^{p_1}(L^{q_1})_\theta}\|g\|_{l^{p_2}(L^{q_2})_\theta},\tag{A.1}$$

where $f*g$ is the convolution of f and g.

Appendix B. (Recursive formula of operators)

In this appendix we introduce some formulas on the operator Ad used in §5–§6.

Lemma B.1 (Lemma 3.1 from [7])**.** *Let X and Y be topological vector spaces, and let A and B be continuous linear operators from X and Y into themselves, respectively. If L is a continuous linear operator from X into Y, then there exists a set of constants $\left\{C(n,m)\middle|\, n \geq 0,\, 0 \leq m \leq n\right\}$ such that*

$$B^n L = \sum_{m=0}^{n} C(n,m) \mathrm{Ad}^m(L) A^{n-m}. \tag{B.1}$$

We discuss about two kind of recursive formulas of operator

$$R_\theta = (\theta H_V + M)^{-1}.$$

Hereafter we put

$$X = \mathscr{D}(\Omega), \quad Y = \mathscr{D}'(\Omega),$$

and

$$A = B = x_i - \theta^{1/2} n_i \quad \text{for some } i \in \{1, \cdots, d\}.$$

Lemma B.2. *The sequence $\{\mathrm{Ad}^k(R_\theta)\}_{k=0}^\infty$ of operators satisfies the following recursive formula:*

$$\mathrm{Ad}^0(R_\theta) = R_\theta, \quad \mathrm{Ad}^1(R_\theta) = -2\theta R_\theta \partial_{x_i} R_\theta, \tag{B.2}$$

$$\mathrm{Ad}^k(R_\theta) = \theta \left\{ -2k \mathrm{Ad}^{k-1}(R_\theta) \partial_{x_i} R_\theta + k(k-1) \mathrm{Ad}^{k-2}(R_\theta) R_\theta \right\}, \quad k \geq 2. \tag{B.3}$$

Proof. When $k = 0$, the first equation in (B.2) is trivial. Hence it is sufficient to prove the lemma for $k > 0$. For the sake of simplicity, we perform a rough argument without considering the domain of operators. The rigorous argument will be given in the final part.

Let us introduce the generalized binomial coefficients $\Gamma(k,m)$ as follows:

$$\Gamma(k,m) = \begin{cases} \dfrac{k!}{(k-m)!m!}, & k \geq m \geq 0, \\ 0, & k < m \text{ or } k < 0. \end{cases}$$

Once the following recursive formula is established:

$$\mathrm{Ad}^k(R_\theta) = -\sum_{m=0}^{k-1} \Gamma(k,m) \mathrm{Ad}^m(R_\theta) \mathrm{Ad}^{k-m}(\theta H_V) R_\theta, \quad k = 1, 2, \cdots, \tag{B.4}$$

(B.2)–(B.3) are an immediate consequence of (B.4), since

$$\mathrm{Ad}^1(\theta H_V) = 2\theta \partial_{x_i}, \quad \mathrm{Ad}^2(\theta H_V) = -2\theta, \quad \mathrm{Ad}^k(\theta H_V) = 0, \quad k \geq 3.$$

Hence, all we have to do is to prove (B.4). We proceed the argument by induction. For $k = 1$, it can be readily checked that

$$\begin{aligned}\mathrm{Ad}^1(R_\theta) =& x_i R_\theta - R_\theta x_i \\ =& R_\theta(\theta H_V + M)x_i R_\theta - R_\theta x_i(\theta H_V + M)R_\theta \\ =& R_\theta\left(\theta H_V x_i - x_i \cdot \theta H_V\right) R_\theta \\ =& - R_\theta \mathrm{Ad}^1(\theta H_V) R_\theta \\ =& - \Gamma(1,0)\mathrm{Ad}^0(R_\theta)\mathrm{Ad}^1(\theta H_V)R_\theta.\end{aligned}$$

