Let $m \leq n$ be positive integers. Let $H$ be the diagonal subgroup of $S_m \times S_m$ regarded as a subgroup of $S_m \times S_n$. Then $H \times S_{n-m}$ can naturally be regarded as a subgroup of $S_m \times S_n$. The permutation character of $S_m \times S_n$ on the cosets of the subgroup $H \times S_{n-m}$ decomposes as follows.

$$1^{S_m \times S_n}_{H \times S_{n-m}} = (1^{S_m \times S_m \times S_{n-m}}_{H \times S_{n-m}})^{S_m \times S_n} = \bigoplus_{\chi \in \text{Irr}(S_m)} ((\chi \otimes \chi) \otimes 1^{S_{n-m}}_{S_n})^{S_m \times S_n} = \bigoplus_{\chi \in \text{Irr}(S_m)} \chi \otimes (\chi \otimes 1^{S_{n-m}}_{S_n})^{S_n}.$$ 

This permutation representation is multiplicity-free. Indeed, by [2, Theorem 2.8.2], we have

$$1^{S_m \times S_n}_{H \times S_{n-m}} = \bigoplus_{\alpha} \bigoplus_{\gamma} \chi_{\alpha} \otimes \chi_{\gamma},$$

where the first sum is over all partitions $\alpha$ of $m$, and the second sum is over all partitions $\gamma$ of $n$ satisfying the condition

$$\gamma_1 \geq \alpha_1 \geq \gamma_2 \geq \alpha_2 \geq \cdots \ (1)$$

Taking the conjugate partitions, this condition is equivalent to

$$\gamma'_i = \alpha'_i \text{ or } \alpha'_i + 1,$$

or equivalently, $\gamma - \alpha$ is a vertical strip. As we shall see later, the set of pairs of such partitions can be shown to be in one-to-one correspondence with the isomorphism type of the graphs arising from orbitals. In fact, one can associate with such a pair $(\gamma', \alpha')$, the disjoint union of graphs with $\gamma'_i + \alpha'_i$ edges which is either a cycle or a path, depending on $\gamma'_i = \alpha'_i$ or $\alpha'_i + 1$. All of these observations reduce to the well-known theory of characters of symmetric groups when $m = n$.

Now let $|M| = m$, $|N| = n$, and let $X$ be the set of all injections from $M$ to $N$. Then $S_m \times S_n$ acts on $X$ by

$$(\sigma, \tau) \psi := \tau \psi \sigma^{-1}, \quad \sigma \in S_m, \ \tau \in S_n, \ \psi \in X.$$
Elements of $X$ can be regarded as maximal matchings of the complete bipartite graph $K_{m,n}$. For $\psi, \phi \in X$, we denote by $\Gamma_{\psi,\phi}$ the edge-colored bipartite graph on $M \cup N$ defined as follows: $(x, y) \in M \times N$ is an edge with color 1 (resp. 2) iff $\psi(x) = y$ (resp. $\phi(x) = y$). Forgetting colors in $\Gamma_{\psi,\phi}$ yields a graph $\Delta_{\psi,\phi}$. The graph $\Delta_{\psi,\phi}$ may have multiple edges. Every vertex in $M$ has degree 2 in $\Delta_{\psi,\phi}$, while vertices in $N$ have degree 0, 1, or 2 in $\Delta_{\psi,\phi}$.

**Theorem 1.** Let $\psi, \psi', \phi, \phi' \in X$. Then there exists $(\sigma, \tau) \in S_m \times S_n$ such that $(\sigma, \tau) \psi = \psi'$ and $(\sigma, \tau) \phi = \phi'$ if and only if $\Delta_{\psi,\phi} \cong \Delta_{\psi',\phi'}$.

**Proof.** It is clear that if $(\sigma, \tau) \psi = \psi'$ and $(\sigma, \tau) \phi = \phi'$, then $\sigma \times \tau : M \times N \to M \times N$ is an isomorphism from $\Delta_{\psi,\phi}$ to $\Delta_{\psi',\phi'}$. To prove the converse, first observe that there exists an isomorphism $\Delta_{\psi,\phi} \cong \Delta_{\psi',\phi'}$, leaving the bipartition invariant. Let $\sigma \times \tau$ be such an isomorphism. Let $C$ be a connected component of $\Delta_{\psi,\phi}$. The restriction of $\sigma \times \tau$ to $C$ may not preserve the edge-coloring, but we claim that $\sigma \times \tau$ can be taken in such a way that it preserves the edge-coloring. If $C$ is an isolated vertex in $N$, then there is nothing to do. If $C$ has an edge, then $C$ is either a cycle or a path, in which edges are colored alternately. If $C$ is a path and $\sigma \times \tau|_C$ does not preserve the edge-coloring, then reflecting the image yields a color-preserving isomorphism. One can argue in a similar manner for cycles, and one obtains a color-preserving isomorphism. Then we have $(\sigma, \tau) \psi = \psi'$ and $(\sigma, \tau) \phi = \phi'$.

Notice that the orbits of $S_m \times S_n$ are self-paired, since $\Delta_{\psi,\phi} \cong \Delta_{\phi,\psi}$. This gives another proof that the permutation character is multiplicity-free.

