On the smallest eigenvalues of the line graphs of some trees

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g_i(x) = (x + 1 - s_{n-i+2})g_{i-1}(x) - s_{n-i+2}g_{i-2}(x).$$
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where $X$ is a $|V| \times m$ matrix,
\[ \Gamma = (V, E) : \text{finite undirected simple connected graph} \]

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where \( X \) is a \(|V| \times m\) matrix, giving a representation:

\[ (u, v) = \begin{cases} 
-\lambda_{\min}(\Gamma) & \text{if } u = v, \\
1 & \text{if } u \sim v, \\
0 & \text{otherwise.} 
\end{cases} \]
The line graph of $\Gamma = (V, E)$ has $E$ as its vertex set, and $e \sim f$ iff there exists a common vertex in $e$ and $f$. 
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$\lambda_{\min}(L(\Gamma)) \geq -2.$

Conversely, all graphs $\Delta$ with $\lambda_{\min}(\Delta) \geq -2$ are essentially known (generalized line graphs + finitely many exceptions).
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This is the corona of $K_n$ and $K_s$. The spectrum can be obtained by a formula of Schwenk (1973).
The corona $K_n \boxtimes K_s$ of $K_n$ and $K_s$ is the line graph of the tree

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Cvetković–Stevanović (2003): are there other family of line graphs of trees with constant smallest eigenvalue?
Let $T_{s_1,s_2,...,s_n}$ be the tree depicted below:

Then $\lambda_{\min}(L(T_{s_1,s_2,...,s_n}))$ is the smallest zero of the polynomial $g_n(x)$, where

$$
g_0(x) = 1, \quad g_1(x) = x + 1,$$

$$
g_i(x) = (x + 1 - s_{n-i+2})g_{i-1}(x) - s_{n-i+2}g_{i-2}(x).$$

In particular, it is independent of $s_1$. 
The characteristic polynomial of \( L(T_{s_1,s_2,...,s_n}) \) is

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\frac{1}{x + 2} \left( g_{n+1}(x) + g_n(x) \right) \prod_{i=1}^{n} g_i(x)^{\sigma_{n-i+1}}
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where

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The characteristic polynomial of $L(T_{s_1, s_2, \ldots, s_n})$ is

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where

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$$m_j = \prod_{i=1}^{j} s_i \quad (j = 0, 1, \ldots, n),$$

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