Small eigenvalues of the rough and Hodge Laplacians under fixed volume

Colette Anné and Junya Takahashi

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Abstract

For each degree p and each natural number $k \geq 1$, we construct on any closed manifold a family of Riemannian metrics, with fixed volume such that the k^{th} positive eigenvalue of the rough or the Hodge Laplacian acting on differential p-forms converge to zero. In particular, on the sphere, we can choose these Riemannian metrics as those of non-negative sectional curvature. This is a generalization of the results by Colbois and Maerten in 2010 to the case of higher degree forms.

Résumé. Pour chaque degré p et chaque entier naturel $k \geq 1$, nous construisons, sur toute variété compacte, une famille de métriques riemanniennes à volume fixé telle que la $k^{\text{ième}}$ valeur propre strictement positive du Laplacien brut ou du Laplacien de Hodge agissant sur les formes différentielles de degré p converge vers zéro. En particulier, sur la sphère, nous pouvons choisir des métriques à courbure sectionnelle positive. Ce résultat généralise aux plus hauts degrés celui de Colbois et Maerten de 2010.

1 Introduction

We study the eigenvalue problems of two elliptic differential operators acting on differential p-forms on a connected oriented closed Riemannian manifold (M^m, g) of dimension $m \geq 2$.

One is the rough Laplacian $\overline{\Delta} = \nabla^* \nabla$, or the connection Laplacian, acting on p-forms on (M, g), where ∇ is the covariant derivative induced from the Levi-Civita connection of the Riemannian metric g. The spectrum of the rough Laplacian consists only of **non-negative** eigenvalues with finite multiplicity. We denote its eigenvalues counted with multiplicity by

$$0 \leq \overline{\lambda}_1^{(p)}(M, g) \leq \overline{\lambda}_2^{(p)}(M, g) \leq \cdots \leq \overline{\lambda}_k^{(p)}(M, g) \leq \cdots .$$

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The other is the Hodge-Laplacian $\Delta = d\delta + \delta d$ acting on p-forms on (M, g), where d is the exterior derivative and δ its formal adjoint with respect to the L^2-inner product. The spectrum of the Hodge-Laplacian consists only of non-negative eigenvalues with finite multiplicity. We also denote its **positive** eigenvalues counted with multiplicity by

$$\underbrace{0 = \cdots = 0}_{b_p(M)} < \lambda_1^{(p)}(M, g) \leq \lambda_2^{(p)}(M, g) \leq \cdots \leq \lambda_k^{(p)}(M, g) \leq \cdots ,$$

where the multiplicity of the eigenvalue 0 is equal to the p-th Betti number $b_p(M)$ of M, by the Hodge-Kodaira-de Rham theory. In particular, it is independent of a choice of Riemannian metrics.

Furthermore, since the Hodge-Laplacian Δ commutes with d and δ, we can define the k-th eigenvalues of the Hodge-Laplacian acting on exact and co-exact p-forms, which are denoted by $\lambda_k'^{(p)}(M, g)$ and $\lambda_k''^{(p)}(M, g)$, respectively. These are always positive. From the Hodge duality, it follows that for any degree p

$$\lambda_k'^{(p)}(M, g) = \lambda_k''^{(m-p)}(M, g) \tag{1.1}$$

for $k = 1, 2, \ldots$. In particular, we see that

$$\begin{aligned} \lambda_1^{(p)}(M, g) &= \min\{\lambda_1'^{(p)}(M, g),\ \lambda_1''^{(p)}(M, g)\} \\ &= \min\{\lambda_1'^{(p)}(M, g),\ \lambda_1'^{(m-p)}(M, g)\}. \end{aligned} \tag{1.2}$$

We are interested in the supremum and the infimum of the k-th eigenvalues under all Riemannian metrics with fixed volume on M. Colbois and Dodziuk [CD94] proved that there exists no universal upper bound of the first positive eigenvalue of the Laplacian acting on functions under fixed volume. Similar results to the rough-Laplacian acting on p-forms with $1 \leq p \leq m-1$ were proved by Colbois and Maerten [CM10, Theorem 1.1], and to the Hodge-Laplacian acting on p-forms for $2 \leq p \leq m-2$ by Gentile and Pagliara [GP95]. But, the case of $p = 1, m-1$ is still unknown (cf. [Tan83], [Ge99]).

There exists no positive universal lower bound of the first positive eigenvalue of the Laplacian acting on functions, if we deform a Riemannian manifold to a dumbbell under fixed volume, which is called the Cheeger dumbbell [Ch70]. Similar results to the rough Laplacian acting on p-forms with $p = 0, 1, m-1, m$ were also proved by Colbois and Maerten [CM10, Theorem 1.2]. On any connected oriented closed manifold M of dimension $m \geq 3$, there exists a one-parameter family of Riemannian metrics $\overline{g}_L$ with volume one such that for $p = 0, 1, m-1, m$ and for any $k \geq 1$,

$$\overline{\lambda}_k^{(p)}(M, \overline{g}_L) \longrightarrow 0 \ \text{ as } \ L \longrightarrow \infty.$$

In the present paper, we prove similar results in the case of all degree p with $1 \leq p \leq m-1$. We also prove them in the case of the Hodge-Laplacian for all degree p

with $1 \leq p \leq m-1$. In the same way as Colbois and Dodziuk [CD94] who deduced their result from that on the spheres, for the odd dimensional spheres by Tanno [Tan79], Bleecker [B83], for the even dimensional spheres by Muto [Mu80], using a spectral analysis of connected sums, we first construct such a family of Riemannian metrics with non-negative sectional curvature on the m-dimensional standard sphere $\mathbb{S}^m$. More precisely,

Theorem 1.1. *For $m \geq 2$ and a given degree p with $1 \leq p \leq m-1$, there exists a one-parameter family of Riemannian metrics $\overline{g}_{p,L}$ on the m-dimensional standard sphere $\mathbb{S}^m$ with volume one and non-negative sectional curvature such that for any integer $k \geq 1$,*

(1) $\overline{\lambda}_k^{(p)}(\mathbb{S}^m, \overline{g}_{p,L}) \longrightarrow 0$;

(2) $\lambda_k^{\prime\prime(p)}(\mathbb{S}^m, \overline{g}_{p,L}) \longrightarrow 0$,

as $L \longrightarrow \infty$.

Furthermore, we give lower bounds of the eigenvalues of the Hodge-Laplacian acting on p-forms on $\mathbb{S}^m$ in Theorem 4.1 and Corollary 4.2.

Next, by gluing this sphere with any closed manifold, we obtain the same result as Theorem 1.1 for any closed manifold, without keeping non-negative sectional curvature.

Theorem 1.2. *Let M^m be a connected oriented closed manifold of dimension $m \geq 2$. For any fixed degree p with $1 \leq p \leq m-1$, any integer $k \geq 1$ and for any $\varepsilon > 0$, there exists a Riemannian metric $\overline{g}_{p,\varepsilon}$ on M with volume one such that*

$$0 < \overline{\lambda}_k^{(p)}(M, \overline{g}_{p,\varepsilon}) < \varepsilon \quad \text{and} \quad \lambda_k^{\prime\prime(p)}(M, \overline{g}_{p,\varepsilon}) < \varepsilon.$$

We note that Riemannian metrics $\overline{g}_{p,L}$ and $\overline{g}_{p,\varepsilon}$ in Theorems 1.1 and 1.2 depend on the degree p. But, by taking connected sums of M and $(m-1)$ spheres at distinct $(m-1)$ points, we obtain a Riemannian metrics $\overline{g}_\varepsilon$ on M with small eigenvalues for all $p = 1, 2, \ldots, m-1$.

Theorem 1.3. *Let M^m be a connected oriented closed manifold of dimension $m \geq 2$. For any $\varepsilon > 0$ and any integer $k \geq 1$, there exists a Riemannian metric $\overline{g}_\varepsilon$ on M with volume one such that for any degree p with $1 \leq p \leq m-1$,*

$$0 < \overline{\lambda}_k^{(p)}(M, \overline{g}_\varepsilon) < \varepsilon \quad \text{and} \quad \lambda_k^{\prime\prime(p)}(M, \overline{g}_\varepsilon) < \varepsilon.$$

Remark 1.4. (*i*) *In the case of $m = 2$, Theorem 1.2 is covered with the result by Colbois and Maerten [CM10].*

(ii) *The same results for the Hodge-Laplacian were obtained from the results by Guerini [Gu04] and Jammes [Ja08], [Ja11]. Although the sectional curvature for their Riemannian metrics on $\mathbb{S}^m$ diverges to $-\infty$, our Riemannian metric constructed in Theorem* 1.1 *has non-negative sectional curvature. This is our advantage.*

(iii) *As a consequence of our result on spheres, Theorem* 1.1*, their exists no lower bound of the positive eigenvalue of degree p with $1 \leq p \leq m-1$ depending only on the dimension, the volume and a lower bound of the sectional curvature. See Remark* 3.3 *below.*

(iv) *Jammes [Ja08] constructed similar Riemannian metrics within a fixed conformal class for $m \geq 5$, except for $p = \frac{m}{2}$ if m is even.*

The present paper is organized as follows: In Section 2, we recall the Weizenböck formula and the properties of parallel forms. In Section 3, we consider the case of the sphere, and give the proof of Theorem 1.1. In Section 4, we give lower bounds for the eigenvalues of the Hodge-Laplacian on the sphere. In Section 5, we consider the case of a general manifold, and give the proof of Theorems 1.2 and 1.3. In Section 6, as an appendix, we prove the convergence theorem of the eigenvalues of the rough Laplacian acting on p-forms, when one side of a connected sum of two closed Riemannian manifolds collapses to a point.

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2 Notations and basic facts

We fix the notations used in the present paper. Let (M^m, g) be a connected oriented closed Riemannian manifold of dimension $m \geq 2$. The metric g defines a volume element $d\mu_g$ and a scalar product on the fibers of any tensor bundle. The L^2-inner product of the space of all smooth p-forms $\Omega^p(M)$ is defined as, for any p-form φ, ψ on M

$$(\varphi, \psi)_{L^2(M,g)} := \int_M \langle \varphi, \psi \rangle d\mu_g \quad \text{and} \quad \| \varphi \|^2_{L^2(M,g)} := (\varphi, \varphi)_{L^2(M,g)}.$$

The space of L^2 p-forms $L^2(\Lambda^p M, g)$ is the completion of $\Omega^p(M)$ with respect to this L^2-norm.