Hence (B.4) is true for $k = 1$. Let us suppose that (B.4) holds for $k = 1, \ldots, \ell$. Writing

$$\mathrm{Ad}^{\ell+1}(R_\theta) = x_i \mathrm{Ad}^\ell(R_\theta) - \mathrm{Ad}^\ell(R_\theta)x_i, \tag{B.5}$$

we see that the first term becomes

$$\begin{aligned}& x_i \mathrm{Ad}^\ell(R_\theta) \\ =& x_i \Big\{ - \sum_{m=0}^{\ell-1} \Gamma(\ell,m)\mathrm{Ad}^m(R_\theta)\mathrm{Ad}^{\ell-m}(\theta H_V)\Big\} R_\theta \\ =& - \sum_{m=0}^{\ell-1} \Gamma(\ell,m)\left\{\mathrm{Ad}^{m+1}(R_\theta)\mathrm{Ad}^{\ell-m}(\theta H_V) + \mathrm{Ad}^m(R_\theta)\mathrm{Ad}^{\ell-m+1}(\theta H_V)\right\} R_\theta \\ & - \sum_{m=0}^{\ell-1} \Gamma(\ell,m)\mathrm{Ad}^m(R_\theta)\mathrm{Ad}^{\ell-m}(\theta H_V)x_i R_\theta \\ =:& I_1 + I_2.\end{aligned}$$

Here I_1 can be written as

$$\begin{aligned}I_1 =& - \sum_{m=1}^{\ell} \Gamma(\ell,m-1)\mathrm{Ad}^m(R_\theta)\mathrm{Ad}^{\ell-m+1}(\theta H_V)R_\theta \\ & - \sum_{m=0}^{\ell-1} \Gamma(\ell,m)\mathrm{Ad}^m(R_\theta)\mathrm{Ad}^{\ell-m+1}(\theta H_V)R_\theta \\ =& - \sum_{m=0}^{\ell} \Gamma(\ell,m-1)\mathrm{Ad}^m(R_\theta)\mathrm{Ad}^{\ell+1-m}(\theta H_V)R_\theta \\ & - \sum_{m=0}^{\ell} \Gamma(\ell,m)\mathrm{Ad}^m(R_\theta)\mathrm{Ad}^{\ell-m+1}(\theta H_V) + \mathrm{Ad}^\ell(R_\theta)\mathrm{Ad}^1(\theta H_V)R_\theta \\ =& - \sum_{m=0}^{\ell} \Gamma(\ell+1,m)\mathrm{Ad}^m(R_\theta)\mathrm{Ad}^{\ell+1-m}(\theta H_V) + \mathrm{Ad}^\ell(R_\theta)\mathrm{Ad}^1(\theta H_V)R_\theta,\end{aligned}$$

where we used

$$\Gamma(\ell,m-1) + \Gamma(\ell,m) = \Gamma(\ell+1,m)$$

in the last step. As to I_2, we can write as

$$\begin{aligned}I_2 =& -\left\{\sum_{m=0}^{\ell-1}\Gamma(\ell,m)\mathrm{Ad}^m(R_\theta)\mathrm{Ad}^{\ell-m}(\theta H_V)R_\theta\right\}(\theta H_V+M)x_iR_\theta\\ =&\mathrm{Ad}^\ell(R_\theta)(\theta H_V+M)x_iR_\theta.\end{aligned}$$

Hence, summarizing the previous equations, we get

$$\begin{aligned}x_i\mathrm{Ad}^\ell(R_\theta) =& -\sum_{m=0}^{\ell}\Gamma(\ell+1,m)\mathrm{Ad}^m(R_\theta)\mathrm{Ad}^{\ell+1-m}(\theta H_V)\\ &+\mathrm{Ad}^\ell(R_\theta)\left\{\mathrm{Ad}^1(\theta H_V)+(\theta H_V+M)x_i\right\}R_\theta.\end{aligned}$$

Therefore, going back to (B.5), and noting

$$\mathrm{Ad}^1(\theta H_V)+(\theta H_V+M)x_i = x_i(\theta H_V+M),$$

we conclude that

$$\begin{aligned}\mathrm{Ad}^{\ell+1}(R_\theta) =& -\sum_{m=0}^{\ell}\Gamma(\ell+1,m)\mathrm{Ad}^m(R_\theta)\mathrm{Ad}^{\ell+1-m}(\theta H_V)\\ &+\mathrm{Ad}^\ell(R_\theta)\left\{\mathrm{Ad}^1(\theta H_V)+(\theta H_V+M)x_i\right\}R_\theta-\mathrm{Ad}^\ell(R_\theta)x_i\\ =& -\sum_{m=0}^{\ell}\Gamma(\ell+1,m)\mathrm{Ad}^m(R_\theta)\mathrm{Ad}^{\ell+1-m}(\theta H_V)\\ &+\mathrm{Ad}^\ell(R_\theta)x_i(\theta H_V+M)R_\theta-\mathrm{Ad}^\ell(R_\theta)x_i\\ =& -\sum_{m=0}^{\ell}\Gamma(\ell+1,m)\mathrm{Ad}^m(R_\theta)\mathrm{Ad}^{\ell+1-m}(\theta H_V).\end{aligned}$$

Hence (B.4) is true for $k=\ell+1$.

