The Terwilliger algebra of a \( Q \)-polynomial distance-regular graph with respect to a set of vertices

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Notation

- \( \Gamma = (X, R) \): a finite connected simple graph
  - \( X \): the vertex set
  - \( R \): the edge set (= a set of 2-element subsets of \( X \))

- \( \partial \): the path-length distance on \( X \)

\[
\partial(x,y) = i
\]

- \( D := \max \{ \partial(x, y) : x, y \in X \} \): the diameter of \( \Gamma \)

- \( \Gamma_i(x) := \{ y \in X : \partial(x, y) = i \} \): the \( i \)th subconstituent
Distance-regular graphs

- \( \Gamma : \text{distance-regular} \)

\[
\iff \exists a_i, b_i, c_i \ (0 \leq i \leq D) \ \text{s.t.} \ \forall x, y \in X : \\
|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = c_i \\
|\Gamma_i(x) \cap \Gamma_1(y)| = a_i \\
|\Gamma_{i+1}(x) \cap \Gamma_1(y)| = b_i
\]

where \( \partial(x, y) = i \).
The adjacency algebra

- **Mat\(_X(\mathbb{C})\)**: the set of square matrices over \(\mathbb{C}\) index by \(X\)
- The **i\(^{th}\) distance matrix** \(A_i \in \text{Mat}\(_X(\mathbb{C})\) is**

\[
(A_i)_{x,y} = \begin{cases} 
1 & \text{if } \partial(x, y) = i \\
0 & \text{otherwise}
\end{cases}
\]

[Note: \(A_0 = I\)]

- \(A_0, A_1, \ldots, A_D\) satisfy the **three-term recurrence**

\[
A_1A_i = b_{i-1}A_{i-1} + a_iA_i + c_{i+1}A_{i+1} \quad (0 \leq i \leq D)
\]

where \(A_{-1} = A_{D+1} = 0\).
Recall the three-term recurrence

\[ A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (0 \leq i \leq D) \]

where \( A_{-1} = A_{D+1} = 0. \)

\( M := \mathbb{C}[A_1] \subseteq \text{Mat}_X(\mathbb{C}) : \text{the adjacency algebra of } \Gamma \)

\( \exists v_i \in \mathbb{Q}[t] \text{ s.t. } \deg v_i = i \text{ and } A_i = v_i (A_1) \quad (0 \leq i \leq D) \)

\( M = \langle A_0, A_1, \ldots, A_D \rangle \)

\( A_1 \text{ has } D + 1 \text{ distinct eigenvalues } \theta_0, \theta_1, \ldots, \theta_D \in \mathbb{R}. \)
The $Q$-polynomial property

- Recall
  - $\theta_0, \theta_1, \ldots, \theta_D \in \mathbb{R}$: the distinct eigenvalues of $A_1$
  - $\Gamma$: regular with valency $k_1 := |\Gamma_1(x)| (= b_0)$

Always set $\theta_0 = k_1$.

- $E_\ell \in \text{Mat}_X(\mathbb{C})$: the orthogonal projection onto the eigenspace of $\theta_\ell$  
  [Note: $E_0 = \frac{1}{|X|} J$ ($J$: the all-ones matrix)]

- $M = \mathbb{C}[A_1] = \langle A_0, A_1, \ldots, A_D \rangle = \langle E_0, E_1, \ldots, E_D \rangle$

- $E_0, E_1, \ldots, E_D$: the primitive idempotents of $M$
The $Q$-polynomial property

- Recall the three-term recurrence

$$A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (0 \leq i \leq D).$$

- $\Gamma : Q$-polynomial w.r.t. $\{E_\ell\}_{\ell=0}^D$

$$\text{def} \quad \exists a_\ell^*, b_\ell^*, c_\ell^* \quad (0 \leq \ell \leq D) \quad \text{s.t.} \quad b_{\ell-1}^* c_\ell^* \neq 0 \quad (1 \leq \ell \leq D) \quad \text{and}$$

$$|X| E_1 \circ E_\ell = b_{\ell-1}^* E_{\ell-1} + a_\ell^* E_\ell + c_{\ell+1}^* E_{\ell+1} \quad (0 \leq \ell \leq D)$$

where $E_{-1} = E_{D+1} = 0$ and $\circ$ is the entrywise product.
We shall assume $\Gamma$ is a $Q$-polynomial DRG.

