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Binomial coefficients

\[
\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots1} = \frac{n!}{r!(n-r)!}
\]

is an integer, because it counts the number of \(r\)-subsets of an \(n\)-set. The middle binomial coefficient

\[
\binom{2n}{n} = \frac{(2n)!}{n!n!}
\]

is not only an integer, but also divisible by \(n + 1\). That is, the Catala number

\[
C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}
\]

is an integer, because . . .
The Catalan number

\[ C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \]

is an integer, because it counts the number of different ways \( n + 1 \) factors can be completely parenthesized:

\[ ((ab)c)d, (a(bc))d, (ab)(cd), a((bc)d), a(b(cd)) \].

\[ C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i} \].
Catalan (1874) also observed

\[ S(m, n) = \frac{(2m)! (2n)!}{m! n! (m+n)!} \in \mathbb{Z}. \]

\[ S(0, n) = \binom{2n}{n}, \quad S(1, n) = 2C_n = \frac{2}{n+1} \binom{2n}{n}. \]

Gessel–Xin (2005) gave a combinatorial reason for \( S(2, n), S(3, n) \in \mathbb{Z} \).

On the other hand, von Szily’s identity (1894)

\[ S(m, n) = \sum_{k \in \mathbb{Z}} (-1)^k \binom{2m}{m+k} \binom{2n}{n-k} \]

implies that \( S(m, n) \in \mathbb{Z} \).
Combinatorial interpretation of von Szily’s identity

\[ S(m, n) = \sum_{k \in \mathbb{Z}} (-1)^k \binom{2m}{m+k} \binom{2n}{n-k} \]

\[ = (-1)^m \sum_{h=0}^{m+n} (-1)^h \binom{2m}{h} \binom{2n}{m+n-h} \quad (h = m+k) \]

\[ S(m, n) = (-1)^m \sum_{h=0}^{m+n} (-1)^h \left| \begin{array}{c} \{ X \} \\
X \subset M \cup N \\
|X| = m+n \\
|X \cap M| = h \end{array} \right|, \]

where \(|M| = 2m, |N| = 2n, M \cap N = \emptyset|\).
Lattice paths

Let $|M| = 2m$, $|N| = 2n$, $M \cap N = \emptyset$, so $|M \cup N| = 2(m + n)$. 

$$\left| \left\{ X \mid X \subset M \cup N \text{ and } |X| = m + n \right\} \right|$$

counts the number of all lattice paths from $(0, 0)$ to $(m + n, m + n)$ consisting of unit steps right or up.
\[ |M| = 2m, \ |N| = 2n, \ M \cap N = \emptyset \]

\[
\left\{ \begin{array}{l}
X \subset M \cup N \\
|X| = m + n \\
|X \cap M| = h
\end{array} \right. 
\]

counts the number of all lattice paths from \((0, 0)\) to \((m + n, m + n)\) consisting of unit steps \(\rightarrow\) or \(\uparrow\), such that the height is \(h\) after the \(2m\)-th step.

\[
m = 6 \\
n = 4 \\
2m = 12 \\
h = 5
\]
\[ |M| = 2m, \ |N| = 2n, \ M \cap N = \emptyset \]

\[ S(m, n) \]

\[ = (-1)^m \sum_{h=0}^{m+n} (-1)^h \binom{2m}{h} \binom{2n}{m+n-h} \]

\[ = (-1)^m \sum_{h=0}^{m+n} (-1)^h \left\{ X \subseteq M \cup N \; \begin{array}{l} |X| = m + n \vspace{0.5em} \\
|X \cap M| = h \end{array} \right\} \]

\[ = (-1)^m \sum_{h=0}^{m+n} (-1)^h \left\{ \text{lattice paths from } (0,0) \text{ to } (m+n, m+n) \text{ consisting of unit steps } \rightarrow \text{ or } \uparrow, \text{ such that the height is } h \text{ after the } 2m\text{-th step} \right\} \]
Let $F_2 = \{0, 1\}$ denote the finite field with two elements (equipped with binary addition and multiplication). For a vector $x = (x_1, \ldots, x_d) \in F_2^d$,

$$\text{supp}(x) = \{i \mid 1 \leq i \leq d, \ x_i = 1\},$$

$$\text{wt}(x) = |\text{supp}(x)|$$

Let $z = (1, \ldots, 1, 0, \ldots, 0)$, $\text{supp}(z) = 2m$. Then

$$\left| \left\{ X \subseteq M \cup N \mid |X| = m + n, |X \cap M| = h \right\} \right| = \sum_{x \in F_2^{2(m+n)}} \frac{1}{\text{wt}(x)=m+n} \text{supp}(x) \cap \text{supp}(z) = h$$

counts the number of all binary vectors $x \in F_2^{2(m+n)}$ of weight $m + n$, such that $\text{supp}(x) \cap \text{supp}(z) = h$. 

