A parametric family of complex Hadamard matrices

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A (real) Hadamard matrix of order $n$ is an $n \times n$ matrix $H$ with entries $\pm 1$, satisfying $HH^\top = nI$.

A complex Hadamard matrix of order $n$ is an $n \times n$ matrix $H$ with entries in $\{\xi \in \mathbb{C} \mid |\xi| = 1\}$, satisfying $HH^* = nl$, where $*$ means the conjugate transpose.
Hadamard matrices and generalizations

\[
H = \begin{bmatrix}
-1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 \\
\end{bmatrix}
\]

\[HH^\top = 4I\]

\[
H = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 \\
1 & \omega^2 & \omega \\
\end{bmatrix}
\]

\[HH^* = 3I\]
Existence and classification

Conjecture

For any $n \equiv 0 \pmod{4}$, a Hadamard matrix of order $n$ exists.

Known for $n \leq 664$. Classified for $n \leq 32$.

For any $n$, a complex Hadamard matrix of order $n$ exists.
An example is given by the character table of an abelian group of order $n$. 
Complex Hadamard matrices

Classified up to order 5 (unique by Haagerup 1996). Open for order $\geq 6$.

**Definition**

Two complex Hadamard matrices $H_1, H_2$ are **equivalent** if $H_1 = PH_2Q$ for some monomial matrices $P, Q$ whose nonzero entries are complex numbers with absolute value 1.

If $n$ is not a prime, then there are uncountably many inequivalent complex Hadamard matrices, up to equivalence.

symmetric regular Hadamard matrix $\iff$ certain strongly regular graph:

$$H = I + A_1 - A_2, \quad J = I + A_1 + A_2.$$ 

$A_1 = \text{adjacency matrix}$

Chan and Godsil (2010):

complex Hadamard matrices $\iff$ certain strongly regular graph

$$H = I + w_1 A_1 + w_2 A_2, \quad J = I + A_1 + A_2.$$
$A_1 = \text{adjacency matrix}$

Chan (2011):

$$H = I + w_1 A_1 + w_2 A_2 + w_3 A_3, \quad J = I + A_1 + A_2 + A_3.$$  

complex Hadamard matrices $\not\Leftarrow$ certain distance-regular graphs of diameter 3

Ikuta and Munemasa (2014+):

complex Hadamard matrices $\Leftarrow$ certain symmetric association scheme of class 3.

$A_i$ are pairwise commutative symmetric disjoint $(0, 1)$-matrices, such that $\langle I, A_1, A_2, A_3 \rangle$ is closed under multiplication (Bose-Menser algebra).
Bose-Mesner algebra

Let $A_1, A_2, A_3$ be pairwise commutative symmetric disjoint $(0, 1)$-matrices satisfying $I + A_1 + A_2 + A_3 = J$, such that $\mathcal{A} = \langle I, A_1, A_2, A_3 \rangle$ is closed under multiplication (Bose-Menser algebra). Then $A_i$ are simultaneously diagonalizable.

Example

Example: Cubic residues in finite fields (Cyclotomic schemes)

$V_0 = \text{Ker}(A_1 - fI) = \text{Ker}(A_2 - fI) = \text{Ker}(A_3 - fI)$

$V_1 = \text{Ker}(A_1 - \theta_1 I) = \text{Ker}(A_2 - \theta_3 I) = \text{Ker}(A_3 - \theta_2 I)$

$V_2 = \text{Ker}(A_1 - \theta_2 I) = \text{Ker}(A_2 - \theta_1 I) = \text{Ker}(A_3 - \theta_3 I)$

$V_3 = \text{Ker}(A_1 - \theta_3 I) = \text{Ker}(A_2 - \theta_2 I) = \text{Ker}(A_3 - \theta_1 I)$
Let $A_1, A_2, A_3$ be pairwise commutative symmetric disjoint $(0, 1)$-matrices satisfying $I + A_1 + A_2 + A_3 = J$, such that $\mathcal{A} = \langle I, A_1, A_2, A_3 \rangle$ is closed under multiplication (Bose-Menser algebra). Then $A_i$ are simultaneously diagonalizable.