For a positive constant $a > 0$, it is easy to see that

$$\begin{aligned} &\overline{\lambda}_k^{(p)}(M, ag) = a^{-1}\overline{\lambda}_k^{(p)}(M, g), \quad \lambda_k^{(p)}(M, ag) = a^{-1}\lambda_k^{(p)}(M, g), \\ &\mathrm{vol}(M, ag) = a^{\frac{m}{2}} \,\mathrm{vol}(M, g), \end{aligned} \tag{2.1}$$

where $\mathrm{vol}(M,g)$ denotes the volume of (M,g). Thus, if we take a new Riemannian metric

$$\overline{g} := \mathrm{vol}(M,g)^{-\frac{2}{m}}\, g, \tag{2.2}$$

then the volume is one: $\mathrm{vol}(M,\overline{g}) = 1$. Therefore, instead of considering a volume normalized metric, we may consider the following invariants

$$\overline{\lambda}_k^{(p)}(M,g)\,\mathrm{vol}(M,g)^{\frac{2}{m}}, \quad \lambda_k^{(p)}(M,g)\,\mathrm{vol}(M,g)^{\frac{2}{m}}.$$

The relation between the rough and Hodge Laplacians on p-forms is given by the Weitzenböck formula: for any p-form φ on M,

$$\Delta\,\varphi = \overline{\Delta}\,\varphi + F_p(\varphi), \tag{2.3}$$

where F_p is the Weitzenböck curvature tensor defined as

$$F_p(\varphi) = -\sum_{i,j=1}^{m} e^i \wedge i_{e_j}(R(e_i,e_j)\,\varphi), \tag{2.4}$$

where R denotes the curvature tensor with respect to the covariant derivative induced from the Levi-Civita connection and i_X denotes the interior product of a vector X, and $\{e_1,\dots,e_m\}$ is a local orthonormal frame and $\{e^1,\dots,e^m\}$ is its dual frame.

Now, after taking the scalar product of φ with (2.3), we obtain the Bochner formula: for each point on M,

$$\begin{aligned}\frac{1}{2}\Delta(|\,\varphi\,|^2) &= -|\nabla\,\varphi\,|^2 + \langle\overline{\Delta}\,\varphi,\varphi\rangle \\ &= \langle\Delta\,\varphi,\varphi\rangle - |\nabla\,\varphi\,|^2 - \langle F_p(\varphi),\varphi\rangle.\end{aligned} \tag{2.5}$$

By the Hodge-Kodaira-de Rham theory, the kernel of the Hodge-Laplacian acting on p-forms consists of harmonic p-forms, whose dimension is equal to the p-th Betti number of M, that is, a topological invariant of M. In contract, the kernel of the rough Laplacian acting on p-forms consists of parallel p-forms, whose dimension is not a topological invariant. In fact, we can kill all parallel p-forms under local perturbation of a Riemannian metric.

Lemma 2.1. *Let (M,g) be a connected oriented closed Riemannian manifold. If F_p is positive definite at one point, there exist no non-zero parallel p-forms on (M,g).*

Proof. We prove this by contradiction. Let φ be a non-zero parallel p-form on M. Then, φ is harmonic and of constant norm. By the assumption, there exists an open subset U of M such that $\langle F_p(\varphi),\varphi\rangle > 0$ on U. From the Bochner formula on U

(2.5) $$\frac{1}{2}\Delta(|\,\varphi\,|^2) = \langle\Delta\,\varphi,\varphi\rangle - |\nabla\,\varphi\,|^2 - \langle F_p(\varphi),\varphi\rangle,$$

we have

$$0 = |\nabla \varphi|^2 = -\langle F_p(\varphi), \varphi \rangle < 0,$$

which is a contradiction. □

Lemma 2.2. *Let (M^m, g) be a connected oriented closed Riemannian manifold. For any open subset U, there exists a Riemannian metric g' on M with $g' = g$ on $M \setminus U$ such that all parallel p-forms with respect to g' are zero.*

Proof. We take any point x_0 in any open subset U of M. On a neighborhood of x_0, we deform the Riemannian metric g to g' such that g' has constant sectional curvature 1. The curvature operator is also 1 on this neighborhood of x_0 (see [Pe16], p.84, Proposition 3.1.3). Since the Weitzenböck curvature tensor F_p is controlled below by the curvature operator (see [GM75] p.264, Corollary 2.6), we see that $F_p \geq p(m-p) > 0$ at x_0. Hence, from Lemma 2.1, we see that (M, g') has no non-zero parallel p-forms. □

3 Small eigenvalues on the sphere $\mathbb{S}^m$

We first consider the case of the m-dimensional standard sphere $\mathbb{S}^m$.

Notations. For a dimension n, Let $g_{\mathbb{S}^n}$ be the Riemannian metric on $\mathbb{S}^n$ of constant sectional curvature one. We denote by $\mathbb{D}^n$ the n-dimensional closed disk, and let $g_{\mathbb{D}^n}$ a fixed Riemannian metric on it, which is identified with $[0,2] \times \mathbb{S}^{n-1}$, of non-negative sectional curvature $K_{g_{\mathbb{D}^n}} \geq 0$. We can, in addition, assume that $g_{\mathbb{D}^n}$ is a product metric near the boundary. In fact, if we take a smooth positive function $f(r)$ on the interval $[0,2]$ satisfying that

$$f(r) = \begin{cases} \sin(r) & \text{on } [0,1], \\ 1 & \text{on } [3/2, 2], \end{cases}$$

(Note that $\sin(1) \sim 0.84$.) and $0 \leq f'(r) \leq 1$ and $f''(r) \leq 0$, then the metric $g_{\mathbb{D}^n}$ is written as

$$g_{\mathbb{D}^n} = dr^2 \oplus f^2(r) g_{\mathbb{S}^{n-1}} \quad \text{on } [0,2] \times \mathbb{S}^{n-1}. \tag{3.1}$$

The sectional curvatures of $g_{\mathbb{D}^n}$ are given, if X and Y are orthonormal vectors tangent to the angle directions, by

$$K(\partial_r, X) = -\frac{f''(r)}{f(r)}, \quad K(X, Y) = \frac{1 - (f'(r))^2}{f^2(r)},$$

both of which are non-negative (e.g., Petersen [Pe16], **4.2.3**, p.121).

Proof of Theorem 1.1. We take any degree p with $1 \leq p \leq m-1$. We consider the decomposition (see Figure 1)

$$\mathbb{S}^m = \left(\mathbb{S}^p \times \mathbb{D}_L^{m-p}\right) \cup_{\mathbb{S}^p \times \mathbb{S}^{m-p-1}} \left(\mathbb{D}^{p+1} \times \mathbb{S}^{m-p-1}\right), \tag{3.2}$$

and set

$$H_1 := \mathbb{S}^p \times \mathbb{D}_L^{m-p} \quad \text{and} \quad H_2 := \mathbb{D}^{p+1} \times \mathbb{S}^{m-p-1}.$$

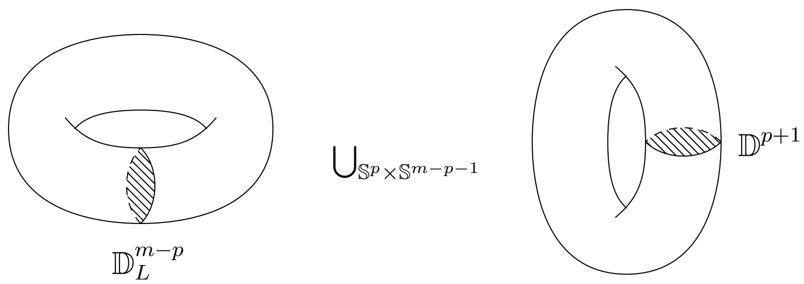

Figure 1: $\mathbb{S}^m \cong \left(\mathbb{S}^p \times \mathbb{D}_L^{m-p}\right) \cup_{\mathbb{S}^p \times \mathbb{S}^{m-p-1}} \left(\mathbb{D}^{p+1} \times \mathbb{S}^{m-p-1}\right)$

For any real number $L > 0$, we construct a one-parameter family of Riemannian metrics $g_{p,L}$ on $\mathbb{S}^m$. First we introduce a one-parameter family of Riemannian metrics $g_{\mathbb{D}^{m-p},L}$ on $\mathbb{D}^{m-p}$ containing a long cylinder as follows (see Figure 2):

$$g_{\mathbb{D}^{m-p},L} := \begin{cases} dr^2 \oplus f^2(r) g_{\mathbb{S}^{m-p-1}} & \text{on } [0,2] \times \mathbb{S}^{m-p-1}, \\ dr^2 \oplus g_{\mathbb{S}^{m-p-1}} & \text{on } [2, L+2] \times \mathbb{S}^{m-p-1}. \end{cases}$$

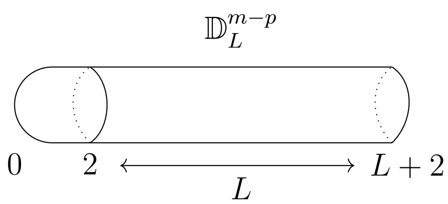

Figure 2: long disk $\mathbb{D}_L^{m-p}$

Then, we define the smooth Riemannian metric $g_{p,L}$ on $\mathbb{S}^m$ as

$$g_{p,L} := \begin{cases} g_{\mathbb{S}^p} \oplus g_{\mathbb{D}^{m-p},L} & \text{on } H_1 = \mathbb{S}^p \times \mathbb{D}_L^{m-p}, \\ g_{\mathbb{D}^{p+1}} \oplus g_{\mathbb{S}^{m-p-1}} & \text{on } H_2 = \mathbb{D}^{p+1} \times \mathbb{S}^{m-p-1}. \end{cases} \tag{3.3}$$

Since

$$\begin{aligned} \mathrm{vol}(\mathbb{S}^p \times \mathbb{D}_L^{m-p}) &= \mathrm{vol}(\mathbb{S}^p) \cdot \mathrm{vol}(\mathbb{D}_L^{m-p}) \\ &= \mathrm{vol}(\mathbb{S}^p) \cdot \left\{ \mathrm{vol}(\mathbb{D}^{m-p}) + \mathrm{vol}(\mathbb{S}^{m-p-1}) \cdot L \right\}, \end{aligned}$$

we can write for some constants $A, B > 0$ independent of L

$$\begin{aligned} \mathrm{vol}(\mathbb{S}^m, g_{p,L}) &= \mathrm{vol}(\mathbb{S}^p \times \mathbb{D}_L^{m-p}) + \mathrm{vol}(\mathbb{D}^{p+1} \times \mathbb{S}^{m-p-1}) \\ &= AL + B. \end{aligned} \tag{3.4}$$

Next, we estimate the eigenvalues of the rough and Hodge Laplacians acting on p-forms from above.

Lemma 3.1. *For any integer $k \geq 1$ and any real number $L > 0$, we have*

(1) $\overline{\lambda}_k^{(p)}(\mathbb{S}^m, g_{p,L}) \leq \dfrac{k^2\pi^2}{L^2}$;

(2) $\lambda_k''^{(p)}(\mathbb{S}^m, g_{p,L}) \leq \dfrac{k^2\pi^2}{L^2}$.