- $Y \subseteq X$: a nonempty subset of $X$
- $\chi \in \mathbb{C}^X$: the characteristic vector of $Y$
- $w = \max\{i : \chi^T A_i \chi \neq 0\}$: the width of $Y$
- $w^* = \max\{\ell : \chi^T E_\ell \chi \neq 0\}$: the dual width of $Y$
We have

\[ w + w^* \geq D. \]

If \( w + w^* = D \) then \( Y \) is completely regular, and the induced subgraph \( \Gamma_Y \) on \( Y \) is a \( Q \)-polynomial DRG with diameter \( w \) provided it is connected.

\( Y \) : a descendent of \( \Gamma \) \( \overset{\text{def}}{\leftrightarrow} \) \( w + w^* = D \)

Some descendents

- $w = 0 : Y = \{x\} \ (x \in X)$
- $w = D : Y = X$
- $w = 1 : \text{Delsarte cliques} \ (\theta_D = \theta_{\text{min}}) \ i.e., \ |Y| = 1 - \frac{k_1}{\theta_D}$
A chain of descendents

$Q_1$  $Q_2$  $Q_3$

$Q_4$
Theorem (T., 2011)

Let \( Y \) be a descendent of \( \Gamma \) and suppose \( \Gamma_Y \) is connected. Then a nonempty subset of \( Y \) is a descendent of \( \Gamma_Y \) if and only if it is a descendent of \( \Gamma \).

- \( \mathcal{L} \) : the set of isomorphism classes of \( Q \)-polynomial DRGs
- \( [\Delta] \preceq [\Gamma] \overset{\text{def}}{\iff} \exists Y : \text{a descendent of } \Gamma \text{ s.t. } [\Delta] = [\Gamma_Y] \)
- \( (\mathcal{L}, \preceq) : \text{a poset} \)
The classification of descendents is complete for the 15 known infinite families of DRGs with unbounded diameter and with classical parameters (BGKM, 2003; T., 2006, 2011).

The ideal $\mathcal{I}_{[\Gamma]} = \{[\Delta] \in \mathcal{L} : [\Delta] \preceq [\Gamma]\}$ is known if $\Gamma$ belongs to one of the above families.
The structure of $(\mathcal{L}, \preceq)$

### Problem
- Determine the filter $\mathcal{F}_{[\Gamma]} = \{ [\Delta] \in \mathcal{L} : [\Gamma] \preceq [\Delta] \}$

- This has been solved at the parameteric level.
- The generic case is described in terms of 5 scalars (besides $D$) $q, r_1, r_2, s, s^*$ where $r_1 r_2 = ss^* q^{D+1}$ (Leonard, 1982).

### Theorem (T., 2009, 2011)
- Suppose $[\Gamma] \preceq [\Delta]$ and $\Delta$ has diameter $C \geq D$. If $D \geq 3$ then the scalars corresponding to $\Delta$ are

$$q, r_1, r_2, sq^{D-C}, s^*.$$
When $\Gamma_Y$ is connected

**Theorem (Brouwer–Godsil–Koolen–Martin, 2003)**

- We have
  \[ w + w^* \geq D. \]
- If \( w + w^* = D \) then \( Y \) is completely regular, and the induced subgraph \( \Gamma_Y \) on \( Y \) is a $Q$-polynomial DRG with diameter \( w \) provided it is connected.

