Vanishing of cohomology groups and large eigenvalues of the Laplacian on p-forms in collapsing

Junya Takahashi

Tôhoku University, Japan

May 2005

Brest in France

1. Introduction

• (M^m, g) : connected oriented closed Riemannian manifold, $\dim M = m$

closed = compact, without boundary

• the de Rham complex

$d : \Omega^p(M) \to \Omega^{p+1}(M), \; d^2 \equiv 0,$

δ : the L^2-adjoint of d

w.r.t. the L^2-inner product,

i.e. $(d\,\varphi, \psi)_{L^2} = (\varphi, \delta\,\psi)_{L^2}$.

the L^2-inner product is defined as

$$(\varphi, \psi)_{L^2} := \int_M \langle \varphi, \psi \rangle_g d\mu_g$$

for any $\varphi, \psi \in \Omega^p(M)$.

- Laplacian on p-forms

$$\Delta := d\,\delta + \delta\,d : \Omega^p(M) \to \Omega^p(M)$$

Spectrum of Δ consists only of eigenvalues, because M is closed and Δ is elliptic.
So, we denote the eigenvalues by

$$0 = \cdots = 0 < \lambda_1^{(p)}(M, g) \leq \lambda_2(M, g) \leq \cdots \to \infty.$$

(1) the number of zero eigenvalues equals $b_p(M) := \dim H^p(M; \mathbb{R})$, which is independent of metric g (de Rham-Hodge-Kodaira) .
(2) the positive eigenvalues depend on metric g.

<u>Basic Problem on Spectral Geometry</u>
Find the geometrical and topological information of $\lambda_k^{(p)}$.

Estimates of $\lambda_1^{(p)}$

• $p = 0$, (functions)

(M, g) : closed Riem. manifold with

$$\begin{cases} \mathrm{Ric} \geq -\kappa^2, \\ 0 < d_1 \leq \mathrm{diam} \leq d_2. \end{cases} \quad \cdots \star$$

$\Rightarrow {}^{\exists}C_1(m, \kappa, d_2), C_2(m, \kappa, d_1) > 0$

s.t. $C_1 \leq \lambda_1^{(0)}(M, g) \leq C_2$.

• $1 \leq p \leq m - 1$

An analogy does not hold !

i.e. ${}^{\exists}\{(M_\alpha, g_{\alpha,i})\}_{i=1,2,\ldots}$: $(\alpha = 1, 2)$ two families of closed Riemannian manifolds with $\star$ s.t.

(1) $\lambda_1^{(p)}(M_1, g_{1,i}) \to 0$,

(small eigenvalues)

(2) $\lambda_1^{(p)}(M_2, g_{2,i}) \to 0$,

(large eigenvalues)

as $i \to \infty$.

Known results

• **Small eigenvalues**

Colbois-Courtois [90, 00], Lott [02, 04], Forman [95], Jammes [03, 04], T— [02], etc.

typical example $\cdots$ the Berger sphere.

• **Large eigenvalues**

Aubry-Colbois-Ghanaat-Ruh [03], Lott [02].

typical example $\cdots$ some nilmanifold.

All known examples are given by collapsing.

• **In this talk, we study large eigenvalues under some collapsing.**

2. Results

• If $|K| \leq bdd.$, Lott [02] completely characterized the large eigenvalues.

$\Rightarrow$ we are interested in $K \geq bdd.$

Roughly speaking,
collapsing under $K \geq bdd.$ consists of gluing of (singular) fiber bundles (Shioya-Yamaguchi [00], Yamaguchi [02]).

But, this case is very difficult in general.

$\Rightarrow$ so, we study a special collapsing of Riemannian manifolds which is gluing of two trivial bundles with boundaries.

Construction

Take two conn. orientable compact Riemannian manifolds with boundaries:

$$\begin{cases} (N^n, g_N) \rightsquigarrow_\partial (Z^{n-1} := \partial N, g_Z), \\ (W^{r+1}, g_W) \rightsquigarrow_\partial (F^r := \partial W, g_F). \end{cases}$$

By taking products of them, we have 2-families of cpt. Riem. manifolds :

$$\begin{cases} M_{1,\varepsilon} & := (F \times N, \varepsilon^2 g_F \oplus g_N), \\ M_{2,\varepsilon} & := (W \times Z, \varepsilon^2 g_W \oplus g_N). \end{cases}$$

Both boundaries are isometric.
So, by gluing them along their boundaries, we obtain the collapsing of closed Riem. manifolds :

$$(M^m, g_\varepsilon) := M_{1,\varepsilon} \cup_\partial M_{2,\varepsilon} \to (N^n, g_N),$$

as $\varepsilon \to 0$, where $m := n + r$.
In general, $K_\varepsilon \to \pm\infty$.

Main Theorem (T — [05])

For our collapsing family $(M, g_\varepsilon) \to (N, g_N)$ with $n \geq 2$, consider the three conditions:

(1) $n < p < r$,
(2) $H^q(F; \mathbb{R}) = 0$
for $p - n \leq {}^\forall q \leq p$,
(3) $H^q(W; \mathbb{R}) = H^q(W, \partial W; \mathbb{R})$
$= 0$ for $p - n + 1 \leq {}^\forall q \leq p$.

Then, we obtain

$$\lambda_1^{(p)}(M, g_\varepsilon) \begin{cases} \to \infty & \text{if (1), (2), (3) hold,} \\ \leq C & \text{otherwise,} \end{cases}$$

as $\varepsilon \to 0$, where $C > 0$ is independent of ε.

- **Remark**

In the case of $n = 1$, we do not need the condition (3).

Special case: Sphere

(D^k, g_{D^k}) : the k-dimensional disk with $K_{D^k} \geq 0$. Then, we take

$$\begin{aligned} (N^n, g_N) \quad &:= (D^n, g_{D^n}), \\ (W^{r+1}, g_W) &:= (D^{r+1}, g_{D^{r+1}}). \end{aligned}$$

Then, we see that our $M \cong_{C^\infty} S^m$. In fact,

$$\begin{aligned} M_\varepsilon^m &= (S_\varepsilon^r \times D^n) \cup_\partial (D_\varepsilon^{r+1} \times S^{n-1}) \\ &\cong \partial(D^{r+1} \times D^n) \cong \partial D^{m+1} \\ &\cong S^m. \end{aligned}$$

Thus, we obtain the collapsing family $(S^m, g_\varepsilon) \to (D^n, g_{D^n})$ with $K_{g_\varepsilon} \geq 0, \sup K_{g_\varepsilon} \to \infty$.

• Remark

S^{2n} does not collapse under $|K| \leq bdd.$, because $\chi(S^{2n}) \neq 0$.

(If possible, $\chi(M) = 0$.)

Corollary. (T — [05])

$\exists$ collapsing $(S^m, g_\varepsilon) \to (D^n, g_{D^n})$ with $K_{g_\varepsilon} \geq 0$ s.t.

$$\lambda_1^{(p)}(S^m, g_\varepsilon) \begin{cases} \to \infty & \text{if } n < p < r, \\ \leq C & \text{otherwise,} \end{cases}$$

as $\varepsilon \to 0$, where $C > 0$ is a constant independent of ε.

The conditions (2), (3) in Main Theorem are automatically satisfied.