Remark 3.2. *We note that, for any metric, the rough and Hodge Laplacians acting on p-forms of $\mathbb{S}^m$ for $1 \leq p \leq m-1$ have no 0 eigenvalues. In fact, from $b_p(\mathbb{S}^m) = 0$ for $1 \leq p \leq m-1$, by the Hodge theory, there exist no non-zero harmonic p-forms on $\mathbb{S}^m$. In particular, there exist no non-zero parallel p-forms.*

Proof. We construct k test p-forms φ_i for the min-max principle. Their behaviour will be like $f_i v_p$, for suitable functions f_i, if v_p is the volume p-form on $(\mathbb{S}^p, g_{\mathbb{S}^p})$, so we can take advantage of the properties of the standard volume form. The functions f_i are constructed as follows: we divide the interval $[2, L+2]$ of length L into k intervals $I_i := [r_{i-1}, r_i]$ $(i = 1, \dots, k)$, where

$$2 = r_0 < r_1 < \cdots < r_k = L + 2 \quad \text{with} \quad r_i := \frac{L}{k}\, i + 2 \quad (i = 0, 1, \dots, k).$$

Let $f_i(r)$ be the first Dirichlet eigenfunction of the Laplacian acting on functions on the interval I_i, that is,

$$f_i(r) = \sin\left((r - r_{i-1})\frac{k\pi}{L}\right) \qquad \text{for} \quad r \in [r_{i-1}, r_i].$$

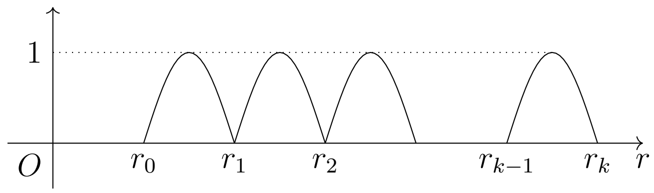

Figure 3: test functions $f_i(r)$

Then, we define k test p-forms φ_i on $\mathbb{S}^m$ as follows (recall that v_p is the volume p-form on $(\mathbb{S}^p, g_{\mathbb{S}^p})$): on $H_1 = \mathbb{S}^p \times \mathbb{D}_L^{m-p}$,

$$\varphi_i := \begin{cases} f_i(r) v_p & \text{on } \mathbb{S}^p \times \Big([r_{i-1}, r_i] \times \mathbb{S}^{m-p-1}\Big), \\ 0 & \text{otherwise}, \end{cases} \tag{3.5}$$

and $\varphi_i \equiv 0$ on $H_2 = \mathbb{D}^{p+1} \times \mathbb{S}^{m-p-1}$.

We remark that the family φ_i $(i = 1, \ldots, k)$ is orthogonal.

(1) We first prove the case of the rough Laplacian acting on p-forms. Since the orthogonal family φ_i $(i = 1, \ldots, k)$ have disjoint support, they are also orthogonal for the quadratic form defining the rough Laplacian, namely, $\|\nabla \varphi\|^2_{L^2(\mathbb{S}^m, g_{p,L})}$. The min-max principle and Remark 3.2 give then

$$0 < \overline{\lambda}_k^{(p)}(\mathbb{S}^m, g_{p,L}) \leq \max_{i=1,2,\ldots,k} \left\{ \frac{\|\nabla \varphi_i\|^2_{L^2(\mathbb{S}^m, g_{p,L})}}{\|\varphi_i\|^2_{L^2(\mathbb{S}^m, g_{p,L})}} \right\}. \tag{3.6}$$

Since v_p is parallel, the numerator of the right-hand side of (3.6) is

$$\|\nabla \varphi_i\|^2_{L^2(\mathbb{S}^m, g_{p,L})} = \|\nabla(f_i v_p)\|^2_{L^2(\mathbb{S}^p \times \mathbb{D}^{m-p}_L)} = \|df_i \otimes v_p\|^2_{L^2(\mathbb{S}^p \times (I_i \times \mathbb{S}^{m-p-1}))}.$$

Since the Riemannian metric on $C_i = I_i \times \mathbb{S}^{m-p-1}$ is product, we have

$$\begin{aligned} \|df_i \otimes v_p\|^2_{L^2(\mathbb{S}^p \times C_i)} &= \int_{\mathbb{S}^p} \int_{I_i \times \mathbb{S}^{m-p-1}} |df_i \otimes v_p|^2 d\mu_{\mathbb{S}^p} d\mu_{C_i} \\ &= \operatorname{vol}(\mathbb{S}^p) \int_{I_i \times \mathbb{S}^{m-p-1}} |df_i|^2 dr d\mu_{\mathbb{S}^{m-p-1}} \\ &= \operatorname{vol}(\mathbb{S}^p) \operatorname{vol}(\mathbb{S}^{m-p-1}) \int_{r_{i-1}}^{r_i} |df_i|^2 dr. \end{aligned}$$

Since $f_i(r)$ is the Dirichlet eigenfunction on the interval $I_i = [r_{i-1}, r_i]$, which is isometric to $[0, \frac{L}{k}]$, we have

$$\int_{r_{i-1}}^{r_i} |df_i|^2 dr = \frac{k^2\pi^2}{L^2} \int_{r_{i-1}}^{r_i} |f_i|^2 dr.$$

Therefore, the numerator of the right-hand side of (3.6) is

$$\|\nabla \varphi_i\|^2_{L^2(\mathbb{S}^m, g_{p,L})} = \frac{k^2\pi^2}{L^2} \operatorname{vol}(\mathbb{S}^p) \operatorname{vol}(\mathbb{S}^{m-p-1}) \int_{r_{i-1}}^{r_i} |f_i|^2 dr. \tag{3.7}$$

On the other hand, the denominator of the right-hand side of (3.6) is

$$\begin{aligned} \|\varphi_i\|^2_{L^2(S^m, g_{p,L})} &= \|f_i v_p\|^2_{L^2(\mathbb{S}^p \times \mathbb{D}^{m-p}_L)} = \|f_i v_p\|^2_{L^2(\mathbb{S}^p \times C_i)} \\ &= \operatorname{vol}(\mathbb{S}^p) \int_{I_i \times \mathbb{S}^{m-p-1}} |f_i|^2 dr d\mu_{\mathbb{S}^{m-p-1}} \\ &= \operatorname{vol}(\mathbb{S}^p) \operatorname{vol}(\mathbb{S}^{m-p-1}) \int_{r_{i-1}}^{r_i} |f_i|^2 dr. \end{aligned} \tag{3.8}$$

Thus, by substituting (3.7) and (3.8) into (3.6), we obtain

$$0 < \overline{\lambda}_k^{(p)}(\mathbb{S}^m, g_{p,L}) \leq \frac{k^2\pi^2}{L^2}.$$

(2) Next, we prove the case of the Hodge Laplacian acting on co-exact p-forms. We use the same test p-forms φ_i constructed in (3.5). For the same reason as before, this family is orthogonal for the quadratic form defining the Hodge Laplacian $\|(d+\delta)\,\varphi\,\|^2_{L^2(\mathbb{S}^m,g_{p,L})}$. Moreover, we note that the test p-forms φ_i are co-closed. Indeed, since the Riemannian metric is product on the support of φ_i,

$$\delta_{g_{p,L}}\,\varphi_i = f(\delta_{g_{p,L}} v_p) - i_{(\mathrm{grad}_{g_{p,L}} f)}(v_p) \equiv 0.$$

Since $\mathbb{S}^m$ has no non-zero harmonic p-forms, all co-closed forms must be co-exact. Therefore, from the min-max principle of the Hodge-Laplacian acting on co-exact p-forms, it follows that

$$0 < \lambda_k^{\prime\prime(p)}(\mathbb{S}^m, g_{p,L}) \le \max_{i=1,2,\dots,k} \left\{ \frac{\|d\,\varphi_i\,\|^2_{L^2(\mathbb{S}^m,g_{p,L})}}{\|\,\varphi_i\,\|^2_{L^2(\mathbb{S}^m,g_{p,L})}} \right\}.$$

By the same calculations as in (1), we obtain the same upper bound

$$\lambda_k^{\prime\prime(p)}(\mathbb{S}^m, g_{p,L}) \le \frac{k^2\pi^2}{L^2}.$$

□

Finally, we normalize the volume to be one. Namely, if we set a new Riemannian metric

$$\overline{g}_{p,L} := \mathrm{vol}(\mathbb{S}^m, g_{p,L})^{-\frac{2}{m}} g_{p,L},$$

then $\mathrm{vol}(\mathbb{S}^m, \overline{g}_{p,L}) \equiv 1$ and still $K_{\overline{g}_{p,L}} \ge 0$. From Lemma 3.1 and (3.4), we have

$$\begin{aligned} \overline{\lambda}_k^{(p)}(\mathbb{S}^m, \overline{g}_{p,L}) &= \overline{\lambda}_k^{(p)}(\mathbb{S}^m, g_{p,L}) \cdot \mathrm{vol}(\mathbb{S}^m, g_{p,L})^{\frac{2}{m}} \\ &\le \frac{k^2\pi^2}{L^2} \cdot (AL+B)^{\frac{2}{m}} \\ &= k^2\pi^2 \cdot \left(\frac{AL+B}{L^m}\right)^{\frac{2}{m}} \longrightarrow 0, \end{aligned} \tag{3.9}$$

and similarly $\lambda_k^{\prime\prime(p)}(\mathbb{S}^m, \overline{g}_{p,L}) \longrightarrow 0$ as $L \longrightarrow \infty$. Thus, we have finished the proof of Theorem 1.1. □

Remark 3.3. *Theorem 1.1 implies that the first positive eigenvalue of the Hodge-Laplacian acting on p-forms cannot be estimated below in terms of dimension, volume and a lower bound of the sectional curvature. For this family $\overline{g}_{p,L}$, the diameter* $\mathrm{diam}(\mathbb{S}^m, \overline{g}_{p,L}) \longrightarrow \infty$ *as $L \to \infty$.*

Lott [Lo04, p.918] conjectured that for given $m \in \mathbb{N}$, $\kappa \in \mathbb{R}$ and $v, D > 0$, there exists a positive constant $C(m,\kappa,v,D) > 0$ such that any connected oriented closed Riemannian manifold (M^m, g) of dimension m with the sectional curvature $K_g \ge \kappa$, the volume $\mathrm{vol}(M,g) \ge v$ *and the diameter* $\mathrm{diam}(M,g) \le D$ *satisfies*

$$\lambda_1^{(p)}(M,g) \ge C(m,\kappa,v,D) > 0.$$

This conjecture is still open. We note that this is a non-collapsing case. In a collapsing case, we do not know anything, but recently Boulanger and Courtois [BC21] proposed a Cheeger constant for coexact 1-forms, whose square gives a lower bound of the first positive eigenvalue of the Hodge-Laplacian acting on 1*-forms for Riemannian manifolds with bounded diameter and bounded sectional curvature.*

4 Lower bounds for the eigenvalues of the Hodge-Laplacian on $\mathbb{S}^m$

We consider lower bounds of the eigenvalues of the Hodge-Laplacian acting on exact q-forms for $1 \leq q \leq m$ for the one-parameter family of volume un-normalized Riemannian metrics $g_{p,L}$ on $\mathbb{S}^m$ constructed in Theorem 1.1.