**Theorem (T., 2011)**

- Let \( Y \) be a descendent of \( \Gamma \). Then \( \Gamma_Y \) is connected if and only if \( q \neq -1 \), or \( q = -1 \) and \( w^* \) is even.
$Y \subseteq X :$ a nonempty subset of $X$

$Y_i = \{z \in X : \partial(z, Y) = i\}$

$\tau = \max\{i : Y_i \neq \emptyset\} :$ the covering radius of $Y$

The distance partition of $X$
The Terwilliger algebra

- $\chi_i \in \mathbb{C}^X$: the characteristic vector of $Y_i$ ($0 \leq i \leq \tau$)
- $E_i^* = \text{Diag}(\chi_i) \in \text{Mat}_X(\mathbb{C})$ ($0 \leq i \leq \tau$),

\[
(E_i^*)_{zz} = \begin{cases} 
1 & \text{if } z \in Y_i, \\
0 & \text{otherwise},
\end{cases} \quad (z \in X).
\]

- $T = T(Y) = \mathbb{C}[A_1, E_0^*, \ldots, E_\tau^*]$: the Terwilliger algebra with respect to $Y$ (Martin–Taylor, 1997; Suzuki, 2005)

- $Y = \{x\} \implies T = T(x)$: the Terwilliger algebra with respect to $x$ (Terwilliger, 1992)
The case when $Y$ is a descendent

- We shall assume $Y$ is a descendent of $\Gamma$.

- We have $\tau = |\{\ell \neq 0 : \chi^T E_{\ell} \chi \neq 0\}| = w^*$.

- $T = \mathbb{C}[A_1, E_0^*, \ldots, E_w^*]$
The dual adjacency matrix

\[ E_i^* A_1^* E_j^* = 0 \text{ if } |i - j| > 1 \]

\[ A_1^* = \frac{|X|}{|C|} \text{Diag}(E_1 \chi) \in \text{Mat}_X(\mathbb{C}) : \text{the dual adjacency matrix} \]

\[ Y : \text{completely regular} \implies A_1^* \in M^* := \langle E_0^*, E_1^*, \ldots, E_w^* \rangle \]

“dual Bose–Mesner algebra”


\[ E_i A_1^* E_j = 0 \text{ if } |i - j| > 1 \]
Tridiagonal pairs

- \( W \): a finite-dimensional complex vector space
- \( \alpha, \alpha^* \in \text{End}(W) \): diagonalizable
- \((\alpha, \alpha^*)\): a tridiagonal pair (Ito–Tanabe–Terwilliger, 2001)

\[ \text{def} \quad \exists W_0, W_1, \ldots, W_d : \text{an ordering of the eigenspaces of } A \text{ s.t.} \]
\[ \alpha^* W_i \subset W_{i-1} + W_i + W_{i+1} \quad (0 \leq i \leq d); \]

\[ \exists W_0^*, W_1^*, \ldots, W_{d^*}^* : \text{an ordering of the eigenspaces of } A^* \text{ s.t.} \]
\[ \alpha W_i^* \subset W_{i-1}^* + W_i^* + W_{i-1}^* \quad (0 \leq i \leq d^*); \]
- \( W \): irreducible as a \( \mathbb{C}[\alpha, \alpha^*] \)-module.

Proposition (Ito–Tanabe–Terwilliger, 2001)

- \( d = d^* \).
Do irreducible $T$-modules afford tridiagonal pairs?

- $E_i^* A_1 E_j^* = 0$ if $|i - j| > 1$
- $E_i A_1^* E_j = 0$ if $|i - j| > 1$

$W$: an irreducible $T$-module

- $A_1 E_i^* W \subset E_{i-1}^* W + E_i^* W + E_{i+1}^* W$
- $A_1^* E_i W \subset E_{i-1} W + E_i W + E_{i+1} W$

If $M^* = \mathbb{C}[A_1^*]$ then $W$ is irreducible as a $\mathbb{C}[A_1, A_1^*]$-module.

