Binary codes related to the moonshine vertex operator algebra

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Group Theory

Starting from very small set of axioms

- $\cdot : G \times G \rightarrow G$, $(a, b) \mapsto a \cdot b$,
- associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
- $\exists$ identity $1 \in G$, $a \cdot 1 = 1 \cdot a = a$,
- $\exists$ inverse: $\forall a \in G$, $\exists b \in G$, $a \cdot b = b \cdot a = 1$.

The goal of finite group theory is to understand the set of all finite $G$ satisfying the axioms, in some reasonable manner.
Finite group theory has its origin in the remarkable work of É. Galois who proved that the occurrence of a non-abelian simple group caused impossibility of solvability by radical of polynomial equations of degree $\geq 5$.

- A group $G$ is simple if $\nexists$ normal subgroup $N$ with $\{1\} \neq N \neq G$,
- $N$ is normal in $G \iff \forall a \in G$, $aN = Na$,
- Example: $A_5 =$ symmetry group of icosahedron

Burnside (1915) further developed finite group theory.
Finite Simple Groups

• Chevalley (1955) systematically constructed finite groups of Lie type. Steinberg, Ree, Suzuki found more families. There are 26 sporadic ones.

• E. Mathieu (1861, 1873), E. Witt (1938): \( \text{Aut}(\text{Steiner system } S(5, 8, 24)) = M_{24} \), M.J.E. Golay (1949): \( \text{Aut}(\text{Golay code}) = M_{24} \)


The smallest among 26 is the Mathieu group \( M_{11} \) of order

\[ 11 \cdot 10 \cdot 9 \cdot 8 = 7920, \]

the largest is \( \mathbb{M} \) of order

\[ 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \]
Decompositions of $V^\natural$

\[ V^\natural = \bigoplus_{n=0}^{\infty} V_n \]

infinite sum of finite-dimensional subspaces.

\[ \dim V_0 = 1, \quad \dim V_1 = 0, \]

$V_2 = \text{Griess algebra}, \quad \dim V_2 = 196884.$

\[ j(\tau) = \frac{1}{q} + 744 + 196884q + \cdots \]

This coincidence lead to Conway–Norton conjecture, proved by R. Borcherds (1992).

The smallest matrix representation of $\mathbb{M}$ has dimension 196883. R. Wilson found an explicit 196882-dimensional matrix representation of $\mathbb{M}$ over $\mathbb{F}_2 = \{0, 1\} = \mathbb{Z}/2\mathbb{Z}.$
Decompositions of $V^\mathbb{H}$

Instead of

$$V^\mathbb{H} = \bigoplus_{n=0}^{\infty} V_n$$

infinite sum of finite-dimensional subspaces,

$$V^\mathbb{H} = \bigoplus_{\alpha \in \mathbb{F}_2^{48}} V^\alpha$$

finite sum of infinite-dimensional subspaces.

Lam–Yamauchi (2008): every Virasoro frame (certain subalgebra of $V^\mathbb{H}$) gives rise to such a decomposition.

$$D = \{ \alpha \in \mathbb{F}_2^{48} \mid V^\alpha \neq 0 \}$$

is called the structure code of the Virasoro frame.

There are only finitely many Virasoro frames, and $D$ is invariant under $\mathcal{M}$. 
Let $m$ be an integer (actually we need only $m = 2$ (binary) and $m = 4$).
A subgroup $C \subset (\mathbb{Z}/m\mathbb{Z})^n$ is called a code of length $n$ over $\mathbb{Z}/m\mathbb{Z}$. The dual $C^\perp$ of a code $C$ is
\[ C^\perp = \{ x \in (\mathbb{Z}/m\mathbb{Z})^n \mid (x, y) = 0 \ (\forall y \in C) \}. \]

- $m = 4$, $C \subset (\mathbb{Z}/4\mathbb{Z})^n$ is type II if $C = C^\perp$, $\sum_{i=1}^{n} x_i^2 \equiv 0 \pmod{8}$ for all $x \in C$,
  and for $n = 24$, $C$ is extremal if $\sum_{i=1}^{n} x_i^2 > 8$,

- $m = 2$, $C \subset (\mathbb{Z}/2\mathbb{Z})^n$ is doubly even if
  $\text{wt}(x) = |\{ i \mid x_i = 1 \}| \equiv 0 \pmod{4}$ for all $x \in C$,

- $m = 2$, $D \subset (\mathbb{Z}/2\mathbb{Z})^n$ is triply even if
  $\text{wt}(x) = |\{ i \mid x_i = 1 \}| \equiv 0 \pmod{8}$ for all $x \in D$.