Theorem 4.1. *Let p be an integer with $1 \leq p \leq m-1$. For the one parameter family of Riemannian metrics $g_{p,L}$ on $\mathbb{S}^m$ constructed in the proof of Theorem 1.1, the eigenvalues of the Hodge-Laplacian acting on exact q-forms for $1 \leq q \leq m$ satisfy the following:*

(1) *For $q \neq 1, p, p+1, m-p-1, m-p, m-1, m$, there exists a positive constant $C > 0$ independent of L such that*

$$\lambda_1'^{(q)}(\mathbb{S}^m, g_{p,L}) \geq C > 0.$$

(2) *For $q = 1, p, p+1, m-p-1, m-p, m-1, m$, there exist positive constants $C_1, C_2 > 0$ independent of L such that for sufficiently large $L > 0$*

$$\lambda_{n_q+1}'^{(q)}(\mathbb{S}^m, g_{p,L}) \geq \frac{1}{C_1 L^2 + C_2}.$$

Here, n_q is given by

$$n_q := \begin{cases} 4 & \text{if } (p,q) = (\frac{m-1}{2}, \frac{m+1}{2}) \text{ and } m \text{ is odd}, \\ 2 & \text{if } q = 1, p+1, m-p, m, \\ & \qquad \text{except for } (p,q) = (\frac{m-1}{2}, \frac{m+1}{2}) \text{ if } m \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

Corollary 4.2. *Let p be an integer with $1 \leq p \leq m-1$. For the one parameter family of the volume normalized Riemannian metrics $\overline{g}_{p,L}$ on $\mathbb{S}^m$, if $q \neq 1, p, p+1, m-p-1, m-p, m-1, m$, we have*

$$\lambda_1^{(q)}(\mathbb{S}^m, \overline{g}_{p,L}) \longrightarrow \infty \quad \textit{as} \quad L \to \infty.$$

Proof of Corollary 4.2. By combining (1) in Theorem 4.1 with the Hodge-duality $\lambda_1''^{(q)} = \lambda_1'^{(m-q)}$, we find that the first positive eigenvalue of the Hodge-Laplacian acting on q-forms for $q \neq 1, p, p+1, m-p-1, m-p, m-1$ have a uniform lower bound in L. That is, there exists a positive constant $C > 0$ independent of L such that

$$\begin{aligned}\lambda_1^{(q)}(\mathbb{S}^m, g_{p,L}) &= \min\{\lambda_1'^{(p)}(\mathbb{S}^m, g_{p,L},\ \lambda_1''^{(p)}(\mathbb{S}^m, g_{p,L})\} \\ &= \min\{\lambda_1'^{(p)}(\mathbb{S}^m, g_{p,L},\ \lambda_1'^{(m-p)}(\mathbb{S}^m, g_{p,L})\} \\ &\geq C > 0.\end{aligned}$$

For the volume normalized metric $\overline{g}_{p,L} = \mathrm{vol}(\mathbb{S}^m, g_{p,L})^{-\frac{2}{m}} g_{p,L}$, in the same way as in (3.9), we have

$$\begin{aligned}\lambda_1^{(q)}(\mathbb{S}^m, \overline{g}_{p,L}) &= \lambda_1^{(q)}(\mathbb{S}^m, g_{p,L}) \cdot \mathrm{vol}(\mathbb{S}^m, g_{p,L})^{\frac{2}{m}} \\ &\geq C(AL+B)^{\frac{2}{m}} \longrightarrow \infty, \quad \text{as } L \to \infty.\end{aligned}$$

□

Remark 4.3. *In the case of $n_q = 2, 4$ for $q = 1, p, p+1, m-p-1, m-p, m-1, m$, we do not know whether or not $\lambda_i^{(q)}(\mathbb{S}^m, g_{p,L})$ for $i = 1, \ldots, n_q$ have positive lower bounds independent of L. A similar problem occurs in [EP17, Remark* 5.7*], p.457.*

We prove Theorem 4.1 in the same way as Gentile and Pagliara [GP95]. In this way, the following result by McGowan [MG93, Lemma 2.3] (see also [GP95], Lemma 1) plays an important rôle. A similar argument was used to prove Corollary 1.5 in [Tak05].

We now denote by $\nu_1'^{(p)}(U, g)$ the first positive eigenvalue of the Hodge-Laplacian acting on exact p-forms on (U, g) with the absolute boundary condition.

Lemma 4.4 (McGowan [MG93])**.** *Let (M^m, g) be a connected oriented closed Riemannian manifold of dimension m. We take a finite open covering $\{U_i\}_{i=1}^K$ of M satisfying $U_i \cap U_j \cap U_k = \emptyset$ and a partition of unity $\{\rho_i\}_{i=1}^K$ subordinated to $\{U_i\}_{i=1}^K$. If we set $n_p := \sum_{i<j} \dim H^{p-1}(U_{ij}; \mathbb{R})$, where $U_{ij} := U_i \cap U_j$, and*

$$C_g(\rho) := \max_{i=1,\ldots,K} \max_{x \in U_i} \left\{ |d\rho_i|_g^2(x) \right\},$$

then we have

$$\lambda_{n_p+1}'^{(p)}(M, g) \geq$$

$$\frac{1}{8 \sum_{i=1}^K \left\{ \frac{1}{\nu_1'^{(p)}(U_i, g)} + \sum_{\substack{j \neq i \\ U_i \cap U_j \neq \emptyset}} \left(\frac{C_g(\rho)}{\nu_1'^{(p-1)}(U_{ij}, g)} + 1 \right) \left(\frac{1}{\nu_1'^{(p)}(U_i, g)} + \frac{1}{\nu_1'^{(p)}(U_j, g)} \right) \right\}}.$$

Proof of Theorem 4.1. We take an open covering $\{U_1, U_2, U_3\}$ of M as follows:

$$\begin{aligned}
U_1 &:= \mathbb{S}^p \times \big([0,3] \times \mathbb{S}^{m-p-1}\big),\\
U_2 &:= \mathbb{S}^p \times \big([2, L+2] \times \mathbb{S}^{m-p-1}\big),\\
U_3 &:= \Big(\mathbb{S}^p \times \big([L+1, L+2] \times \mathbb{S}^{m-p-1}\big)\Big) \cup \Big(\mathbb{D}^{p+1} \times \mathbb{S}^{m-p-1}\Big).
\end{aligned}$$

Then,

$$\begin{aligned}
U_{12} &= U_1 \cap U_2 = \mathbb{S}^p \times \big([2,3] \times \mathbb{S}^{m-p-1}\big),\\
U_{13} &= U_1 \cap U_3 = \emptyset,\\
U_{23} &= U_2 \cap U_3 = \mathbb{S}^p \times \big([L+1, L+2] \times \mathbb{S}^{m-p-1}\big)
\end{aligned}$$

and $U_{123} := U_1 \cap U_2 \cap U_3 = \emptyset$. Since both U_{12} and U_{23} are isometric to $\mathbb{S}^p \times \big([0,1] \times \mathbb{S}^{m-p-1}\big)$, their eigenvalues do not depend on L. Therefore, only $\nu_1'^{(p)}(U_2, g_{p,L})$ depends on L.

We can take a partition of unity $\{\rho_i\}_{i=1,2,3}$ subordinate to this open covering $\{U_1, U_2, U_3\}$ such that the supports of $d\rho_i$ are in U_{12} or U_{23}. Since the C^0-norms of $d\rho_i$ are independent of L, the constant $C_g(\rho)$ is also independent of L.

Now, we compute n_q. Since both U_{12} and U_{23} are homotopy equivalent to $\mathbb{S}^p \times \mathbb{S}^{m-p-1}$, by the Künneth formula, we have

$$\begin{aligned}
n_q &= \dim H^{q-1}(U_{12}; \mathbb{R}) + \dim H^{q-1}(U_{23}; \mathbb{R})\\
&= 2 \dim H^{q-1}(\mathbb{S}^p \times \mathbb{S}^{m-p-1}; \mathbb{R})\\
&= \begin{cases} 4 & \text{if } (p,q) = (\frac{m-1}{2}, \frac{m+1}{2}) \text{ and } m \text{ is odd},\\ 2 & \text{if } q = 1, p+1, m-p, m,\\ & \qquad \text{except for } (p,q) = (\frac{m-1}{2}, \frac{m+1}{2}) \text{ if } m \text{ is odd},\\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Next, we estimate the eigenvalue $\nu_1'^{(q)}(U_2, g_{p,L})$ from below. If 0 is the eigenvalue for s-forms on (N,h), we denote it by $\lambda_0^{(s)}(N,h)$. From the Künneth formula for the eigenvalues of the Hodge-Laplacian (e.g., [GLP99], p.38, Example 1.5.7), we have

$$\begin{aligned}
\nu_1'^{(q)}(U_2, g_{p,L}) \geq \nu_1^{(q)}(U_2, g_{p,L}) &= \nu_1^{(q)}([0,L] \times \mathbb{S}^p \times \mathbb{S}^{m-p-1})\\
&= \min_{\substack{a+b=q\\ i+j\geq 1}} \Big\{\nu_i^{(a)}([0,L]) + \lambda_j^{(b)}(\mathbb{S}^p \times \mathbb{S}^{m-p-1})\Big\}\\
&= \min_{\substack{a+b=q\\ i+j\geq 1}} \Big\{L^{-2}\, \nu_i^{(a)}([0,1]) + \lambda_j^{(b)}(\mathbb{S}^p \times \mathbb{S}^{m-p-1})\Big\}.
\end{aligned} \tag{4.1}$$

To proceed with the calculation, we consider whether or not 0 is the eigenvalue on $[0,1]$ and $\mathbb{S}^p \times \mathbb{S}^{m-p-1}$. By the Hodge theory, this follows from the cohomology groups

$$\begin{aligned} H^a([0,1];\mathbb{R}) &\begin{cases} \neq 0 & \text{if } a=0, \\ =0 & \text{if } a=1, \end{cases} \\ H^b(\mathbb{S}^p\times\mathbb{S}^{m-p-1};\mathbb{R}) &\begin{cases} \neq 0 & \text{if } b=0,p,m-p-1,m-1, \\ =0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.2}$$

If $q=p,m-p-1,m-1$, by (4.2), 0 is the eigenvalue for q-forms on $\mathbb{S}^p\times\mathbb{S}^{m-p-1}$. Hence, we have for sufficiently large L,

$$\begin{aligned} \nu_1'^{(q)}(U_2,g_{p,L}) &\geq \min_{\substack{a+b=q\\ i+j\geq 1}} \left\{ L^{-2}\,\nu_i^{(a)}([0,1]) + \lambda_j^{(b)}(\mathbb{S}^p\times\mathbb{S}^{m-p-1}) \right\} \\ &= \min_{j\geq 0} \Big\{ L^{-2}\,\underbrace{\nu_0^{(0)}([0,1])}_{=0} + \lambda_1^{(q)}(\mathbb{S}^p\times\mathbb{S}^{m-p-1}),\ L^{-2}\nu_1^{(0)}([0,1]) + \underbrace{\lambda_0^{(q)}(\mathbb{S}^p\times\mathbb{S}^{m-p-1})}_{=0}, \\ &\qquad L^{-2}\nu_1^{(1)}([0,1]) + \lambda_j^{(q-1)}(\mathbb{S}^p\times\mathbb{S}^{m-p-1}) \Big\} \\ &\geq L^{-2}\min\Big\{ \nu_1^{(0)}([0,1]),\ \nu_1^{(1)}([0,1]) \Big\}. \end{aligned}$$

