Variations of two results of Jungnickel–Tonchev on projective spaces

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Two results of Jungnickel–Tonchev (1999, 2009)

- **PG(d, q)**

  (1999) Construction of a balanced generalized weighing matrix of order \(\frac{q^{d+1}-1}{q-1}\), weight \(q^d\)

  \[\rightarrow\] weighing matrix of order \(2\frac{q^{d+1}-1}{q-1}\), weight \(q^d\) if \(q \equiv 1 \pmod{4}\)

- **PG(2e, q)**

  (2009) Distorting blocks of 2-(\(\frac{q^{2e+1}-1}{q-1}\), \(\frac{q^{e+1}-1}{q-1}\), \([2e-1]\)) design

  \[\rightarrow\] twisted Grassmann graph of E. van Dam and J. Koolen (joint work with V. Tonchev).
Weighing matrices

Definition

A **weighing matrix** $W$ of order $n$ and weight $k$ is an $n \times n$ matrix $W$ with entries $1, -1, 0$ such that $WW^T = kI$.

- A Hadamard matrix is a $W(n, n)$.
- We write “$W$ is $W(n, k)$” for short.
- $W(n_1, k) \oplus W(n_2, k) = W(n_1 + n_2, k)$.

Chan–Rodger–Seberry (1985) classified weighing matrices of small $n$ or $k$.
Notably, a $W(12, 5)$ was missing, which is a signed incidence matrix of a semibiplane.
Balanced generalized weighing matrices

- **$G$**: a finite group (multiplicatively written), $\bar{G} = G \cup \{0\}$.

An $n \times n$ matrix $B = (b_{ij})$ with entries from $\bar{G}$ is a balanced generalized weighing matrix (written $\text{BGW}(n, k, \mu)$) over $G$, if

- each row of $B$ contains exactly $k$ nonzero entries,
- for any $i \neq i'$, the multiset

$$\{g_{ij}g_{i'j}^{-1} \mid 1 \leq j \leq n, g_{ij} \neq 0, g_{i'j} \neq 0\}$$

represents every element of $G$ exactly $\frac{\mu}{|G|}$ times.

If $G = \{\pm 1\}$, then $\text{BGW}(n, k, \mu) \iff \text{W}(n, k)$

If $G = \{1\}$, then $W$ is just an incidence matrix of a symmetric 2-$(n, k, \mu)$ design.
Jungnickel–Tonchev (1999) (also Jungnickel (1982)).

\[ \exists \text{BGW}(\frac{q^{d+1} - 1}{q - 1}, q^d, q^d - q^{d-1}) \text{ over } \text{GF}(q)^\times. \]

Complements of intersection hyperplanes

\[ G \to 1: \text{incidence matrix of the symmetric design whose blocks are complements of hyperplanes.} \]
- $B$: $\text{BGW}(n, k, \mu)$ over $G$,

- $\chi : G \to H$: surjective homo. Define $\chi(0) = 0$.

Then $\chi(B)$: $\text{BGW}(n, k, \mu)$ over $H$. In particular,

- $G \to \{\pm 1\}$ surjective $\implies \text{BGW}(n, k, \mu) \to W(n, k)$.

- If $q \equiv 1 \pmod{4}$, then $\text{GF}(q)^\times \to \{\pm 1, \pm i\}$ (surjective).

$$\exists \text{BGW}\left(\frac{q^{d+1} - 1}{q - 1}, q^d, q^d - q^{d-1}\right) \text{ over } \{\pm 1, \pm i\}$$
Lemma

\[ B = X + iY: \text{BGW}(n, k, \mu) \text{ over } \{\pm1, \pm i\}, \text{ where } X \text{ and } Y \text{ are } (0, \pm1)\text{-matrices. Then} \]

\[ W = \begin{bmatrix} X & Y \\ -Y & X \end{bmatrix} \]

is a \( W(2n, k) \).

Proof.

In \( M_n(\mathbb{C}) \), \( BB^* = kI \implies WW^T = kI \).

Thus

\[ \exists W(2 \frac{q^{d+1} - 1}{q - 1}, q^d) \text{ if } q \equiv 1 \pmod{4}. \]

This gives the \( W(12, 5) \) missed by Chan–Rodger–Seberry.
Distorting blocks of $\text{PG}_e(2e, q)$

- $V = V(2e + 1, q)$, $\text{PG}(2e, q) = \binom{V}{1}$,
- Geometric design $\text{PG}_e(2e, q)$ has blocks $\binom{V}{e+1}$.

$$2\left(\frac{q^{2e+1} - 1}{q - 1}, \frac{q^{e+1} - 1}{q - 1}, \binom{2e - 1}{e - 1}\right)$$

**Distorting** (Jungnickel–Tonchev, 2009): fix $H \in \binom{V}{2e}$ and a polarity $\sigma$ on $H$ ($\sigma$ permutes $\binom{H}{e}$).