Theorem

*Every* irreducible $T$-module affords a tridiagonal pair if and only if $q \neq -1$, or $q = -1$ and $\omega$ is even.
Some general results

- We shall assume $q \neq -1$.

- $W$ : an irreducible $T$-module

- $\rho = \min\{i : E_i^* W \neq 0\}$ : the endpoint of $W$

- $\rho^* = \min\{\ell : E_\ell W \neq 0\}$ : the dual endpoint of $W$

- $d = |\{i : E_i^* W \neq 0\}| = |\{\ell : E_\ell W \neq 0\}|$ : the diameter of $W$

- $\{i : E_i^* W \neq 0\} = \{\rho, \ldots, \rho + d\} \subset \{0, 1, \ldots, w^*\}$

- $\{\ell : E_\ell W \neq 0\} = \{\rho^*, \ldots, \rho^* + d\} \subset \{0, 1, \ldots, D\}$

Proposition (cf. Caughman, 1999)

- $2\rho + d \geq w^*$

- $2\rho^* + d \geq w^*$
Some general results

- $\rho + d \leq w^*$
- $\rho^* + d \leq D$
- $2\rho + d \geq w^*$
- $2\rho^* + d \geq w^*$

\[ \eta := \rho + \rho^* + d - w^* : \text{the displacement of } W \]
\[ \wedge \wedge \]
\[ w^* \quad D \]

- $0 \leq \eta \leq D$

We may generalize the displacement and split decompositions of $\mathbb{C}^X$ due to Terwilliger (2005).

In particular, it is likely that $U_q(\hat{sl}_2) \rightarrow \boxtimes_q \rightarrow \mathcal{T}$ when $\Gamma$ is a forms graph (cf. Ito–Terwilliger, 2009).

“$q$-tetrahedron algebra”
Some general results

- $W$: thin $\overset{\text{def}}{\iff} \dim E_i^*W \leq 1 \ (0 \leq i \leq D)\iff$ the associated tridiagonal pair is a Leonard pair

**Theorem (Hosoya–Suzuki, 2007)**

- There are precisely $w + 1$ inequivalent irreducible $T$-modules in $\mathbb{C}^X$ with $\rho = 0$.
- Each of such modules is thin and is generated by an eigenvector of $\Gamma_Y$ in $\mathbb{C}^Y = E_0^*\mathbb{C}^X$.

A $Q$-polynomial DRG with diameter $w$
Hamming graphs

- \([q] = \{0, 1, \ldots, q - 1\} \ (q \geq 2)\)
- \(X = [q]^D\)
- \(y \sim z \iff |\{i : y_i \neq z_i\}| = 1\)
- \(\Gamma = H(D, q) : \text{the Hamming graph}\)

- The structure of \(T(x)\) has been well studied.
- \(H(D, 2) = Q_D \implies U(\mathfrak{sl}_2) \xrightarrow{\exists} T(x)\) (Go, 2002)
- \(H(D, q) \ (q \geq 3) \implies \text{The method for the Doob graphs (Tanabe, 1997) works as well.}\)
\( n \in \{0, 1, \ldots, D\} \)

\( Y = \{z \in X : z_1 = \cdots = z_n = 0\} \): a descendent with \( w = D - n \), \( w^* = n \)

\( z = (0, \ldots, 0 | *, \ldots, *) \)

\( \Gamma_Y \cong H(D - n, q) \)

**Theorem (Brouwer–Godsil–Koolen–Martin, 2003)**

*Every descendent of \( \Gamma = H(D, q) \) with \( w^* = n \) is isomorphic (under \( \text{Aut} \Gamma \)) to \( Y \) above.*
Hamming graphs

\[ z = (0, 0, 0, \ldots, 0|*, \ldots, *) \in Y = Y_0 \]
\[ z = (1, 0, 0, \ldots, 0|*, \ldots, *) \in Y_1 \]
\[ z = (1, 1, 0, \ldots, 0|*, \ldots, *) \in Y_2 \]

\[ Y_i = \Gamma'_i(0) \times [q]^{D-n} \quad (0 \leq i \leq n) \]

where \( \Gamma' = H(n, q) \) and \( 0 = (0, \ldots, 0) \)
Hamming graphs

- $\Gamma' = H(n, q), \quad \Gamma'' = H(D - n, q)$
- Use ' (resp. '') to denote objects associated with $\Gamma'$ (resp. $\Gamma''$).