Similarly, if $q=1,p+1,m-p,m$, by (4.2), 0 is the eigenvalue for $(q-1)$-forms on $\mathbb{S}^p\times\mathbb{S}^{m-p-1}$. Hence, we have for sufficiently large L,

$$\begin{aligned} \nu_1'^{(q)}(U_2,g_{p,L}) &\geq \min_{\substack{a+b=q\\ i+j\geq 1}} \left\{ L^{-2}\,\nu_i^{(a)}([0,1]) + \lambda_j^{(b)}(\mathbb{S}^p\times\mathbb{S}^{m-p-1}) \right\} \\ &= \min_{j\geq 0} \Big\{ L^{-2}\,\underbrace{\nu_0^{(0)}([0,1])}_{=0} + \lambda_1^{(q)}(\mathbb{S}^p\times\mathbb{S}^{m-p-1}),\ L^{-2}\nu_1^{(0)}([0,1]) + \lambda_j^{(q)}(\mathbb{S}^p\times\mathbb{S}^{m-p-1}), \\ &\qquad L^{-2}\nu_1^{(1)}([0,1]) + \underbrace{\lambda_0^{(q-1)}(\mathbb{S}^p\times\mathbb{S}^{m-p-1})}_{=0} \Big\} \\ &\geq L^{-2}\min\Big\{ \nu_1^{(0)}([0,1]),\ \nu_1^{(1)}([0,1]) \Big\}. \end{aligned}$$

Therefore, if $q=1,p,p+1,m-p-1,m-p,m-1,m$, from Lemma 4.4, there exist positive constants $C_1,C_2>0$ independent of L such that

$$\lambda_{n_q+1}'^{(q)}(\mathbb{S}^m,g_{p,L}) \geq \frac{1}{C_1L^2+C_2}.$$

If $q\neq 1,p,p+1,m-p-1,m-p,m-1,m$, by (4.2), 0 is neither eigenvalue for $(q-1)$-forms nor for q-forms on $\mathbb{S}^p\times\mathbb{S}^{m-p-1}$. Hence, for any $L>0$, we have

$$\begin{aligned} \nu_1'^{(q)}(U_2,g_{p,L}) &\geq \min\Big\{ \lambda_1^{(q)}(\mathbb{S}^p\times\mathbb{S}^{m-p-1}),\ L^{-2}\nu_1^{(0)}([0,1]) + \lambda_1^{(q)}(\mathbb{S}^p\times\mathbb{S}^{m-p-1}), \\ &\qquad L^{-2}\nu_1^{(1)}([0,1]) + \lambda_1^{(q-1)}(\mathbb{S}^p\times\mathbb{S}^{m-p-1}) \Big\} \\ &\geq \min\Big\{ \lambda_1^{(q)}(\mathbb{S}^p\times\mathbb{S}^{m-p-1}),\ \lambda_1^{(q-1)}(\mathbb{S}^p\times\mathbb{S}^{m-p-1}) \Big\} > 0. \end{aligned}$$

In this case, we have $n_q = 0$. Thus, we obtain a lower bound $C > 0$ independent of L such that

$$\lambda_1'^{(q)}(\mathbb{S}^m, g_{p,L}) \geq C > 0.$$

$\square$

5 General manifold

5.1 Gluing theorem

To prove Theorem 1.2, we need a gluing theorem for the eigenvalues on a connected sum. The gluing theorem we use here is obtained from the convergence of the eigenvalues of the Laplacian, when one side of a connected sum of two closed Riemannian manifolds collapses to a point. We call it collapsing of connected sums. This was studied in the case of the Laplacian acting on functions in [Tak02], and in the case of the Hodge-Laplacian acting on p-forms in [Tak03], [AT12]. We recall it.

Let (M_i, g_i), $i = 1, 2$, be connected oriented closed Riemannian manifolds of the same dimension m ($m \geq 2$). For simplicity, we suppose that each metric g_i is flat on the geodesic ball $B(x_i, r_i)$ with the radius $r_i > 0$ centered at $x_i \in M_i$, where r_i is smaller than the injectivity radius of (M_i, g_i). By changing the scale of g_2, we may suppose $r_2 = 2$. Set $M_i(r) := M_i \setminus B(x_i, r)$. For any $\varepsilon > 0$ with $0 < \varepsilon < \min\{r_1, 1\}$, since both boundaries of $\partial(M_1(\varepsilon), g_1)$ and $\partial(M_2(1), \varepsilon^2 g_2)$ are isometric to the $(m-1)$-dimensional sphere of radius ε in $\mathbb{R}^m$, we glue $(M_1(\varepsilon), g_1)$ to $(M_2(1), \varepsilon^2 g_2)$ along their boundaries. After deforming g_2 on a neighborhood of $\partial M_2(1)$, we obtain the new closed **smooth** Riemannian manifold

$$(M, g_\varepsilon) := (M_1(\varepsilon), g_1) \cup_\partial (M_2(1), \varepsilon^2 g_2). \tag{5.1}$$

If we choose suitable orientations of M_1 and M_2, M is also oriented.

From the construction of (M, g_ε), it is easy to see that

$$\lim_{\varepsilon \to 0} \mathrm{vol}(M, g_\varepsilon) = \mathrm{vol}(M_1, g_1). \tag{5.2}$$

In our previous works [AT12], [Tak02], we have the following convergence theorem for the eigenvalues of the Hodge-Laplacian acting on exact and co-exact p-forms. In fact, by considering the convergence of eigenforms, we find that all the eigenvalues for exact and co-exact forms still converge.

Lemma 5.1. *For all $k = 1, 2, \ldots,$ we have*

$$\lim_{\varepsilon \to 0} \lambda_k'^{(p)}(M, g_\varepsilon) = \lambda_k'^{(p)}(M_1, g_1),$$
$$\lim_{\varepsilon \to 0} \lambda_k''^{(p)}(M, g_\varepsilon) = \lambda_k''^{(p)}(M_1, g_1).$$

We also have the convergence of the eigenvalues of the rough Laplacian acting on p-forms.

Theorem 5.2. *For all $k = 1, 2, \ldots,$ we have*

$$\lim_{\varepsilon\to 0} \overline{\lambda}_k^{(p)}(M, g_\varepsilon) = \overline{\lambda}_k^{(p)}(M_1, g_1).$$

In fact, in the same way as the proof of Theorem 4.4 in [Tak03], p.21, we see the upper bound for the eigenvalues of the rough Laplacian acting on p-forms.

Lemma 5.3. *For all $k = 1, 2, \ldots,$ we have*

$$\limsup_{\varepsilon\to 0} \overline{\lambda}_k^{(p)}(M, g_\varepsilon) \leq \overline{\lambda}_k^{(p)}(M_1, g_1).$$

On the other hand, we will give the proof of the lower bound for the eigenvalues of the rough Laplacian acting on p-forms in Section 6, Appendix.

Proof of Lemma 5.3. To prove Lemma 5.3, we use a standard cut-off argument for the min-max principle for eigenvalues of the rough Laplacian. Let $\{\varphi_1, \ldots, \varphi_k\}$ be an $L^2(M_1, g_1)$-orthonormal system of the eigen p-forms of the rough Laplacian on (M_1, g_1) associated with the eigenvalue $\overline{\lambda}_j^{(p)}(M_1, g_1)$ for $j = 1, 2, \ldots, k$. We take a cut-off function $\chi_\varepsilon(r)$ on M_1 defined as

$$\chi_\varepsilon(r) := \begin{cases} 0 & (0 \leq r \leq \varepsilon), \\ -\dfrac{2}{\log\varepsilon}\log\left(\dfrac{r}{\varepsilon}\right) & (\varepsilon \leq r \leq \sqrt{\varepsilon}), \\ 1 & (\sqrt{\varepsilon} \leq r), \end{cases} \tag{5.3}$$

where r is the Riemannian distance from $x_1 \in M_1$ with respect to g_1. We take a linear subspace E_ε in $H^1(\Lambda^p M, g_1)$ spanned by $\{\chi_\varepsilon\varphi_1, \ldots, \chi_\varepsilon\varphi_k\}$, and we see $\dim E_\varepsilon = k$. If we take this subspace E_ε as a test k-dimensional subspace for the min-max principle for the eigenvalues of the rough Laplacian acting on p-forms, we obtain

$$\overline{\lambda}_k^{(p)}(M, g_\varepsilon) \leq \sup_{\varphi_\varepsilon \neq 0 \in E_\varepsilon} \left\{ \frac{\| \nabla \varphi_\varepsilon \|^2_{L^2(M, g_\varepsilon)}}{\| \varphi_\varepsilon \|^2_{L^2(M, g_\varepsilon)}} \right\} \leq \overline{\lambda}_k^{(p)}(M_1, g_1) + \delta(\varepsilon), \tag{5.4}$$

where $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. For details, see the proof of Theorem 4.4 in [Tak03], p.21. □

5.2 Proof of Theorem 1.2

We prove Theorem 1.2 for a general manifold M. The main idea is to perform a connected sum of this manifold M and the sphere $\mathbb{S}^m$ equipped with the Riemannian metric constructed in Theorem 1.1.

Proof of Theorem 1.2. Let M^m be a connected oriented closed C^∞-manifold of dimension $m \geq 2$. We fixed a degree p with $1 \leq p \leq m$. We take any smooth Riemannian metric g_2 on M such that g_2 is flat on the geodesic ball $B(x_2, 2)$ with the radius 2 centered at $x_2 \in M$.

For any $\eta > 0$ and any index $k \geq 1$, from Theorem 1.1, there exists some $L_p > 0$ such that for all $L > L_p$,

$$\overline{\lambda}_k^{(p)}(\mathbb{S}^m, \overline{g}_{p,L}) < \frac{\eta}{2} \quad \text{and} \quad \lambda_k''^{(p)}(\mathbb{S}^m, \overline{g}_{p,L}) < \frac{\eta}{2}, \tag{5.5}$$

where $\overline{g}_{p,L}$ is the volume normalized Riemannian metric on $\mathbb{S}^m$ constructed in Theorem 1.1. By the continuity of the eigenvalues of the rough and Hodge Laplacians acting on co-exact forms in the C^0-topology of Riemannian metrics (see [Do82], [CPR01, p.297]), after C^0-perturbation of $\overline{g}_{p,L}$, we may suppose that $\overline{g}_{p,L}$ is flat on a small geodesic ball $B(x_1, r_1)$ with the radius $r_1 > 0$ centered at $x_1 \in \mathbb{S}^m$ such that (5.5) still holds.