For $W \in \binom{V}{e+1}$ with $W \cap H \in \binom{H}{e}$, replace $W \cap H$ by $\sigma(W \cap H)$.

$\Rightarrow$ 2-design with the same parameters but not isomorphic as the geometric design.
Let $V = V(n, q)$. The Grassmann graph $J_q(n, d)$ has vertex set $= \binom{V}{d}$. The adjacency is defined as follows:

$$W_1 \sim W_2 \iff \dim W_1 \cap W_2 = d - 1.$$ 

Then $J_q(n, d)$ is a distance-transitive graph, with intersection array

$$b_i = q^{2i+1} \frac{(q^{d-i} - 1)(q^{n-d-i} - 1)}{(q - 1)^2}, \quad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}^2.$$
The intersection array

\[ \Gamma_i(x) = \{ \text{vertices at distance } i \text{ from } x \} \ni y. \]

\[ c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(y)|. \quad b_i = |\Gamma_{i+1}(x) \cap \Gamma_1(y)|. \]
Let $V = V(n, q)$. The Grassmann graph $J_q(n, d)$ has vertex set $= \binom{V}{d}$. The adjacency is defined as follows:

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Characterization (Metsch, 1995): $J_q(n, d)$ is characterized by the intersection array, in many cases, but $(n, d) = (2e + 1, e + 1)$ was left open.
Twisted Grassmann graph (Van Dam–Koolen, 2005)

\[ V = V(2e + 1, q), \ H \in \binom{V}{2e}. \]

Define

\[ \mathcal{A} = \{ W \in \binom{V}{e + 1} | W \not\subset H \}, \]

\[ \mathcal{B} = \begin{bmatrix} H \\ e - 1 \end{bmatrix}. \]

The adjacency on \( \mathcal{A} \cup \mathcal{B} \) is defined as follows:

\[ W_1 \sim W_2 \iff \dim W_1 \cap W_2 - \frac{1}{2}(\dim W_1 + \dim W_2) + 1 = 0. \]

This graph has the same intersection array as the Grassmann graph \( J_q(2e + 1, e + 1) \) with vertex set \( \binom{V}{e+1} \).
Let $\sigma$ be a polarity of $H$. Points are $\text{PG}(2e, q)$. Blocks are

$$\mathcal{A}' = \{(W \setminus H) \cup \sigma(W \cap H) \mid W \in \begin{bmatrix} V \\ e+1 \end{bmatrix}, \ W \not\subset H\},$$

$$\mathcal{B}' = \begin{bmatrix} H \\ e+1 \end{bmatrix}.$$  

This design has the same parameters, $q$-rank, and block intersection numbers as the geometric design whose blocks are $\begin{bmatrix} V \\ e+1 \end{bmatrix}$. 
The isomorphism

\[ A = \left\{ W \in \begin{bmatrix} V \\ e+1 \end{bmatrix} \mid W \not\subset H \right\} , \quad \mathcal{B} = \begin{bmatrix} H \\ e-1 \end{bmatrix} , \]

\[ A' = \left\{ (W \setminus H) \cup \sigma(W \cap H) \mid W \in A \right\} , \quad \mathcal{B}' = \begin{bmatrix} H \\ e+1 \end{bmatrix} . \]

**Lemma**

Define \( f : A \cup \mathcal{B} \to A' \cup \mathcal{B}' \) by

\[ f(W) = \begin{cases} (W \setminus H) \cup \sigma(W \cap H) & \text{if } W \in A, \\ \sigma(W) & \text{if } W \in \mathcal{B}. \end{cases} \]

Then for \( W_1, W_2 \in A \cup \mathcal{B} \), the blocks \( f(W_1) \) and \( f(W_2) \) meet at

\[ \left\lfloor \dim W_1 \cap W_2 - \frac{\dim W_1 + \dim W_2}{2} + 1 + e \right\rfloor \text{ points.} \]
The twisted Grassmann graph is the block graph

**Theorem (M.–Tonchev)**

The twisted Grassmann graph, is isomorphic to the block graph of the distorted design \((\text{PG}(2e, q), \mathcal{A}' \cup \mathcal{B}')\), where two blocks are adjacent iff they have the largest possible intersection: \(\left[\begin{array}{c} e \\ 1 \end{array}\right]\).

**Corollary**

The automorphism group of the distorted design is the same as that of the twisted Grassmann graph, which is the stabilizer of \(H\) in \(\text{PGL}(2e + 1, q)\).

**Proof.**