Let $H$ be a Hadamard matrix of order 24 whose first row is the all-ones vector. Let $C_3(H)$ denote the ternary code generated by the rows of $H$:

$$C_3(H) = \mathbb{F}_3^{24}H.$$ 

Let $B = \frac{1}{2}(H + J)$ be the binary Hadamard matrix associated to $H$, and let $C_2(H)$ denote the binary code generated by the rows of $B$:

$$C_2(H) = \mathbb{F}_2^{24}B.$$ 

Then

- $C_3(H)$ is a self-dual code with minimum weight 6 or 9.
- $C_2(H)$ is a doubly even self-dual code with minimum weight 4 or 8.

As a consequence of complete classification of Hadamard matrices of order 24 (there are 60 of them, up to equivalence, see [4, 5]), Assmus and Key [1] observed the following:

**Observation.** $C_2(H)$ has minimum weight 8 if and only if $C_3(H^T)$ has minimum weight 9.

Note that $C_2(H)$ has minimum weight 8 if and only if it is equivalent to the extended binary Golay code, while $C_3(H^T)$ has minimum weight 9 if and only if it is an extremal ternary self-dual [24, 12, 9] codes which were classified
in [6]. I found in January 2007, a proof which does not use the classification.
In March 2007, I found another proof using lattices. In this report, I give
the latter proof.

Let
\[
\Lambda(\mathbf{H}) = \frac{1}{2\sqrt{2}} \mathbb{Z}^{49} \begin{bmatrix} \mathbf{H} + \mathbf{J} \\ 8\mathbf{I} \\ 4\mathbf{e}_1 + 1 \end{bmatrix} \subset \mathbb{R}^{24}.
\]

Then \(\Lambda(\mathbf{H})\) is an even unimodular lattice, and its minimum norm is 2 or 4.

**Theorem 1.** Let \(\mathbf{H}\) be a Hadamard matrix of order 24 whose first row is the
all-ones vector. The following statements are equivalent:

(i) \(C_2(\mathbf{H})\) has minimum weight 8,

(ii) \(C_3(\mathbf{H}^T)\) has minimum weight 9,

(iii) \(\Lambda(\mathbf{H})\) has minimum norm 4.

Note that there are exactly two Hadamard matrices satisfying the condi-
tions of Theorem 1. For one of these Hadamard matrices, the code \(C_3(\mathbf{H}^T)\)
is the quadratic residue code, and for the other Hadamard matrix, \(C_3(\mathbf{H}^T)\)
is the Pless symmetry code. For both of these Hadamard matrices, \(C_2(\mathbf{H})\) is
the binary extended Golay code, and \(\Lambda(\mathbf{H})\) is the Leech lattice.

An interesting point is how the transpose of \(\mathbf{H}\) come into play in (ii).
The equivalence of (i) and (ii) can be shown directly (indeed this was the
proof I discovered in January 2007), and that explains why \(\mathbf{H}^T\) appears in
(ii). The other proof involving (iii) also explains why \(\mathbf{H}^T\) appears, as you
will see below.

To prove Theorem 1, set
\[
\Lambda_0(\mathbf{H}) = \frac{1}{2\sqrt{2}} \mathbb{Z}^{49} \begin{bmatrix} \mathbf{H} + \mathbf{J} \\ 8\mathbf{I} \end{bmatrix},
\]

and observe

(iii) \(\Leftrightarrow\) \(\Lambda(\mathbf{H})\) has minimum norm > 2

\(\Leftrightarrow\) \(\Lambda_0(\mathbf{H})\) has minimum norm > 2.

Indeed, since \(\Lambda(\mathbf{H}) \setminus \Lambda_0(\mathbf{H}) \subset \frac{1}{2\sqrt{2}}(1 + 2\mathbb{Z})^{24}\), \(\Lambda(\mathbf{H}) \setminus \Lambda_0(\mathbf{H})\) has minimum
norm at least 3.