Equivalence: permutation of coordinates, and multiplication by $-1$ on some coordinates ($m = 4$).
Factorization of the polynomial $X^{23} - 1$

$$(X - 1)(X^{22} + X^{21} + \cdots + X + 1) \quad \text{over } \mathbb{Z}$$

$$= (X - 1)(X^{11} + X^{10} + \cdots + 1)$$
$$\times (X^{11} + X^9 + \cdots + 1) \quad \text{over } \mathbb{F}_2$$

$$= (X - 1)(X^{11} - X^{10} + \cdots - 1)$$
$$\times (X^{11} + 2X^{10} - X^9 + \cdots - 1) \quad \text{over } \mathbb{Z}/4\mathbb{Z}$$

(by Hensel’s lemma).
\[ X^{23} - 1 = (X - 1) f(X) g(X) \] over \( \mathbb{Z}/4\mathbb{Z} \)

An extremal type II code of length 24 over \( \mathbb{Z}/4\mathbb{Z} \) is generated by the rows of:

\[
\begin{bmatrix}
1 & f(X) \\
1 & f(X) \\
1 & f(X) \\
\vdots & \vdots \\
1 & f(X)
\end{bmatrix}, \quad 23 \times 24 \text{ matrix}
\]


\( \overline{f}(X) = f(X) \mod 2 \). Golay code (a doubly even code of length 24) is generated by the rows of:

\[
\begin{bmatrix}
1 & \overline{f}(X) \\
1 & \overline{f}(X) \\
1 & \overline{f}(X) \\
\vdots & \vdots \\
1 & \overline{f}(X)
\end{bmatrix}, \quad \text{over } \mathbb{F}_2 \quad \text{(mod 2 reduction)}
\]
Codes and Virasoro frames

**Theorem (Dong–Mason–Zhu (1994))**

\[
\{ \text{extremal type II code of length 24 over } \mathbb{Z}/4\mathbb{Z} \} \rightarrow \{ \text{Virasoro frames } V^\flat \} 
\]

**Theorem (Lam–Yamauchi (2008))**

\[
\{ \text{Virasoro frames of } V^\flat \} \xrightarrow{\text{str}} \{ \text{binary triply even codes of length 48} \} \quad V^\flat = \bigoplus_{\alpha \in D} V^\alpha
\]

- Actually these mapping induce mappings of equivalence classes.
- What happens if we compose these two mappings?
Composition of the two mappings gives a mapping from codes to codes

\[ \{ \text{Virasoro frames of } V^h \} \overset{\text{str}}{\rightarrow} \{ \text{binary \triply\ even codes of length 48} \} \]

\[ \uparrow \text{DMZ} \]

\[ \{ \text{extremal type II codes of length 24 over } \mathbb{Z}/4\mathbb{Z} \} \]

There must be a easier description of the composition mapping.
Commutative Diagram

Lam–Yamauchi (2008): \( \text{str} \circ \text{DMZ} = \mathcal{D} \circ \text{Res} \).

\[ \{ \text{Virasoro frames of } V^h \} \xrightarrow{\text{str}} \{ \text{binary triply even codes of length 48} \} \]

\[ \uparrow \text{DMZ} \]

\[ \{ \text{extremal type II codes of length 24 over } \mathbb{Z}/4\mathbb{Z} \} \xrightarrow{\text{Res}} \{ \text{binary doubly even codes of length 24} \} \]

\[ \uparrow \mathcal{D} \text{ (doubling)} \]
**Res(C) and D**

If $C$ is a code over $\mathbb{Z}/4\mathbb{Z}$, then its modulo 2 reduction is called the residue code and is denoted by

$$\text{Res}(C) \subset (\mathbb{Z}/2\mathbb{Z})^n = \mathbb{F}_2^n.$$ 

Let $C = \text{Span}_{\mathbb{F}_2}(A)$ be the binary code of length $n$ spanned by the row vectors of a $k \times n$ matrix $A$. The doubling of $C$ is defined by

$$D(C) = \text{Span}_{\mathbb{F}_2} \begin{bmatrix} A & A \\ 1_n & 0 \\ 0 & 1_n \end{bmatrix},$$

where $1_n = (1, 1, \ldots, 1)$.

