Partial collapsing and the spectrum of the Hodge-de Rham operator

Junya Takahashi

This talk is based on a joint work with Colette Anné in Université de Nantes, France ([AT12] and [AT13]).

Let M_1 and M_2 be two connected compact manifolds with the same boundary Σ, a compact manifold of dimension $n \geq 2$. We denote by $m = n + 1$ the dimension of M_1 and M_2. We endow Σ with a fixed metric h.

Let $\overline{M}_1$ be the compact manifold with conical singularity obtained from M_1 by gluing M_1 to the Euclidean cone $\mathcal{C}(\Sigma) = [0,1] \times \Sigma \big/ \{0\} \times \Sigma$. There exists a Riemannian metric $\bar{g}_1$ on $\overline{M}_1 = M_1 \cup_\Sigma \mathcal{C}(\Sigma)$ which is written by $dr^2 \oplus r^2 h$ on the smooth part of the cone $\mathcal{C}(\Sigma)$.

We choose a metric g_2 on M_2 which is 'trumpet like', i.e., M_2 is isometric near the boundary to $[0, \frac{1}{2}] \times \Sigma$ with the conical metric $ds^2 \oplus (1-s)^2 h$, where s is the coordinate from the boundary at $s = 0$.

For any $\varepsilon > 0$, we define

$$\mathcal{C}_{\varepsilon,1}(\Sigma) = \{(r, y) \in \mathcal{C}(\Sigma) \,|\, r > \varepsilon\} \quad \text{and} \quad M_1(\varepsilon) = M_1 \cup_\Sigma \mathcal{C}_{\varepsilon,1}(\Sigma).$$

Our purpose is to determine the limit spectrum of the Hodge-de Rham operator $\Delta = \delta d + d\delta$ acting on the differential forms of the closed Riemannian manifold

$$(M, g_\varepsilon) = (M_1(\varepsilon), g_1) \cup_{(\Sigma, \varepsilon^2 h)} (M_2, \varepsilon^2 g_2)$$

which is obtained by gluing together $(M_1(\varepsilon), g_1)$ and $(M_2, \varepsilon^2 g_2)$, as ε goes to 0 (see Figure 1). We remark that, by construction, these two manifolds have isometric boundary and that the metric g_ε obtained on M_ε is smooth.

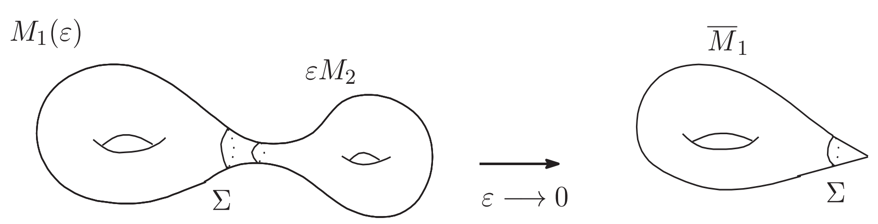

Figure 1: Partial collapsing of M_ε

2010 *Mathematics Subject Classification.* Primary 58J50; Secondary 35P15, 53C23, 58J32.

We can describe the limit spectrum as follows: it has two parts. One comes from the non-collapsing part $\overline{M_1}$, and is expressed by the spectrum of a suitable extension of the Hodge-de Rham operator on this manifold with conical singularity. This extension is self-adjoint and comes from an extension of the Gauß-Bonnet operators. All these extensions are classified by the subspaces W of the total eigenspace corresponding to the eigenvalues within $(-\frac{1}{2}, \frac{1}{2})$ of an elliptic operator A acting on the boundary Σ.

The other part comes from the collapsing part M_2, where the limit Gauß-Bonnet operator satisfies boundary conditions of the Atiyah-Patodi-Singer type. This operator, denoted by $\mathcal{D}_2$, can also be seen on the quasi-asymptotically conical space $\widetilde{M_2}$, which is obtained from gluing M_2 to the infinite cone:

$$\big(\widetilde{M_2}, \widetilde{g}_2\big) := \big(M_2, g_2\big) \cup_\Sigma \big([1, \infty) \times \Sigma,\, dr^2 + r^2 h\big).$$

Our main theorem consists of two parts:

Main Theorem A. *If the limit of the spectrum of M_ε is positive, then it belongs to the positive spectrum of the Hodge-de Rham operator $\Delta_{1,W}$ on $\overline{M_1}$, where $\Delta_{1,W}$ is a closed extension obtained from the subspace*

$$W \subset \bigoplus_{|\gamma|<\frac{1}{2}} \mathrm{Ker}(A - \gamma).$$

This subspace W consists of extended solutions, not including L^2, on the complete non-compact Riemannian manifold $\widetilde{M_2}$.

Main Theorem B. *The multiplicity of 0 in the limit spectrum is given by*

$$\dim \mathrm{Ker}(\Delta_{1,W}) + \dim \mathrm{Ker}(\mathcal{D}_2) + \dim \mathcal{I}_{\frac{1}{2}},$$

where $\mathcal{I}_{\frac{1}{2}}$ denotes the linear subspace of $\mathrm{Ker}(A - \frac{1}{2})$ spanned by the boundary values of extended solutions on $\widetilde{M_2}$ in the sense of the trace.

Furthermore, there will appear eigenforms whose norms concentrate on the singularity of the cone $\mathcal{C}(\Sigma)$ in the limit. Such eigenforms correspond to the elements in $\mathcal{I}_{\frac{1}{2}}$. This is an entirely new phenomenon.

References

[AT12] C. Anné and J. Takahashi, p-spectrum and collapsing of connected sums, Trans. Amer. Math. Soc. **364** no. 4 (2012), 1711-1735.

[AT13] C. Anné and J. Takahashi, Partial collapsing and the spectrum of the Hodge-de Rham operator, preprint, (2013).

Junya Takahashi
Division of Mathematics, Graduate School of Information Sciences,
Tôhoku University, Aoba 6–3–09, Sendai, 980–8579, Japan.
e-mail: t-junya@math.is.tohoku.ac.jp