There are two unimodular lattices containing \(\Lambda_0(\mathbf{H})\), other than \(\Lambda(\mathbf{H})\).
These are given by the following lemma.
Lemma 2. The lattice $\Lambda_0(H)$ is a sublattice of index 2 in $\Lambda(H)$. The two unimodular lattices containing $\Lambda_0(H)$, other than $\Lambda(H)$ are:

$$\Lambda'(H) = \frac{1}{2\sqrt{2}} \mathbb{Z}^{48} \left[ \begin{array}{c} H \\ 8I \end{array} \right], \quad \Lambda''(H) = \frac{1}{\sqrt{2}} \mathbb{Z}^{48} \left[ \frac{1}{2} (H + J) \right].$$

In particular,

$$\Lambda_0(H) = \{ x \in \Lambda'(H) \mid \|x\|^2 \equiv 0 \pmod{2} \}$$  \hspace{1cm} (1)

$$= \{ x \in \Lambda''(H) \mid \frac{1}{\sqrt{2}} x \cdot 1 \equiv 0 \pmod{2} \}.$$  \hspace{1cm} (2)

**Proof.** Since

$$\mathbb{Z}^{48} \left[ \begin{array}{c} H + J \\ 8I \end{array} \right] \subset (2\mathbb{Z})^{24}$$

and $4e_1 + 1 \notin (2\mathbb{Z})^{24}$, we have $\Lambda(H) \supseteq \Lambda_0(H)$. Since

$$2(4e_1 + 1) = 8e_1 + (H + J)1 \in \mathbb{Z}^{48} \left[ \begin{array}{c} H + J \\ 8I \end{array} \right],$$

we have $|\Lambda(H) : \Lambda_0(H)| = 2$.

Clearly, $\Lambda_0(H) \subset \Lambda'(H) \cap \Lambda''(H)$. Since $\Lambda_0(H)$ is even and $\Lambda'(H)$ is odd, $\Lambda_0(H) \not\subseteq \Lambda'(H)$. Since $\Lambda'(H) = \langle \Lambda_0(H), \frac{1}{2\sqrt{2}} 1 \rangle$ and $2 \cdot \frac{1}{2\sqrt{2}} 1 \in \Lambda_0(H)$, we obtain $|\Lambda'(H) : \Lambda_0(H)| = 2$. Since $\Lambda''(H)$ is even, we have $\Lambda'(H) \neq \Lambda''(H)$ and since

$$\Lambda''(H) \subset \frac{1}{2\sqrt{2}} (2\mathbb{Z})^{24} \not\supseteq \Lambda(H),$$

we have $\Lambda''(H) \neq \Lambda(H)$.

Since $F_2^{24}(\frac{1}{2} (H + J))$ is a self-dual code, $\Lambda''(H)$ is unimodular. Thus det $\Lambda''(H) = 1 = \det \Lambda(H)$, and hence by [2, p.2],

$$|\Lambda''(H) : \Lambda_0(H)|^2 = \frac{\det \Lambda_0(H)}{\det \Lambda''(H)} = \frac{\det \Lambda_0(H)}{\det \Lambda(H)} = |\Lambda(H) : \Lambda_0(H)|^2 = 2^2.$$

The two expressions (1) and (2) for $\Lambda_0(H)$ follows by observing that each forms a proper sublattice of $\Lambda(H)$ containing $\Lambda_0(H)$. \hfill $\Box$
By (1), we find

\[ \min \Lambda_0(H) = \frac{1}{8} \min \{ \|x\|^2 \mid 0 \neq x \in \mathbb{Z}^{48} \begin{bmatrix} H \\ 8I \end{bmatrix}, \frac{1}{\sqrt{24}} H^T, \|x\|^2 \equiv 0 \pmod{16} \} \]

\[ = \frac{1}{3} \min \{ \|x\|^2 \mid 0 \neq x \in \mathbb{Z}^{48} \begin{bmatrix} H^T \\ 3I \end{bmatrix}, \|y\|^2 \equiv 0 \pmod{16} \} \]

\[ = \begin{cases} 2 & \text{if } C_3(H^T) \text{ has a codeword of weight 6}, \\ 4 & \text{otherwise}. \end{cases} \]

Also, by (2), we find

\[ \min \Lambda_0(H) = \frac{1}{8} \min \{ \|x\|^2 \mid 0 \neq x \in \mathbb{Z}^{48} \begin{bmatrix} H + J \\ 4I \end{bmatrix}, x \cdot 1 \equiv 0 \pmod{8} \} \]

\[ = \frac{1}{2} \min \{ \|y\|^2 \mid 0 \neq y \in \mathbb{Z}^{48} \begin{bmatrix} B \\ 2I \end{bmatrix}, y \cdot 1 \equiv 0 \pmod{4} \} \]

\[ = \begin{cases} 2 & \text{if } C_2(H) \text{ has a codeword of weight 4}, \\ 4 & \text{otherwise}. \end{cases} \]

This completes the proof of Theorem 1.

**References**