If $C$ is doubly even and $8 | n$, then $D(C)$ is a triply even code of length $2n$. In particular,

$$\{\text{doubly even code of length 24}\} \xrightarrow{D} \{\text{triply even code of length 48}\}$$
Commutative Diagram

\{\text{Virasoro frames of } V^h\} \xrightarrow{\text{difficult}} \begin{cases} \text{binary triply even codes} \\ \text{of length 48} \end{cases}

\uparrow \text{DMZ}

\begin{cases} \text{extremal type II codes} \\ \text{of length 24} \\ \text{over } \mathbb{Z}/4\mathbb{Z} \end{cases} \xrightarrow{\text{Res}} \begin{cases} \text{binary doubly even codes} \\ \text{of length 24} \end{cases}

\uparrow \mathcal{D} \text{ (doubling)}

Harada–Lam–M. (2010):

\str^{-1}(\mathcal{D}(\{\text{doubly even}\})) \overset{(\cong)}{=} \str^{-1}(\mathcal{D} \circ \text{Res}(\{\text{extremal type II}\}))

\overset{(\cong)}{=} \text{DMZ}(\{\text{extremal type II}\})

all coincide.
Virasoro frames of $V^h$ 

<table>
<thead>
<tr>
<th>difficult</th>
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</table>

\[
\uparrow \text{DMZ} \quad \uparrow \mathcal{D} \quad \text{(doubling)}
\]

\[
\begin{cases}
\text{extremal type II codes} \\
of \text{length 24} \\
\text{over } \mathbb{Z}/4\mathbb{Z}
\end{cases}
\]

\[
\begin{cases}
\text{binary} \\
\text{doubly even codes} \\
of \text{length 24}
\end{cases}
\]

\[
\begin{cases}
\text{triply even codes} \\
of \text{length 48}
\end{cases}
\]

Pless–Sloane (1975) enumerated maximal (all have dimension 12) members of

\[
\{\text{binary doubly even codes of length 24}\}.
\]
Virasoro frames of $V^h$ difficult

$\uparrow$ DMZ

$\begin{align*}
\{ & \text{extremal type II codes of length 24} \\
& \text{of length 24 over } \mathbb{Z}/4\mathbb{Z} \} \\
\{ & \text{binary doubly even codes of length 24} \}
\end{align*}$

$\begin{align*}
\{ & \text{binary triply even codes of length 48} \}
\end{align*}$

Betsumiya–M. (2010) enumerated maximal (dimension $\{9, 13, 14, 15\}$) members of

$\{\text{binary triply even codes of length 48}\}$. 
Theorem (Betsumiya–M., 2010)

Let $D$ be a maximal binary triply even code of length 48. Then

- $\exists$ doubly even codes $C_1, C_2$ of length 24,
- $\exists$ linear isomorphism $f : C_1/R_1 \rightarrow C_2/R_2$, where

\[
R_i = \{ x \in (C_i \ast C_i)^\perp \mid \text{wt}(x) \equiv 0 \pmod{8} \} \subset C_i \quad (i = 1, 2),
\]

satisfying

\[
\begin{align*}
\mathbf{x}_1 \in C_1, \quad \mathbf{x}_2 + R_2 & \in f(\mathbf{x}_1 + R_1) \quad \implies \quad \text{wt}(\mathbf{x}_1) \equiv \text{wt}(\mathbf{x}_2) \pmod{8},
\end{align*}
\]

such that

\[
D \cong \{ (\mathbf{x}_1, \mathbf{x}_2) \mid \mathbf{x}_1 \in C_1, \ \mathbf{x}_2 + R_2 \in f(\mathbf{x}_1 + R_1) \}.
\]

Remark Taking $C_1 = C_2$, $f = \text{identity}$ gives $\mathcal{D}(C_1)$. 
Theorem (Betsumiya–M., 2010)