Now, we set $(M_1, g_1) := (\mathbb{S}^m, \overline{g}_{p,L})$ and $(M_2, g_2) := (M, g)$. By the construction of collapsing of the connected sum (M, g_ε) from (M_1, g_1) and (M_2, g_2) as in Subsection 5.1 (5.1), we obtain a family of Riemannian metrics g_ε on the connected sum $M \cong \mathbb{S}^m \sharp M$ such that

$$(M, g_\varepsilon) := (\mathbb{S}^m(\varepsilon), \overline{g}_{p,L}) \cup_\partial (M(1), \varepsilon^2 g).$$

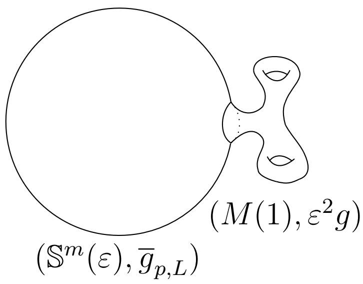

Figure 4: $(M, g_{p,\varepsilon})$

From Lemmas 5.1 and 5.3, there exists some $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$,

$$\begin{aligned} 0 \leq \overline{\lambda}_k^{(p)}(M, g_\varepsilon) &\leq \overline{\lambda}_k^{(p)}(\mathbb{S}^m, \overline{g}_{p,L}) + \frac{\eta}{2}, \\ 0 < \lambda_k''^{(p)}(M, g_\varepsilon) &\leq \lambda_k''^{(p)}(\mathbb{S}^m, \overline{g}_{p,L}) + \frac{\eta}{2}. \end{aligned} \tag{5.6}$$

By substituting (5.5) to (5.6), we obtain

$$0 \leq \overline{\lambda}_k^{(p)}(M, g_\varepsilon) < \eta \quad \text{and} \quad 0 < \lambda_k''^{(p)}(M, g_\varepsilon) < \eta.$$

Finally, we normalize a Riemannian metric g_ε on M. If we set a new Riemannian metric

$$\overline{g}_\varepsilon := \mathrm{vol}(M, g_\varepsilon)^{-\frac{2}{m}} g_\varepsilon,$$

then $\mathrm{vol}(M, \overline{g}_\varepsilon) \equiv 1$. From the volume convergence (5.2), there exists smaller $\varepsilon_1 \leq \varepsilon_0$, if necessary, such that for all $\varepsilon < \varepsilon_1$, $\mathrm{vol}(M, g_\varepsilon) \leq 2$. Therefore, we have

$$\overline{\lambda}_k^{(p)}(M, \overline{g}_\varepsilon) = \overline{\lambda}_k^{(p)}(M, g_\varepsilon) \cdot \mathrm{vol}(M, g_\varepsilon)^{\frac{2}{m}} < \eta \cdot 2^{\frac{2}{m}},$$

and similarly $\lambda_k''^{(p)}(M, \overline{g}_\varepsilon) < \eta \cdot 2^{\frac{2}{m}}$. Since $\eta > 0$ is arbitrary, we have finished the proof of Theorem 1.2. □

5.3 Proof of Theorem 1.3

Proof of Theorem 1.3. We prove it in the same way as in the proof of Theorem 1.2. Let M^m be a connected oriented closed C^∞-manifold of dimension $m \geq 2$. We take a smooth Riemannian metric g on M such that g is flat on $(m-1)$ disjoint geodesic balls $B(x_i, 2)$ for $i = 1, 2, \ldots, m-1$ with the radius 2, centered at distinct $(m-1)$ points $x_i \in M$. By rescaling of g, we can do this.

For any $\eta > 0$ and any index $k \geq 1$, as in the proof of Theorem 1.2, we can take a positive number $L_p > 0$ safisfying that the inequalities (5.5) hold. For all $L > \max_{p=1,2,\ldots,m-1} L_p$, the inequalities (5.5) hold uniformly for all degrees $p = 1, 2, \ldots, m-1$.

We consider now the $(m-1)$ spheres $(\mathbb{S}^m, \overline{g}_{1,L})$, $(\mathbb{S}^m, \overline{g}_{2,L})$, $\ldots$, $(\mathbb{S}^m, \overline{g}_{m-1,L})$ and fix a point $x_0 \in \mathbb{S}^m$, it defines on each of them a point $x_{0,p}$. From the continuity of the eigenvalues of the rough and Hodge Laplacians acting on co-exact forms in the C^0-topology of Riemannian metrics, we may suppose that each metric $\overline{g}_{p,L}$ on $\mathbb{S}^m$ for $p = 1, 2, \ldots, m-1$ is flat on each geodesic ball $B(x_{0,p}, r_1)$ centered at $x_{0,p}$ with radius $r_1 > 0$ small enough, such that the inequalities (5.5) still hold.

We now perform the connected sum of M and these $(m-1)$ spheres $(\mathbb{S}^m, \overline{g}_{1,L})$, $(\mathbb{S}^m, \overline{g}_{2,L})$, $\ldots$, $(\mathbb{S}^m, \overline{g}_{m-1,L})$, where $x_p \in M$ is related to $x_{0,p} \in \mathbb{S}^m$ equipped with the metric $\overline{g}_{p,L}$ (See Figure 5). We denote the resulting manifold by

$$\begin{aligned}(M, g_\varepsilon) := \Big((\mathbb{S}^m(\varepsilon), \overline{g}_{1,L}) \sqcup (\mathbb{S}^m(\varepsilon), \overline{g}_{2,L}) \sqcup \cdots \sqcup (\mathbb{S}^m(\varepsilon), \overline{g}_{m-1,L})\Big)& \\ \cup_\partial \left(M \setminus \sqcup_{p=1}^{m-1} B(x_p, 1), \varepsilon^2 g\right)&,\end{aligned}$$

where $\mathbb{S}^m(\varepsilon) := \mathbb{S}^m \setminus B(x_0, \varepsilon)$. After smoothing the Riemannian metric g on each neighborhood of $\partial(M \setminus B(x_p, 1))$ for $p = 1, 2, \ldots, m-1$, we obtain a family of closed **smooth** Riemannian manifolds (M, g_ε) for $\varepsilon > 0$.

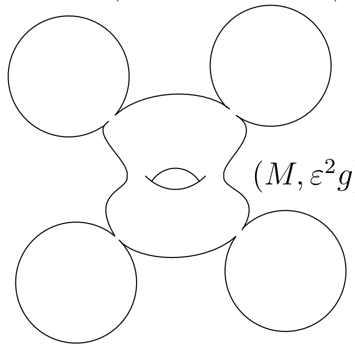

Figure 5: (M, g_ε)

For this (M, g_ε), we find that the same statement as in Lemma 5.3 holds for the Hodge-Laplacian and rough Laplacian. In fact, we take the same cut-off function as in (5.3) for each component, and estimate the Rayleigh-Ritz quotients from above. Since a contribution from each cut-off function is within its own component, we obtain the same estimate as in (5.4).

Therefore, there exists some $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ and $p = 1, 2, \ldots, m-$
1

$$\begin{aligned} 0 \leq \overline{\lambda}_k^{(p)}(M, g_\varepsilon) &\leq \overline{\lambda}_k^{(p)}(\mathbb{S}^m, \overline{g}_{p,L}) + \frac{\eta}{2}, \\ 0 < \lambda_k''^{(p)}(M, g_\varepsilon) &\leq \lambda_k''^{(p)}(\mathbb{S}^m, \overline{g}_{p,L}) + \frac{\eta}{2}. \end{aligned} \tag{5.7}$$

By substituting the same inequalities as in (5.5) to (5.7), for small $\varepsilon > 0$, we obtain

$$0 \leq \overline{\lambda}_k^{(p)}(M, g_\varepsilon) < \eta \quad \text{and} \quad 0 < \lambda_k''^{(p)}(M, g_\varepsilon) < \eta$$

for all $p = 1, 2, \ldots, m-1$. Since $\mathrm{vol}(\mathbb{S}^m, \overline{g}_{p,L})$ is alomost 1 (because of small perturbation around one point), we find

$$\mathrm{vol}(M, g_\varepsilon) \leq \sum_{p=1}^{m-1} \mathrm{vol}(\mathbb{S}^m, \overline{g}_{p,L}) + \mathrm{vol}(M, \varepsilon^2 g) \leq m.$$

After normalization of Riemannian metric g_ε on M, we can find a Riemannian metric $\overline{g}_\varepsilon$ on M such that $\mathrm{vol}(M, \overline{g}_\varepsilon) \equiv 1$ and

$$\overline{\lambda}_k^{(p)}(M, \overline{g}_\varepsilon) < \eta \cdot m^{\frac{2}{m}} \quad \text{and} \quad \lambda_k''^{(p)}(M, \overline{g}_\varepsilon) < \eta \cdot m^{\frac{2}{m}}$$

for all $p = 1, 2, \ldots, m-1$.

Thus, we have finished the proof of Theorem 1.3. □

6 Appendix: Convergence of the eigenvalues of the rough Laplacian

We study here the convergence of the eigenvalues of the rough Laplacian acting on p-forms, when one side of a connected sum of two closed Riemannian manifolds collapses to a point. Our setting is the same as in the beginning of Subsection 5.1.

Theorem 6.1. *For all p with $0 \le p \le m$ and for all $k = 1, 2, \dots,$ we have*

$$\lim_{\varepsilon\to 0} \overline{\lambda}_k^{(p)}(M, g_\varepsilon) = \overline{\lambda}_k^{(p)}(M_1, g_1).$$

To prove this Theorem 6.1, we follow the schema of [AT12] which dealt with the Hodge Laplacian, but with less difficulties: when working with the rough Laplacian the related quadratic form is exactly the one involved in the H^1-norm and we are rather in the situation of [Tak02] which dealt with functions. Nevertheless notice that we use here cut-off functions, while [Tak02] used a technique by means of the harmonic extension.

Let φ_ε be a normalized eigen p-form of the rough Laplacian associated with the eigenvalue $\overline{\lambda}_k^{(p)}(\varepsilon) = \overline{\lambda}_k^{(p)}(M, g_\varepsilon)$:

$$\overline{\Delta}\,\varphi_\varepsilon = \overline{\lambda}_k^{(p)}(\varepsilon)\,\varphi_\varepsilon \quad \text{and} \quad \|\,\varphi_\varepsilon\,\|_{L^2(M,g_\varepsilon)} \equiv 1.$$

By Lemma 5.3, we already know that the family $\{\overline{\lambda}_k^{(p)}(\varepsilon)\}_{\varepsilon>0}$ is bounded. So, we set $\overline{\lambda}_k^{(p)} = \liminf_{\varepsilon\to 0} \overline{\lambda}_k^{(p)}(M, g_\varepsilon)$, and decompose the eigen p-form φ_ε on the connected sum into

$$\varphi_{j,\varepsilon} = \left(\varphi_\varepsilon^1,\, \varepsilon^{p-\frac{m}{2}}\,\varphi_\varepsilon^2\right) \text{ with } \varphi_\varepsilon^1 \in H^1(\Lambda^p M_1(\varepsilon), g_1),\ \varphi_\varepsilon^2 \in H^1(\Lambda^p M_2(1), g_2).$$

Then, these satisfy

$$\begin{aligned} &\|\,\varphi_\varepsilon^1\,\|^2_{L^2(M_1(\varepsilon),g_1)} + \|\,\varphi_\varepsilon^2\,\|^2_{L^2(M_2(1),g_2)} \equiv 1, \\ &\varphi_\varepsilon^2 = \varepsilon^{\frac{m}{2}-p}\,\varphi_\varepsilon^1 \text{ on the boundary.} \end{aligned} \tag{6.1}$$

Furthermore, since φ_ε is a normalized eigenform, we have

$$\begin{aligned} \overline{\lambda}_k^{(p)}(M, g_\varepsilon) &= \int_{M_1(\varepsilon)} |\nabla\,\varphi_\varepsilon^1\,|^2 d\mu_{g_1} + \frac{1}{\varepsilon^2}\int_{M_2(1)} |\nabla\,\varphi_\varepsilon^2\,|^2 d\mu_{g_2} \\ &= \|\nabla\,\varphi_\varepsilon^1\,\|^2_{L^2(M_1(\varepsilon),g_1)} + \frac{1}{\varepsilon^2}\|\nabla\,\varphi_\varepsilon^2\,\|^2_{L^2(M_2(1),g_2)}. \end{aligned} \tag{6.2}$$

From (6.1) and (6.2), it follows that

$$\begin{aligned} \|\,\varphi_\varepsilon^2\,\|^2_{H^1(M_2(1),g_2)} &= \|\,\varphi_\varepsilon^2\,\|^2_{L^2(M_2(1),g_2)} + \|\nabla\,\varphi_\varepsilon^2\,\|^2_{L^2(M_2(1),g_2)} \\ &\le 1 + \varepsilon^2\overline{\lambda}_k^{(p)}(M, g_\varepsilon). \end{aligned}$$

By Lemma 5.3, we see that the family $\{\varphi_\varepsilon^2\}_{\varepsilon>0}$ is bounded in $H^1(\Lambda^p M_2(1), g_2)$. Since $M_2(1)$ is compact, there exists a subsequence $\{\varphi_{\varepsilon_i}^2\}_{i=1}^\infty$ which converges weakly to φ^2 in $H^1(\Lambda^p M_2(1), g_2)$ and strongly in $L^2(\Lambda^p M_2(1), g_2)$.