- $Y_i = \Gamma'_i(0) \times [q]^{D-n} \quad (0 \leq i \leq n)$
- $E^*_i = E^*_{i'} \otimes I'' \in T'(0) \otimes M'' \quad (0 \leq i \leq n)$

- $A_1 = A'_1 \otimes I'' + I' \otimes A''_1 \in T'(0) \otimes M''$
- $T \subset T'(0) \otimes M''$

**Theorem**

- Every irreducible $(T'(0) \otimes M'')$-module is a thin irreducible $T$-module.
Johnson graphs

- Use $\sim$ to denote objects associated with $Q_v = H(v, 2)$ ($v \geq 2D$).
- $X = \tilde{\Gamma}_D(0) = \{z \in [2]^v : \partial(0, z) = D\}$ where $0 = (0, \ldots, 0)$
  - in bijection with $\binom{[v]}{D}$
- $y \sim z \iff \partial(y, z) = 2$
- $\Gamma = J(v, D)$: the Johnson graph
Johnson graphs

- \( n \in \{0, 1, \ldots, D\} \)
- \( u \in \tilde{\Gamma}_n(0) \leftrightarrow \binom{[v]}{n} \)
- \( Y = \{ z \in X : \partial(u, z) = D - n \} : \text{a descendent with } w = D - n, w^* = n \)

\[
\begin{align*}
  u &= \left(1, \ldots, 1 \mid 0, \ldots, 0, 0, \ldots, 0\right) \\
  z &= \left(1, \ldots, 1 \mid 1, \ldots, 1, 0, \ldots, 0\right) \\
  D - n
\end{align*}
\]

- \( \Gamma_Y \cong J(v - n, D - n) \)

**Theorem (Brouwer–Godsil–Koolen–Martin, 2003)**

- Every descendent of \( \Gamma = J(v, D) \) with \( w^* = n \) is isomorphic (under \( \text{Aut}\ \Gamma \)) to \( Y \) above.
Johnson graphs

\[ u = (1, \ldots, 1, 1, 1 | 0, \ldots, 0, 0, 0, 0, 0, \ldots, 0) \]
\[ z = (1, \ldots, 1, 1, 1 | 1, \ldots, 1, 0, 0, 0, \ldots, 0) \in Y = Y_0 \]
\[ z = (1, \ldots, 1, 1, 0 | 1, \ldots, 1, 1, 0, 0, \ldots, 0) \in Y_1 \]
\[ z = (1, \ldots, 1, 0, 0 | 1, \ldots, 1, 1, 1, 0, \ldots, 0) \in Y_2 \]

\[ Y_i = \Gamma'_{n-i}(0) \times \Gamma''_{D-n+i}(0) (0 \leq i \leq n) \]

where \( \Gamma' = \mathcal{Q}_n \) and \( \Gamma'' = \mathcal{Q}_{v-n} \)
Johnson graphs

- $\Gamma' = Q_n$, $\Gamma'' = Q_{v-n}$
- Use $'$ (resp. $''$) to denote objects associated with $\Gamma'$ (resp. $\Gamma''$).
- $Y_i = \Gamma'_{n-i}(0) \times \Gamma''_{D-n+i}(0)$ ($0 \leq i \leq n$)
- $E^*_i = E^*_{n-i} \otimes E^*_{D-n+i} \in \tilde{E}^*_D (T' \otimes T'') \tilde{E}^*_D$
- $A_1 = \tilde{E}^*_D \tilde{A}_2 \tilde{E}^*_D \in \tilde{E}^*_D (T' \otimes T'') \tilde{E}^*_D$
- $T \subset \tilde{E}^*_D (T' \otimes T'') \tilde{E}^*_D$