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Super Catalan numbers
\[ S(m, n) = (-1)^m \sum_{h=0}^{m+n} (-1)^h \left\{ \begin{array}{c} X \\ |X| = m + n \\ |X \cap M| = h \end{array} \right\} \]
\[ = (-1)^m \sum_{h=0}^{m+n} (-1)^h \sum_{\mathbf{x} \in \mathbb{F}_2^{2(m+n)}} 1 \]
\[ \text{with } \mathbf{x} \in \mathbb{F}_2^{2(m+n)} \]
\[ \text{wt}(\mathbf{x}) = m + n \]
\[ |\text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{z})| = h \]
\[ = (-1)^m \sum_{h=0}^{m+n} (-1)^{|\text{supp}(\mathbf{x}) \cap \text{supp}(\mathbf{z})|} \]
\[ \sum_{\mathbf{x} \in \mathbb{F}_2^{2(m+n)}} 1 \]
\((-1)^m S(m, n) = \sum_{\substack{x \in \mathbb{F}_2^{2(m+n)} \\ \text{wt}(x) = m+n}} (-1)^{\langle x, z \rangle}.
\]

Krawtchouk polynomial \(K_d^j(x)\) is defined by
\[
K_d^j(z) = \sum_{\substack{x \in \mathbb{F}_2^d \\ \text{wt}(x) = j}} (-1)^{\langle x, z \rangle}, \quad \text{where } \text{wt}(z) = z.
\]
\[
= \sum_{h=0}^{j} (-1)^h \binom{z}{h} \binom{d-z}{j-h}.
\]

Then
\[
(-1)^m S(m, n) = K_{m+n}^{2(m+n)}(2m).
\]
\[-1\]m S(m, n) = K_{m+n}^{2(m+n)}(2m)

Krawtchouk polynomials are the eigenvalues of the distance-\(j\) graph of the \(d\)-cube. More precisely,

\[
\{(-1)^m S(m, n) \mid m, n \geq 0, m + n = d\} \cup \{0\}
\]

coincides with the set of eigenvalues of the distance-\(d\) graph of the \(2d\)-cube, which is known as the orthogonality graph.

vertices = \(\{\pm 1\}^{2d}\),

\(x \sim y \iff \langle x, y \rangle = 0\).
MacWilliams identities $C \subset \mathbb{F}_2^d$, $C^\perp \subset \mathbb{F}_2^d$

$$|\{x \in C^\perp \mid \text{wt}(x) = j\}|$$

$$= \sum_{x \in \mathbb{F}_2^d} \frac{1}{|C|} \sum_{z \in C} (-1)^{\langle x, z \rangle}$$

$$= \frac{1}{|C|} \sum_{z=0}^{d} \sum_{z \in C} \sum_{x \in \mathbb{F}_2^d} (-1)^{\langle x, z \rangle}$$

$$= \frac{1}{|C|} \sum_{z=0}^{d} \sum_{z \in C} K_j^d(z)$$

$$= \frac{1}{|C|} \sum_{z=0}^{d} K_j^d(z) |\{z \in C \mid \text{wt}(z) = z\}|.$$
\( S(m, n) \) is the size of a set?

\[
S(m, n) = (-1)^m \sum_{h=0}^{m+n} (-1)^h \left| \left\{ X \mid X \subset M \cup N, |X| = m + n, |X \cap M| = h \right\} \right|
\]

\[
= \sum_{0 \leq h \leq m+n} \left\{ X \mid X \subset M \cup N, |X| = m + n, |X \cap M| = h \right\} \left| \begin{array}{c}
X
\end{array} \right|
\]

\[
- \sum_{0 \leq h \leq m+n} \left\{ X \mid X \subset M \cup N, |X| = m + n, |X \cap M| = h \right\} \left| \begin{array}{c}
X
\end{array} \right|
\]
Problem

Find an injection from

\[
\bigcup_{0 \leq h \leq m+n \atop h+m: \text{odd}} \left\{ X \mid X \subset M \cup N, \quad |X| = m+n, \quad |X \cap M| = h \right\}
\]

to

\[
\bigcup_{0 \leq h \leq m+n \atop h+m: \text{even}} \left\{ X \mid X \subset M \cup N, \quad |X| = m+n, \quad |X \cap M| = h \right\}
\]

and describe the complement of the image.