The above proof is formal in the sense that the domain of operators is never taken into account in the argument. In fact, even for $f\in C_0^\infty(\Omega)$, each $x_iR_\theta f$ does not necessarily belong to the domain of H_V, since we only know the fact $R_\theta f\in\mathcal{D}(H_V)=H_0^1(\Omega)\cap\{H_Vf\in L^2(\Omega)\}$ and $x_iR_\theta f$ may not be in $L^2(\Omega)$ for unbounded domains Ω at least. Therefore, we should perform the argument by using a quadratic form in a rigorous way. We may prove the lemma only for $k=1$. For, as to the case $k>1$, the argument can be done in a similar manner. Now we write

$$\begin{aligned}{}_{\mathscr{D}'}\langle\mathrm{Ad}^1(R_\theta)f,g\rangle_{\mathscr{D}} &= \langle R_\theta f, x_ig\rangle_{L^2(\Omega)}-\langle x_if,R_\theta g\rangle_{L^2(\Omega)}\\ &=: I-II\end{aligned}$$

for $f,g\in C_0^\infty(\Omega)$. Since $R_\theta f, R_\theta g\in H_0^1(\Omega)$, there exist two sequences $\{f_n\}_n$, $\{g_m\}_m$ in $C_0^\infty(\Omega)$ such that

$$f_n\to R_\theta f\quad\text{and}\quad g_m\to R_\theta g\quad\text{in } H^1(\Omega)\quad(n,m\to\infty).$$

Hence we obtain by $x_i f_n, x_i g_m \in C_0^\infty(\Omega)$,

$$\begin{aligned} I &= \lim_{n\to\infty} \langle f_n, x_i g\rangle_{L^2(\Omega)} \\ &= \lim_{n\to\infty} \langle x_i f_n, (H_V+M)R_\theta g\rangle_{L^2(\Omega)} \\ &= \lim_{n\to\infty} \left\{ \langle \nabla(x_i f_n), \nabla R_\theta g\rangle_{L^2(\Omega)} + \int_\Omega (V+M) x_i f_n \overline{R_\theta g}\, dx \right\} \\ &= \lim_{n,m\to\infty} \left\{ \langle \nabla(x_i f_n), \nabla g_m\rangle_{L^2(\Omega)} + \int_\Omega (V+M) x_i f_n \overline{g_m}\, dx \right\} \\ &= \lim_{n,m\to\infty} \left\{ \langle f_n, \partial_{x_i} g_m\rangle_{L^2(\Omega)} + \langle x_i \nabla f_n, \nabla g_m\rangle_{L^2(\Omega)} + \int_\Omega (V+M) x_i f_n \overline{g_m}\, dx \right\}, \end{aligned}$$

$$\begin{aligned} II &= \lim_{m\to\infty} \langle x_i f, g_m\rangle_{L^2(\Omega)} \\ &= \lim_{m\to\infty} \langle (H_V+M)R_\theta f, x_i g_m\rangle_{L^2(\Omega)} \\ &= \lim_{m\to\infty} \left\{ \langle \nabla R_\theta f, \nabla(x_i g_m)\rangle_{L^2(\Omega)} + \int_\Omega (V+M) x_i R_\theta f\, \overline{g_m}\, dx \right\} \\ &= \lim_{n,m\to\infty} \left\{ \langle \nabla f_n, \nabla(x_i g_m)\rangle_{L^2(\Omega)} + \int_\Omega (V+M) x_i f_n \overline{g_m}\, dx \right\} \\ &= \lim_{n,m\to\infty} \left\{ \langle \partial_{x_i} f_n, g_m\rangle_{L^2(\Omega)} + \langle x_i \nabla f_n, \nabla g_m\rangle_{L^2(\Omega)} + \int_\Omega (V+M) x_i f_n \overline{g_m}\, dx \right\}. \end{aligned}$$