Theorem

Every irreducible $(\tilde{E}^*_D (T' \otimes T'') \tilde{E}^*_D)$-module is a thin irreducible $T$-module.
Grassmann graphs

- $\mathcal{V} = \mathbb{F}_q^v (v \geq 2D)$
- $X = \left[ \begin{array}{c} \mathcal{V} \\ D \end{array} \right]_q \quad \text{the set of } D\text{-dimensional subspaces of } \mathcal{V}$
- $y \sim z \overset{\text{def}}{\iff} \dim(y \cap z) = D - 1$
- $\Gamma = J_q(v, D) : \text{the Grassmann graph}$
Grassmann graphs

- \( n \in \{0, 1, \ldots, D\} \)
- \( u \in \binom{V}{n}_q \)

- \( Y = \{z \in X : u \leq z\} : \text{a descendent} \)
  with \( w = D - n, w^* = n \)

- \( \Gamma_Y \cong J_q(v - n, D - n) \)

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**Theorem (T., 2006)**

*Every descendent of \( \Gamma = J_q(v, D) \) with \( w^* = n \) is isomorphic (under \( \text{Aut} \Gamma \)) to \( Y \) above.*

- \( Y_i = \{z \in X : \dim(u \cap z) = n - i\} \) (0 \( \leq i \leq n\))
Grassmann graphs

- $P(V) = \bigcup_{i=0}^{v} [V]_q^i$ : the set of subspaces of $V$
- $G = \text{GL}(V) \curvearrowright P(V)$
- $K = G_u = \{ g \in G : gu = u \}$
- $\mathcal{H} = \{ B \in \text{End}(\mathbb{C}P(V)) : gB = Bg \text{ for } \forall g \in K \}$

Dunkl (1978) decomposed $\mathbb{C}P(V)$ into irreducible $K$-modules, and computed all the spherical functions, i.e., the structure of $\mathcal{H}$ is (essentially) known.
Grassmann graphs

- $\mathcal{H} = \{ B \in \text{End}(\mathbb{C}^{P(V)}) : gB =Bg \text{ for } \forall g \in K \}$ ← known
- $K \acts X = [V_D]$
- $\mathcal{H}_X = \{ B \in \text{End}(\mathbb{C}^X) : gB =Bg \text{ for } \forall g \in K \}$ ← known
- $Y_i = \{ z \in X : \dim(u \cap z) = n - i \} \ (0 \leq i \leq n)$
- $K \cdot Y_i = Y_i \implies E_i^* \in \mathcal{H}_X$
- $T \subset \mathcal{H}_X$

Theorem

**Every irreducible $\mathcal{H}_X$-module is a thin irreducible $T$-module.**
Semilattice-type DRGs

- \( \Gamma \): a Johnson, Hamming, Grassmann, bilinear forms, or a dual polar graph
- \((\mathcal{P}, \preceq)\): the associated semilattice
- \(u \in \mathcal{P}: \text{rank } n\)
- \(Y = \{z \in X: u \preceq z\}: \text{a descendent with } w = D - n, w^* = n\)

**Theorem (BGKM, 2003; T., 2006)**

Every descendent of \( \Gamma \) with \( w^* = n \) is isomorphic (under \( \text{Aut} \Gamma \)) to \( Y \) above.
The bipartite $Q$-polynomial DRGs

- Suppose $\Gamma$ is bipartite.

**Theorem (Caughman, 1999)**

- The structure of $T(x)$ depends only on the parameters of $\Gamma$.
- The dual polar graphs $[D_D(q)]$ and the Hemmeter graphs $\text{Hem}_D(q)$ have the same parameters.

- $Y$ : an edge of $\Gamma$; a descendent with $w = 1$, $w^* = D - 1$

**Problem**

- Study $T(Y)$.