A characterization of quasi-line graphs

Akihiro Munemasa\textsuperscript{1}

\textsuperscript{1}Graduate School of Information Sciences
Tohoku University
(joint work with Tetsuji Taniguchi)

July 10, 2010
Kyushu University
A graph $G = (V, E)$ consists of a finite set $V$ together with a set $E$ of two-element subsets of $V$.

- $V$: vertices
- $E$: edges

The line graph $L(G)$ of a graph $G$ has $E$ as the set of vertices and its set of edges is

$$\{\{e, f\} \mid e, f \in E, |e \cap f| = 1\}.$$
Properties of Line Graphs

$G$: a graph
$\Gamma = L(G)$: the line graph of $G$.

(i) The neighborhood of every vertex in $\Gamma$ is a union of two subsets, each of which is a clique.

(ii) The graph $\Gamma$ admits a representation by vectors of squared norm 2 in $\mathbb{Z}^n$, where $n = |V|$.

Here, a representation by vectors of squared norm 2 means a mapping $\phi : V(\Gamma) \rightarrow \mathbb{Z}^n$ such that $\|\phi(e)\|^2 = 2$ for $e \in E(\Gamma)$, $(\phi(e), \phi(f)) = 1$ or 0, according as $\{e, f\} \in E(\Gamma)$ or not. A graph satisfying (i) is called a quasi-line graph, while a graph satisfying (ii) is called a generalized line graph. (ii) means the image of $\phi$ is contained in the root system of type $D$. 
Representation of Graphs by Vectors

The incidence matrix:

\[
\begin{align*}
\text{edges} &= \text{vertices of } L(G) = \\
&= \begin{pmatrix}
(1 & 1 & 0 & 0 & 0) \\
(0 & 1 & 1 & 0 & 0) \\
(1 & 0 & 0 & 1 & 0) \\
(0 & 0 & 1 & 1 & 0) \\
(0 & 0 & 1 & 0 & 1)
\end{pmatrix}
\end{align*}
\]

Allowing \(-1\)

\[
\begin{pmatrix}
( -1 & 1 & 0 & 0 & 0 & 0) \\
( 0 & 1 & -1 & 0 & 0) \\
( -1 & 0 & 0 & 1 & 0) \\
( 0 & 0 & -1 & 1 & 0) \\
( 0 & 0 & -1 & 0 & 1)
\end{pmatrix} \subseteq \text{root system of type } A
\]
Examples

A claw can be represented by the vectors of squared norm 2:

\[
\begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & -1 & 0
\end{pmatrix}
\]

hence it is a generalized line graph, but it is not a quasi-line graph. The graph with adjacency matrix

\[
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}
\]

is a quasi-line graph, but not a generalized line graph.
Exceptional Root Systems

If

\[
A = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
\end{bmatrix}
\]

then \( A + 2I \) is a Gram matrix of the \( E_6 \)-lattice, which is known not to be contained in \( D_n \) for any \( n \). Hence the graph \( \Gamma \) with adjacency matrix \( A \) is not a generalized line graph. However, \( \Gamma \) is a quasi-line graph.

\[
\{ \text{line graph} \} \subset \{ \text{generalized line graph} \} \subset \{ \text{quasi-line graph} \} \subset \{ \text{claw-free graph} \}
\]
Chudnovsky–Seymour gave a structural characterization of claw-free graphs, and also of quasi-line graphs. We wish to characterize quasi-line graphs using the concept of sums of Hoffman graphs.

**Definition**

A **Hoffman graph** is a graph $H$ with vertex labeling $V(H) \rightarrow \{s, f\}$, satisfying the following conditions:

(i) every vertex with label $f$ is adjacent to at least one vertex with label $s$;

(ii) vertices with label $f$ are pairwise non-adjacent.

$s = \text{“slim,”} \quad f = \text{“fat.”}$

$V_s(H) (V_f(H)) = \text{the set of slim (fat) vertices of } H.$

An ordinary graph $= \text{Hoffman graph without fat vertex.}$
Sums of Hoffman graphs

Definition
Let $H$ be a Hoffman graph, and let $H^i$ ($i = 1, 2, \ldots, n$) be a family of subgraphs of $H$. We write

$$H = \bigcup_{i=1}^{n} H^i$$

if

(i) $V(H) = \bigcup_{i=1}^{n} V(H^i)$;

(ii) $V_s(H^i) \cap V_s(H^j) = \emptyset$ if $i \neq j$;

(iii) if $x \in V_s(H^i)$ and $\alpha \in V_f(H)$ are adjacent, then $\alpha \in V(H^i)$;

(iv) if $x \in V_s(H^i)$, $y \in V_s(H^j)$ and $i \neq j$, then $x$ and $y$ have at most one common fat neighbour, and they have one if and only if they are adjacent.
Given a bipartite graph $G$ with bipartition $V_1 \cup V_2$,

$\hat{G} = \text{Hoffman graph obtained from } G \text{ by making every pair of vertices in } V_i \text{ adjacent, attaching a common fat neighbor } f_i \text{ to } V_i$, for $i = 1, 2$
A construction of quasi-line graphs

Suppose

\[
V(Q) = \bigcup_{i=1}^{n} V_i \quad \text{(disjoint cliques)}
\]

\[
V_i = \bigcup_{j=1}^{n} V_{ij}, \quad V_{ii} = \emptyset,
\]

\[
x \in V_{ij}, \quad y \notin V_i, \quad x \sim y \implies y \in V_{ji}.
\]

Then \(Q\) is a quasi-line graph which can be expressed as a sum of Hoffman graphs of the form \(\hat{G}\).