Every maximal member of

\[
\begin{cases}
\text{binary triply even} \\
\text{code of length 48}
\end{cases}
\]

is

- \(\mathcal{D}(C)\) for some doubly even code \(C\) of length 24, or
- decomposable (only two such codes, one of the form \(\mathcal{D}(C_1) \oplus \mathcal{D}(C_2) \oplus \mathcal{D}(C_3)\), another of the form \(\mathcal{D}(C_1) \oplus \mathcal{D}(C_2)\)), or
- a code of dimension 9 obtained from the triangular graph \(T_{10}\) on \(45 = |S_{10} : S_2 \times S_8|\) vertices.
\{\text{Virasoro frames of } V^\mathfrak{h}\}\ \xrightarrow{\text{str}} \left\{ \begin{array}{l}
\text{binary triply even codes of length 48} \\
\uparrow \text{DMZ} \\
\uparrow \mathcal{D} \,(\text{doubling})
\end{array} \right\}

\left\{ \begin{array}{l}
\text{extremal type II codes of length 24} \\
\text{over } \mathbb{Z}/4\mathbb{Z}
\end{array} \right\} \xrightarrow{\text{Res}} \left\{ \begin{array}{l}
\text{binary doubly even codes of length 24}
\end{array} \right\}

Betsumiya created database of \{\text{binary triply even codes of length 48}\}.

http://www.st.hirosaki-u.ac.jp/~betsumi/triply-even/
Virasoro frames of \( V^q \) difficult

\[ \uparrow \text{DMZ} \]

\[ \begin{array}{l}
\{ \text{extremal type II codes} \\
of \text{length 24} \\
\text{over } \mathbb{Z}/4\mathbb{Z} \} \\
\end{array} \]

\[ \xrightarrow{\text{str}} \]

\[ \{ \text{binary triply even codes} \\
of \text{length 48} \} \]

\[ \xrightarrow{\text{Res}} \]

\[ \{ \text{binary doubly even codes} \\
of \text{length 24} \} \]

\( \uparrow \mathcal{D} \) (doubling)
extremal type II codes of length 24 over $\mathbb{Z}/4\mathbb{Z}$ \[\mapsto\] binary doubly even codes of length 24

For each binary doubly even $C$, classify $C$ such that $\text{Res} C = C$. The map $\text{Res}$ is neither injective nor surjective.

- Calderbank–Sloane (with Young) (1997): $\dim C = 12 \implies C \in \text{image of } \text{Res}$.
- Rains (1999) determined the preimage for $C = \text{Golay}$.
- The image was determined by Harada–Lam–M. (2010), but not preimages.
Theorem (Rains, 1999)

Given a doubly even code $C$ of length $n$, dimension $k$, $\ni 1$.

- the set of all type II $\mathbb{Z}_4$-codes $C$ with $\text{Res}(C) = C$ has a structure as an affine space of dimension $(k - 2)(k + 1)/2$ over $\mathbb{F}_2$ (due to Gaborit, 1996),
- the group $\{\pm 1\}^n \rtimes \text{Aut}(C)$ acts as an affine transformation group,
- two codes $C, C'$ are equivalent if and only if they are in the same orbit under this group.
The number of doubly even codes $C \subset \mathbb{F}_2^{24}$ containing 1 and $C^\perp$ has minimum weight $\geq 4$, and the number of extremal type II codes $C \subset (\mathbb{Z}/4\mathbb{Z})^{24}$ with $\text{Res} C = C$.

<table>
<thead>
<tr>
<th>dim</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>doubly even</td>
<td>1</td>
<td>7</td>
<td>32</td>
<td>60</td>
<td>49</td>
<td>21</td>
<td>9</td>
</tr>
<tr>
<td>extremal type II</td>
<td>1</td>
<td>5</td>
<td>31</td>
<td>178</td>
<td>764</td>
<td>1886</td>
<td>1903</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>13</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{extremal type II codes} & \quad \longrightarrow \quad \text{binary doubly even codes} \\
\text{of length} \ 24 & \quad \text{of length} \ 24 \\
\text{over} \ \mathbb{Z}/4\mathbb{Z} & \quad &
\end{align*}
\]