**Definition**

$\mathcal{A}$ is called pseudocyclic if the $\mathbb{R}^n = V_0 \oplus V_1 \oplus V_2 \oplus V_3$ : common eigenspace decomposition, such that $\downarrow$

\[
\begin{align*}
V_0 &= \ker(A_1 - fI) = \ker(A_2 - fI) = \ker(A_3 - fI) \\
V_1 &= \ker(A_1 - \theta_1 I) = \ker(A_2 - \theta_3 I) = \ker(A_3 - \theta_2 I) \\
V_2 &= \ker(A_1 - \theta_2 I) = \ker(A_2 - \theta_1 I) = \ker(A_3 - \theta_3 I) \\
V_3 &= \ker(A_1 - \theta_3 I) = \ker(A_2 - \theta_2 I) = \ker(A_3 - \theta_1 I)
\end{align*}
\]
Pseudocyclic Bose-Mesner algebra

Conjecture

Given a pseudocyclic Bose-Mesner algebra \( \langle I, A_1, A_2, A_3 \rangle \) of order \( n = 3f + 1 \) with eigenvalues \( f, \theta_1, \theta_2, \theta_3 \), TFAE:

(i) there are infinitely many complex Hadamard matrices of the form \( I + w_1A_1 + w_2A_2 + w_3A_3 \),

(ii) \( \theta_1, \theta_2, \theta_3 \) are not distinct.

(ii) \( \iff \) amorphic. We show (ii) \( \implies \) (i).

\[
\begin{align*}
V_0 &= \text{Ker}(A_1 - fl) = \text{Ker}(A_2 - fl) = \text{Ker}(A_3 - fl) \\
V_1 &= \text{Ker}(A_1 - \theta_1 l) = \text{Ker}(A_2 - \theta_3 l) = \text{Ker}(A_3 - \theta_2 l) \\
V_2 &= \text{Ker}(A_1 - \theta_2 l) = \text{Ker}(A_2 - \theta_1 l) = \text{Ker}(A_3 - \theta_3 l) \\
V_3 &= \text{Ker}(A_1 - \theta_3 l) = \text{Ker}(A_2 - \theta_2 l) = \text{Ker}(A_3 - \theta_1 l)
\end{align*}
\]
Amorphic Bose-Mesner algebra

\[ H = I + w_1 A_1 + w_2 A_2 + w_3 A_3 : \text{ order } n = q^2, \]
\[ H^* = I + \overline{w_1} A_1 + \overline{w_2} A_2 + \overline{w_3} A_3, \]
\[ H^{(-)} = I + \frac{1}{w_1} A_1 + \frac{1}{w_2} A_2 + \frac{1}{w_3} A_3, \]
\[ e_1 = w_1 + w_2 + w_3, \quad e_2 = w_1 w_2 + w_2 w_3 + w_3 w_1, \]
\[ e_3 = w_1 w_2 w_3. \]

**Proposition**

Assume \( q \geq 4 \).

\[ HH^{(-)} = nl \iff e_1 = -3/(q - 1), \quad e_2 = e_1 e_3. \]
\[ e_1 = -3/(q - 1), \quad e_2 = e_1 e_3 \]

\[
(x - w_1)(x - w_2)(x - w_3) = x^3 - e_1 x^2 + e_2 x - e_3.
\]

\[
|w_1| = |w_2| = |w_3| = 1 \iff |e_3| = 1.
\]

Cohn 1922 gave a general condition for a polynomial equation to have all of its roots on the unit circle (a simpler one by Lakatos and Losoncz, 2009).

For any \( e_3 \) with absolute value 1, the complex numbers \( w_1, w_2, w_3 \) defined by the cubic equation above, gives a complex Hadamard matrix

\[
I + w_1 A_1 + w_2 A_2 + w_3 A_3,
\]

of order \( n = q^2 \), where \( A_1, A_2, A_3 \) are the adjacency matrices in an amorphic pseudocyclic Bose-Mesner algebra.