Lemma 6.2. *The sequence $\{\varphi^2_{\varepsilon_i}\}$ converges strongly to φ^2 in $H^1(\Lambda^p M_2(1), g_2)$, and the limit φ^2 is parallel on $(M_2(1), g_2)$.*

Proof. From the lower semi-continuity of the weak limit and Lemma 5.3, it follows that

$$\begin{aligned}\|\nabla \varphi^2\|^2_{L^2(M_2(1),g_2)} &\le \liminf_{\varepsilon\to 0} \|\nabla \varphi^2_\varepsilon\|^2_{L^2(M_2(1),g_2)} \\ &\le \liminf_{\varepsilon\to 0} \varepsilon^2 \overline{\lambda}^{(p)}_k(M, g_\varepsilon) = 0,\end{aligned}$$

that is, φ^2 is parallel on $(M_2(1), g_2)$. Therefore, we have

$$\begin{aligned}\|\varphi^2_{\varepsilon_i} - \varphi^2\|^2_{H^1(M_2(1),g_2)} &= \|\varphi^2_{\varepsilon_i} - \varphi^2\|^2_{L^2(M_2(1),g_2)} + \|\nabla \varphi^2_{\varepsilon_i}\|^2_{L^2(M_2(1),g_2)} \\ &\longrightarrow 0 \quad (i \to \infty).\end{aligned}$$

□

The following boundary value estimate is crucial in our argument.

Lemma 6.3. *There exists a constant $C > 0$ such that for any r with $\varepsilon \le r \le r_1$, $\varphi \in H^1(M_1(r), g_1)$ satisfies*

$$\|\varphi\restriction_{\partial M_1(r)}\|^2_{L^2(\partial M_1(r),g_{1,\partial})} \le \begin{cases} \dfrac{Cr}{m-2}\|\varphi\|^2_{H^1(M_1(r),g_1)} & \text{if } m \ge 3, \\ Cr\big|\log r\big|\,\|\varphi\|^2_{H^1(M_1(r),g_1)} & \text{if } m = 2. \end{cases}$$

Note that since $\varphi \in H^1(M_1(r), g_1)$, the boundary value $\varphi\restriction_{\partial M_1(r)}$ on $\partial M_1(r)$ is considered in the sense of the trace operator $H^1(\Lambda^p M_1(r), g_1) \longrightarrow L^2(\Lambda^p M_1(r), g_1)\restriction_{\partial M_1(r)}$.

Proof. We may assume that φ is smooth. By using the polar coordinates (r, θ) on the geodesic ball $B(x_1, r_1)$, we denote a p-form $\varphi = \alpha + dr \wedge \beta$ and the metric $g_1 = dr^2 + r^2 h$, where h is the standard metric of $\mathbb{S}^{m-1}$. Then, the point-wise norm of φ at (r, θ) is expressed as

$$|\varphi(r,\theta)|^2_{g_1} = r^{-2p}|\alpha(r,\theta)|^2_h + r^{-2p+2}|\beta(r,\theta)|^2_h.$$

We take a cut-off function χ on the ball $B(x_1, r_1)$ satisfying $\chi(s) = 1$ for $s \le r$ and $\chi(r_1) = 0$ (We may take $r_1 < 1$, if necessary). From the Kato inequality $|\nabla|\varphi|| \le |\nabla \varphi|$ and the Schwarz inequality, it follows that

$$\begin{aligned}|\varphi(r,\theta)|_{g_1} &= \int_r^{r_1} \partial_s\big(|\chi\varphi(s,\theta)|_{g_1}\big)\,ds \le \int_r^{r_1} |\nabla(\chi\varphi)(s,\theta)|_{g_1}\,ds \\ &\le \sqrt{\int_r^{r_1} s^{1-m}\,ds} \cdot \sqrt{\int_r^{r_1} |\nabla(\chi\varphi)(s,\theta)|^2_{g_1} s^{m-1}\,ds}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\| \varphi \restriction_{\partial M_1(r)} \|^2_{L^2(\partial M_1(r), g_{1,\partial})} &= \int_{\partial M_1(r)} |\varphi(r,\theta)|^2_{g_1}\, d\mu_{r^2h} = r^{m-1} \int_{\mathbb{S}^{m-1}} |\varphi(r,\theta)|^2_{g_1}\, d\mu_h \\
&\leq r^{m-1} \int_r^{r_1} s^{1-m}\, ds \cdot \int_r^{r_1} \int_{\mathbb{S}^{m-1}} |\nabla(\chi\,\varphi)(s,\theta)|^2_{g_1} s^{m-1}\, ds\, d\mu_h \\
&= r^{m-1} \int_r^{r_1} s^{1-m}\, ds \cdot \int_r^{r_1} \int_{\mathbb{S}^{m-1}} |\nabla\chi \otimes \varphi + \chi\nabla\, \varphi\, |^2_{g_1} s^{m-1}\, ds\, d\mu_h \\
&\leq C\, r^{m-1} \int_r^{r_1} s^{1-m}\, ds \cdot \|\, \varphi\, \|^2_{H^1(M_1(r), g_1)},
\end{aligned}$$

where C is a positive constant depending only on χ and $\nabla\chi$. By combining this with

$$\int_r^{r_1} s^{1-m}\, ds \leq \begin{cases} \dfrac{r^{2-m}}{m-2} & \text{if } m \geq 3, \\ \big|\log r\big| & \text{if } m = 2, \end{cases}$$

we obtain the boundary value estimate. □

Lemma 6.4. *The limit $\varphi^2 = 0$ a.e. $(M_2(1), g_2)$.*

Proof. Since φ^2 is parallel, it is sufficient to prove that the boundary value of φ^2 to $\partial M_2(1)$ in the sense of the trace is zero. Since the trace operator is continuous and $\varphi^2_{\varepsilon_i} \longrightarrow \varphi^2$ strongly in $H^1(M_2(1), g_2)$, we have

$$\|\, \varphi^2_{\varepsilon_i} \restriction_{\partial M_2(1)} - \varphi^2 \restriction_{\partial M_2(1)} \|^2_{L^2(\partial M_2(1), g_{2,\partial})} \leq C \|\, \varphi^2_{\varepsilon_i} - \varphi^2\, \|^2_{H^1(M_2(1), g_2)} \to 0 \quad (i \to \infty).$$

Thus, we see the norm convergence:

$$\|\, \varphi^2 \restriction_{\partial M_2(1)} \|_{L^2(\partial M_2(1), g_{2,\partial})} = \lim_{i\to\infty} \|\, \varphi^2_{\varepsilon_i} \restriction_{\partial M_2(1)} \|_{L^2(\partial M_2(1), g_{2,\partial})}.$$

From the gluing condition (6.1) at the boundary, we have

$$\begin{aligned}
\|\, \varphi^2_{\varepsilon_i} \restriction_{\partial M_2(1)} \|^2_{L^2(\partial M_2(1), g_{2,\partial})} &= \int_{\partial M_2(1)} |\, \varphi^2_{\varepsilon_i} \restriction_{\partial M_2(1)} |^2_{g_2}\, d\mu_h \\
&= \int_{\partial M_1(1)} |\varepsilon_i^{\frac{m}{2}-p}\, \varphi^1_{\varepsilon_i} \restriction_{\partial M_1(\varepsilon_i)} |^2_{g_2}\, d\mu_h \quad \text{(by (6.1))} \\
&= \varepsilon_i \int_{\partial M_1(\varepsilon_i)} \varepsilon_i^{-2p} |\, \varphi^1_{\varepsilon_i} \restriction_{\partial M_1(\varepsilon_i)} |^2_{g_2}\, d\mu_{\varepsilon_i^2 h} \\
&= \varepsilon_i \int_{\partial M_1(\varepsilon_i)} |\, \varphi^1_{\varepsilon_i} \restriction_{\partial M_1(\varepsilon_i)} |^2_{g_1}\, d\mu_{\varepsilon_i^2 h} \\
&= \varepsilon_i \|\, \varphi^1_{\varepsilon_i} \restriction_{\partial M_1(\varepsilon_i)} \|^2_{L^2(\partial M_1(\varepsilon_i), g_{1,\partial})}.
\end{aligned}$$

By Lemma 6.3, we find that the boundary value of φ^2 is zero. Therefore, the limit φ^2 must be zero. □

We take again the cut-off function χ_ε on M_1 as in (5.3), and set

$$\psi_\varepsilon := \chi_\varepsilon \varphi^1_\varepsilon \quad \text{on } M_1. \tag{6.3}$$

Lemma 6.5. *The family $\{\psi_\varepsilon\}_{\varepsilon>0}$ is bounded in $H^1(\Lambda^p M_1, g_1)$.*

Proof. It is easy to see that the L^2-norm of ψ_ε is bounded by 1. Now, we have

$$\begin{aligned}
\int_{M_1} |\nabla(\chi_\varepsilon \varphi^1_\varepsilon)|^2_{g_1} d\mu_{g_1} &= \int_{M_1} |\nabla\chi_\varepsilon \otimes \varphi^1_\varepsilon + \chi_\varepsilon \nabla \varphi^1_\varepsilon|^2_{g_1} d\mu_{g_1} \\
&\leq 2\Big(\frac{2}{\log\varepsilon}\Big)^2 \int_\varepsilon^{\sqrt{\varepsilon}} \int_{\mathbb{S}^{m-1}} |\varphi^1_\varepsilon(r,\theta)|^2_{g_1} r^{m-3}\, dr\, d\mu_h + 2\int_{M_1(\varepsilon)} |\nabla \varphi^1_\varepsilon|^2_{g_1} d\mu_{g_1} \\
&= \frac{8}{|\log\varepsilon|^2} \int_\varepsilon^{\sqrt{\varepsilon}} \|\varphi^1_\varepsilon\|^2_{L^2(\partial M_1(r), g_{1,\partial})} r^{-2}\, dr + 2\|\nabla \varphi^1_\varepsilon\|^2_{L^2(M_1(\varepsilon), g_1)}.
\end{aligned}$$

For the first term, by applying Lemma 6.3, we have, if $m \geq 3$,

$$\begin{aligned}
\frac{8}{|\log\varepsilon|^2} \int_\varepsilon^{\sqrt{\varepsilon}} \|\varphi^1_\varepsilon\|^2_{L^2(\partial M_1(r), g_{1,\partial})} r^{-2}\, dr &\leq \frac{C}{m-2} \cdot \frac{8}{|\log\varepsilon|^2} \|\varphi^1_\varepsilon\|^2_{H^1(M_1(r), g_1)} \int_\varepsilon^{\sqrt{\varepsilon}} r^{-1}\, dr \\
&\leq \frac{C}{m-2} \cdot \frac{4}{|\log\varepsilon|} \|\varphi^1_\varepsilon\|^2_{H^1(M_1(r), g_1)},
\end{aligned}$$

and if $m = 2$,

$$\begin{aligned}
\frac{8}{|\log\varepsilon|^2} \int_\varepsilon^{\sqrt{\varepsilon}} \|\varphi^1_\varepsilon\|^2_{L^2(\partial M_1(r), g_{1,\partial})} r^{-2}\, dr &\leq \frac{8C}{|\log\varepsilon|^2} \|\varphi^1_\varepsilon\|^2_{H^1(M_1(r), g_1)} \int_\varepsilon^{\sqrt{\varepsilon}} |\log r|\, r^{-1}\, dr \\
&\leq \frac{8C}{|\log\varepsilon|} \|\varphi^1_\varepsilon\|^2_{H^1(M_1(r), g_1)} \int_\varepsilon^{\sqrt{\varepsilon}} r^{-1} dr \\
&= 4C\, \|\varphi^1_\varepsilon\|^2_{H^1(M_1(r), g_1)}.
\end{aligned}$$

For the second term, we have

$$\|\nabla \varphi_\varepsilon\|^2_{L^2(M_1(\varepsilon), g_1)} \leq \|\nabla \varphi_\varepsilon\|^2_{L^2(M, g_\varepsilon)} = \overline{\lambda}^{(p)}_k(M, g_\varepsilon),$$

which is uniformly bounded by Lemma 5.3.