Let $x_{ij}$ denote the function on $X$ defined by

$$x_{ij}(\psi) = \begin{cases} 1 & \text{if } \psi(i) = j, \\ 0 & \text{otherwise}. \end{cases}$$

Every complex-valued function on $X$ can be expressed in terms of a polynomial in $x_{ij}$ ($i \in M$, $j \in N$). Indeed, the characteristic function of $\{\psi\}$ is given by $\prod_{i \in M} x_{i,\psi(i)}$. Every monomial of degree greater than $m$ is zero when regarded as a function on $X$. We wish to decompose the space of polynomial functions on $X$ into irreducible submodules. In particular, we are concerned with the function space spanned by the polynomials of the form

$$\sum_{\tau : T \to T'} \prod_{i \in T} x_{i,\tau(i)}$$

where $T$ is a $t$-element subset of $M$, $T'$ is a $t$-element subset of $N$, the sum is taken over all bijections $\tau : T \to T'$. A motivation of doing this is an algebraic approach to the combinatorial object called perpendicular arrays.

**Definition 1.** Let $X$ be the set of all injections from $M$ to $N$, where $M, N$ are finite sets. A perpendicular array of strength $t$ is a subset $Y$ of $X$ satisfying the following property: there exists a positive integer $\lambda$ such that

$$|\{\psi \in Y \mid \psi(T) = T'\}| = \lambda.$$
for any $T \subset M$, $T' \subset N$ with $|T| = |T'| = t$.

The definition looks very similar to that of orthogonal arrays. Much less is known in the theory of perpendicular arrays than in the theory of orthogonal arrays (see [1]).

In terms of the polynomial function (2), the condition (3) can be written as

$$ \sum_{\psi \in Y} \sum_{\tau: T \rightarrow T'} \prod_{i \in T} x_{i, \tau(i)}(\psi) = \lambda. $$

In the following, we give an irreducible decomposition of the space of polynomial functions of the form (2) when $|T| = |T'| = 1$. It would be nice if we can obtain an irreducible decomposition for arbitrary values of $t = |T| = |T'|$.

Let $E = \langle e_1, \ldots, e_m \rangle$, $F = \langle f_1, \ldots, f_n \rangle$ be the permutation modules for $S_m$, $S_n$, respectively. Let $H = \langle x_{ij} | i \in M, j \in N \rangle$. Then there is a surjection $E \otimes F \rightarrow H$ defined by $e_i \otimes f_j \mapsto x_{ij}$ which commutes with the action of $S_m \times S_n$. The module $E \otimes F$ decomposes as an $S_m \times S_n$-module:

$$ E \otimes F = \langle \left( \sum_{i \in M} e_i \right) \otimes \left( \sum_{j \in N} f_j \right) \rangle $$

$$ \oplus \langle \left( \sum_{i \in M} e_i \right) \otimes (f_1 - f_j) | j \in N \rangle $$

$$ \oplus \langle (e_1 - e_i) \otimes \left( \sum_{j \in N} f_j \right) | i \in M \rangle $$

$$ \oplus \langle (e_1 - e_i) \otimes (f_1 - f_j) | i \in M, j \in N \rangle. $$

Note that each of the four submodules are irreducible. Note also that

$$(e_1 - e_i) \otimes \left( \sum_{j \in N} f_j \right) \mapsto \sum_{j \in N} x_{1j} - \sum_{j \in N} x_{ij} = 0$$

while

$$(\sum_{i \in M} e_i) \otimes (f_1 - f_j) \mapsto \sum_{i \in M} x_{i1} - \sum_{i \in M} x_{ij},$$

$$(e_1 - e_2) \otimes (f_1 - f_2) \mapsto x_{11} + x_{22} - x_{12} - x_{21} \neq 0$$

since $(x_{11} + x_{22} - x_{12} - x_{21})(1) = 2$, where 1 denotes the mapping $M \rightarrow N$ defined by $1(i) = i$ for $i \in M$. Observe $\sum_{i \in M} x_{i1} - \sum_{i \in M} x_{ij} = 0$ if and only if $m = n$. Therefore

$$ H = \sum_{i \in M} \sum_{j \in N} x_{ij} \oplus \langle x_{11} + x_{ij} - x_{ij} - x_{i1} | i \in M, j \in N \rangle $$

if $m = n$,

$$ H = \sum_{i \in M} \sum_{j \in N} x_{ij} \oplus \langle x_{11} + x_{ij} - x_{ij} - x_{i1} | i \in M, j \in N \rangle $$

$$ \oplus \langle \sum_{i \in M} (x_{i1} - x_{ij}) | j \in N \rangle $$

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if $m < n$. In particular,

$$\dim H = \begin{cases} 
1 + (m - 1)^2 & \text{if } m = n, \\
mn - m + 1 & \text{if } m < n.
\end{cases}$$

For $\psi, \phi \in X$, define $\rho(\psi, \phi)$ by

$$\rho(\psi, \phi) = |\{ i \in M | \psi(i) = \phi(i) \}|.$$

Note that if we regard $X$ as a subset of the Cartesian power $N^M$, then $\rho$ is the restriction of the Hamming distance in $N^M$ to $X$. Then $(X, \rho)$ becomes a spherical polynomial space in the sense of Conder and Godsil [3].

**References**