Combining the above limits, we obtain

$$\begin{aligned} {}_{\mathscr{D}'}\langle \mathrm{Ad}^1(R_\theta) f, g\rangle_{\mathscr{D}} &= \lim_{n,m\to\infty} \left\{ \langle f_n, \partial_{x_i} g_m\rangle_{L^2(\Omega)} - \langle \partial_{x_i} f_n, g_m\rangle_{L^2(\Omega)} \right\} \\ &= \lim_{n,m\to\infty} \langle -2\partial_{x_i} f_n, g_m\rangle_{L^2(\Omega)} \\ &= \langle -2\partial_{x_i} R_\theta f, R_\theta g\rangle_{L^2(\Omega)} \end{aligned}$$

for any $f, g \in C_0^\infty(\Omega)$. Thus (B.2) is valid in a distributional sense. In a similar way, (B.3) can be also shown in a distributional sense. The proof of Lemma B.2 is finished. □

Lemma B.3. *Let A, B and L be as in Lemma B.2. The following formula holds for each $t > 0$:*

$$\mathrm{Ad}^1(e^{-itR_\theta}) = -i \int_0^t e^{-isR_\theta} \mathrm{Ad}^1(R_\theta) e^{-i(t-s)R_\theta}\, ds. \tag{B.6}$$

Furthermore, the following formulas hold for $k > 1$:

$$\begin{aligned} &\mathrm{Ad}^{k+1}(e^{-itR_\theta}) \\ &= -i \int_0^t \sum_{k_1+k_2+k_3=k} \Gamma(k_1,k_2,k_3) \mathrm{Ad}^{k_1}(e^{-isR_\theta}) \mathrm{Ad}^{k_2+1}(R_\theta) \mathrm{Ad}^{k_3}(e^{-i(t-s)R_\theta})\, ds, \end{aligned} \tag{B.7}$$

where the constants $\Gamma(k_1,k_2,k_3)$ are trinomial coefficients:

$$\Gamma(k_1,k_2,k_3) = \frac{k!}{k_1!k_2!k_3!}$$

with $k = k_1 + k_2 + k_3$ *and* $k_1, k_2, k_3 \geq 0$.

Proof. It is sufficient to prove the lemma without taking account of the domain of operators as in the proof of Lemma B.2. We can write

$$\begin{aligned}
\mathrm{Ad}^1(e^{-itR_\theta}) &= x_i e^{-itR_\theta} - e^{-itR_\theta} x_i \\
&= -\int_0^t \frac{d}{ds}\big(e^{-isR_\theta} x_i e^{-i(t-s)R_\theta}\big)\, ds \\
&= -i\int_0^t e^{-isR_\theta}(x_i R_\theta - R_\theta x_i) e^{-i(t-s)R_\theta}\, ds \\
&= -i\int_0^t e^{-isR_\theta} \mathrm{Ad}^1(R_\theta) e^{-i(t-s)R_\theta}\, ds.
\end{aligned}$$

This proves (B.6). The proof of (B.7) is similar to that of Lemma B.2. So we may omit the details. The proof of Lemma B.3 is complete. □

References

[1] T. Cazenave and A. Haraux, An Introduction to Semilinear Evolution Equations, Oxford Lecture Series in Mathematics and its Applications, Vol. 13, The Clarendon Press Oxford University Press, New York, 1998. Translated from the 1990 French original by Yvan Martel and revised by the authors.

[2] P. D'Ancona and V. Pierfelice, *On the wave equation with a large rough potential*, J. Functional Analysis **227** (2005), 30–77.

[3] J.J.F. Fournier and J. Stewart, *Amalgam of L^p and ℓ^q*, Bull. Amer. Math. Soc. **13** (1985), 1–21.

[4] V. Georgiev and N. Visciglia, *Decay estimates for the wave equation with potential*, Comm. Partial Differential Equations **28** (2003), 1325–1369.

[5] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order., Berlin: Classics in Mathematics; Springer-Verlag, 2001.

[6] A. Jensen and S. Nakamura, *Mapping properties of functions of Schrödinger operators between L^p–spaces and Besov spaces*, Spectral and scattering theory and applications, Vol. 23, Tokyo: Adv. Stud. Pure Math., Mathematical Society of Japan, 435–460, 1994.

[7] A. Jensen and S. Nakamura, *L^p–mapping properties of functions of Schrödinger operators and their applications to scattering theory*, J. Math. Soc. Japan **47** (1995), 253–273.

[8] M. Reed and B. Simon, Methods of modern mathematical physics, I, Functional Analysis, Academic Press, New York, 1975.

[9] B. Simon, *Schrödinger semigroup*, Bulletin of the American Mathematical Society **7** (1982), 447–526, .

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