Therefore, we find that $\{\psi_\varepsilon\}_{\varepsilon>0}$ is bounded in $H^1(\Lambda^p M_1, g_1)$. □

The following lemma is obtained from the same method as in [AT12, Corollary 15], p.1732.

Lemma 6.6. *We can extract a subsequence $\{\psi_{\varepsilon_i}\}$ which converges weakly to ψ in $H^1(\Lambda^p M_1, g_1)$ and strongly in $L^2(\Lambda^p M_1, g_1)$ such that*

$$\overline{\Delta}_{M_1}\psi = \overline{\lambda}^{(p)}_k \psi \quad \textit{and} \quad \|\psi\|_{L^2(M_1, g_1)} = 1.$$

Proof. From Lemma 6.5, a family $\{\psi_\varepsilon\}$ is uniformly bounded in $H^1(\Lambda^p M_1, g_1)$. By the weak compactness for a Hilbert space and the Rellich-Kondrachov theorem, there exist a subsequence $\{\psi_{\varepsilon_i}\}_i$ and the limit $\psi \in H^1(\Lambda^p M_1, g_1)$ such that $\psi_{\varepsilon_i} \to \psi$ weakly in $H^1(\Lambda^p M_1, g_1)$ and strongly in $L^2(\Lambda^p M_1, g_1)$ as $i \to \infty$.

For any smooth p-form $\omega \in \Omega_0^p(M_1 \setminus \{x_1\})$, there exists $\varepsilon_0 > 0$ such that the support of ω is in $M_1 \setminus B(x_1, 2\sqrt{\varepsilon_0})$. So on this support we have $\psi_{\varepsilon_i} = \varphi^1_{\varepsilon_i} = \varphi_{\varepsilon_i}$ as far as $\varepsilon_i < \varepsilon_0$. We label with $(\star)$ when we use this fact. By Lemma 6.4, we have

$$\begin{aligned}
(\psi, \overline{\Delta}_{g_1}\omega)_{L^2(M_1,g_1)} &= \lim_{i\to\infty} (\psi_{\varepsilon_i}, \overline{\Delta}_{g_1}\omega)_{L^2(M_1,g_1)} \\
&\underset{(\star)}{=} \lim_{i\to\infty} (\varphi_{\varepsilon_i}, \overline{\Delta}_{g_{\varepsilon_i}}\omega)_{L^2(M_1(\varepsilon_i),g_1)} \\
&= \lim_{i\to\infty} (\varphi_{\varepsilon_i}, \overline{\Delta}_{g_{\varepsilon_i}}\omega)_{L^2(M,g_{\varepsilon_i})} \quad \text{(by Lemma 6.4)} \\
&= \lim_{i\to\infty} \overline{\lambda}_k^{(p)}(M, g_{\varepsilon_i})(\varphi_{\varepsilon_i}, \omega)_{L^2(M,g_{\varepsilon_i})} = \overline{\lambda}_k^{(p)} \lim_{i\to\infty} (\varphi^1_{\varepsilon_i}, \omega)_{L^2(M_1(\varepsilon_i),g_1)} \\
&\underset{(\star)}{=} \overline{\lambda}_k^{(p)} \lim_{i\to\infty} (\psi_{\varepsilon_i}, \omega)_{L^2(M_1,g_1)} = \overline{\lambda}_k^{(p)} (\psi, \omega)_{L^2(M_1,g_1)}.
\end{aligned}$$

Since $m \geq 2$, $\Omega_0^p(M_1 \setminus \{x_1\})$ is dense in $H^1(\Lambda^p M_1, g_1)$, and we conclude that

$$\overline{\Delta}_{g_1}\psi = \overline{\lambda}_k^{(p)}\psi \quad \text{weakly.}$$

Furthermore, by the regularity theorem of weak solutions to elliptic equations, the limit ψ in fact is a smooth p-form on M_1.

Next, from the normalization $\| \varphi_{\varepsilon_i} \|_{L^2(M,g_{\varepsilon_i})} \equiv 1$ and Lemma 6.4, we have $\|\psi\|_{L^2(M_1,g_1)} = 1$. Hence, the limit ψ is a non-zero smooth eigenform on (M_1, g_1) with the eigenvalue $\overline{\lambda}_k^{(p)}$. □

To complete the proof of Theorem 6.1, we have only to prove the following lemma.

Lemma 6.7. *Let $\{\varphi_{1,\varepsilon_i}, \dots, \varphi_{k,\varepsilon_i}\}$ be $L^2(M, g_{\varepsilon_i})$-orthonormal eigenforms on (M, g_ε) associated with the eigenvalues $\overline{\lambda}_1^{(p)}(M, g_{\varepsilon_i}), \dots, \overline{\lambda}_k^{(p)}(M, g_{\varepsilon_i})$, and let $\{\psi_1, \dots, \psi_k\}$ be the limits obtained from $\{\varphi_{1,\varepsilon_i}, \dots, \varphi_{k,\varepsilon_i}\}$. Then, $\{\psi_1, \dots, \psi_k\}$ are also $L^2(M_1, g_1)$-orthonormal eigenforms on (M_1, g_1) associated with the eigenvalues $\overline{\lambda}_1^{(p)}(M_1, g_1), \dots, \overline{\lambda}_k^{(p)}(M_1, g_1)$.*

Proof. We first calculate:

$$
\begin{aligned}
\|(\chi_{\varepsilon_i}-1)\,\varphi_{\varepsilon_i}\|^2_{L^2(M_1(\varepsilon),g_1)} &\leq \int_{\varepsilon_i}^{\sqrt{\varepsilon_i}}\int_{\mathbb{S}^{m-1}} |\,\varphi_{\varepsilon_i}\,|^2_g\, r^{m-1} dr d\mu_h \\
&\leq C \begin{cases} \dfrac{1}{m-2}\displaystyle\int_{\varepsilon_i}^{\sqrt{\varepsilon_i}} r\,dr\,\|\,\varphi_{\varepsilon_i}\,\|^2_{H^1(M_1(\varepsilon_i),g_1)} & \text{if } m\geq 3, \\ |\log\varepsilon_i|\displaystyle\int_{\varepsilon_i}^{\sqrt{\varepsilon_i}} r\,dr\,\|\,\varphi_{\varepsilon_i}\,\|^2_{H^1(M_1(\varepsilon_i),g_1)} & \text{if } m=2 \end{cases} \\
&\leq C \begin{cases} \dfrac{\varepsilon_i}{m-2}\,\|\,\varphi_{\varepsilon_i}\,\|^2_{H^1(M_1(\varepsilon_i),g_1)} & \text{if } m\geq 3, \\ |\log\varepsilon_i|\varepsilon_i\,\|\,\varphi_{\varepsilon_i}\,\|^2_{H^1(M_1(\varepsilon_i),g_1)} & \text{if } m=2 \end{cases} \\
&\longrightarrow 0 \quad (i\to\infty).
\end{aligned}
\tag{6.4}
$$

Then, from $\lim_{i\to\infty}\varphi^2_{j,\varepsilon_i}=0$ by Lemma 6.4 and (6.4), it follows that for all $j,l=1,\dots,k$,

$$
\begin{aligned}
(\psi_j,\psi_l)_{L^2(M_1,g_1)} &= \lim_{i\to\infty}(\chi_{\varepsilon_i}\,\varphi_{j,\varepsilon_i},\chi_{\varepsilon_i}\,\varphi_{l,\varepsilon_i})_{L^2(M_1,g_1)} \\
&= \lim_{i\to\infty}\left\{(\varphi^1_{j,\varepsilon_i},\varphi^1_{l,\varepsilon_i})_{L^2(M_1(\varepsilon_i),g_1)} + ((\chi^2_{\varepsilon_i}-1)\,\varphi^1_{j,\varepsilon_i},\varphi^1_{l,\varepsilon_i})_{L^2(M_1(\varepsilon_i),g_1)}\right\} \\
&= \lim_{i\to\infty}\left\{(\varphi^1_{j,\varepsilon_i},\varphi^1_{l,\varepsilon_i})_{L^2(M_1(\varepsilon_i),g_1)} + (\varphi^2_{j,\varepsilon_i},\varphi^2_{l,\varepsilon_i})_{L^2(M_2(1),g_2)}\right\} \\
&\quad + \lim_{i\to\infty}((\chi^2_{\varepsilon_i}-1)\,\varphi^1_{j,\varepsilon_i},\varphi^1_{l,\varepsilon_i})_{L^2(M_1(\varepsilon_i),g_1)} \\
&= \lim_{i\to\infty}(\varphi_{j,\varepsilon_i},\varphi_{l,\varepsilon_i})_{L^2(M,g_{\varepsilon_i})} + \lim_{i\to\infty}((\chi^2_{\varepsilon_i}-1)\,\varphi^1_{j,\varepsilon_i},\varphi^1_{l,\varepsilon_i})_{L^2(M_1(\varepsilon_i),g_1)} \\
&= \delta_{jl}.
\end{aligned}
$$

Here, the last equality follows from (6.4). Therefore, we conclude that $\overline{\lambda}^{(p)}_j = \lim_{i\to\infty}\overline{\lambda}^{(p)}_j(M,g_{\varepsilon_i})$ for $j=1,\dots,k$ belong to the set of all eigenvalues of the rough Laplacian acting on p-forms on (M_1,g_1). Hence, we have finished the proof of Theorem 6.1. □

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Colette Anné
Laboratoire de Mathématiques Jean Leray,
Université de Nantes, CNRS, Faculté des Sciences,
BP 92208, 44322 Nantes, France.
colette.anne@univ-nantes.fr

Junya Takahashi
Research Center for Pure and Applied Mathematics,
Graduate School of Information Sciences,
Tôhoku University,
6–3–09 Aoba, Sendai 980 − 8579, Japan.
e-mail: t-junya@tohoku.ac.jp