Self-orthogonal designs and equitable partitions

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September 20, 2015
International Workshop on Algebraic Combinatorics
Zhejiang University
Theorem

The following are equivalent:

1. ∃ a self-orthogonal 3-(16, 8, 3µ) design,
2. ∃ an equitable partition of the folded halved 8-cube with quotient matrix

\[
\begin{bmatrix}
4(\mu - 1) & 4(8 - \mu) \\
4\mu & 4(7 - \mu)
\end{bmatrix}.
\]

3. µ ∈ \{1, 2, 3, 4, 5\}. 

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Self-orthogonal designs
A $t$-$(\nu, k, \lambda)$ design $(X, \mathcal{B})$

e.g., $3$-$(16, 8, 3\mu)$ design

- $X$ is a finite set, $|X| = \nu,$
- $\mathcal{B} \subset \binom{X}{k} = \{k$-element subsets of $X\},$
- $\forall T \in \binom{X}{t},$

$$\lambda = |\{B \in \mathcal{B} \mid B \supset T\}|.$$

Elements of $X$ are called “points”, elements of $\mathcal{B}$ are called “blocks”.
A $t-(v, k, \lambda)$ design $(X, \mathcal{B})$ is **self-orthogonal** if

$$|B \cap B'| \equiv 0 \pmod{2} \quad (\forall B, B' \in \mathcal{B}).$$

In particular $k \equiv 0 \pmod{2}$. 
A $t-(v, k, \lambda)$ design $(X, \mathcal{B})$ is self-orthogonal if

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Let $M$ be the block-point incidence matrix. Then

self-orthogonal $\iff MM^\top = 0$ over $\mathbb{F}_2$.

We call the row space $C$ of $M$ the (binary) code of the design. Then $C \subset C^\perp$.

(Often $C \subset D = D^\perp \subset C^\perp$.)
If $H$ is a Hadamard matrix of order $8n$, i.e., $H$ is a $8n \times 8n$ matrix with entries in $\{\pm 1\}$ satisfying $HH^\top = 8nI$, 

$\implies$ a self-orthogonal $3-(8n, 4n, 2n - 1)$ design.
Hadamard 3-designs

If $H$ is a Hadamard matrix of order $8n$, i.e., $H$ is a $8n \times 8n$ matrix with entries in $\{\pm 1\}$ satisfying $HH^\top = 8nI$,

$\implies$ a self-orthogonal 3-$(8n, 4n, 2n - 1)$ design.

Indeed, after normalizing $H$ so that its first row is 1:

$$H = \left[ \begin{array}{c} 1 \\ H_1 \end{array} \right],$$

an incidence matrix is given by

$$M = \frac{1}{2} \left[ \begin{array}{c} J - H_1 \\ J + H_1 \end{array} \right].$$
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3-$(16, 8, 3)$ Hadamard design is self-orthogonal. Do there exist 3-$(16, 8, 3\mu)$ designs for $\mu > 1$?
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3-$(16, 8, 3)$ Hadamard design is self-orthogonal. Do there exist 3-$(16, 8, 3\mu)$ designs for $\mu > 1$? (take union?)
Existence problem

Given \( t, v, k, \lambda \), does there exist a \( t-(v, k, \lambda) \) design?

Before Teirlinck (1987), only a few \( t \)-designs with \( t \geq 5 \) were known.
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In our work we only consider orthogonality mod 2. The $5-(12, 6, 1)$ design of Witt (1938) is not self-orthogonal.
A $k$-dimensional subspace of $\mathbb{F}_2^n$ is called an $[n, k]$ code. For an $[n, k]$ code $C$, its minimum weight is

$$\min C = \min\{\text{wt}(x) \mid 0 \neq x \in C\}.$$ 

and $C$ is called an $[n, k, d]$ code if $d = \min C$. 
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and $C$ is called an $[n, k, d]$ code if $d = \min C$. A code $C$ is doubly even if

$$\text{wt}(x) \equiv 0 \pmod{4} \quad (\forall x \in C),$$

self-orthogonal if

$$C \subset C^\perp,$$

self-dual if

$$C = C^\perp.$$
A consequence of the Assmus–Mattson theorem: Doubly even self-dual \([24m, 12m, 4m + 4]\) code \(\rightarrow 5-(24m, 4m + 4, \lambda)\) design.
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- \(m = 1\): Witt design; related designs were characterized by Tonchev (1986)
- \(m = 2\): Harada-M.-Tonchev (2005)
- \(m \geq 3\): existence is unknown: Harada-M.-Kitazume (2004), \(m = 4\) by Harada (2005), \(m \geq 5\) by de la Cruz and Willems (2012).
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- \(m = 1\): Witt design; related designs were characterized by Tonchev (1986)
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For \(m \geq 3\), existence is unknown:

- \(m = 3\) by Harada-M.-Kitazume (2004),
- \(m = 4\) by Harada (2005),
- \(m \geq 5\) by de la Cruz and Willems (2012).
Lalaude-Labayle (2001), determined binary self-orthogonal codes of min. wt. $k$ whose min. wt. codewords support:

- 3-design for $k \leq 10$,
- 5-design for $k \leq 18$. 

Motivated by spherical analogue:

Venkov’s theorem on spherical designs supported by an even unimodular lattice

Martinet (2001): lattices of min $\leq 3$ with spherical 5-design, min $\leq 5$ with spherical 7-design

Nossek (2014): lattices of min $\leq 7$ with spherical 9-design, min $\leq 9$ with spherical 11-design, $\not\exists$ lattices of min $\leq 11$ with spherical 13-design.
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Design theoretic viewpoint

Instead of classifying self-orthogonal codes $C$ of min. wt. $k$ such that

$$B = \{ \text{supp}(x) \mid x \in C, \text{wt}(x) = k \}$$

forms a $t$-design,
Design theoretic viewpoint

Instead of classifying self-orthogonal codes \( C \) of min. wt. \( k \) such that

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\mathcal{B} = \{ \text{supp}(x) \mid x \in C, \ \text{wt}(x) = k \}
\]

forms a \( t \)-design,

we hope to classify self-orthogonal designs:

\[
\mathcal{B} \subset \{ x \in C \mid \text{wt}(x) = k \} \subset C \subset C^\perp.
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Note: $k$ is even.

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More generally, we assume

$$t \geq \left\lceil \frac{k}{4} \right\rceil + 1.$$ 

Note: $k$ is even.

Mendelsohn equations are “overdetermined” system.
Let $(X, \mathcal{B})$ be a $t-(v, k, \lambda)$ design, $S \subset X$.

$$n_j = \left| \{ B \in \mathcal{B} \mid j = |B \cap S| \} \right|.$$ 

Then

$$\sum_{j \geq 1} \binom{j}{i} n_j = \lambda_i \binom{|S|}{i} \quad (i = 1, \ldots, t),$$

a system of $t$ linear equations in unknowns $n_1, n_2, \ldots$ (at most $\min\{k, |S|\}$).
Mendelsohn equations

Let \((X, \mathcal{B})\) be a \(t-(v, k, \lambda)\) design, \(S \subset X\).

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If \(S \in \mathcal{C}^\perp\), then \(n_j = 0\) for \(j\) odd.

If \(S \in \mathcal{B}\) and \(k = \min \mathcal{C}^\perp\), then \(n_j = 0\) for \(j > k/2\), so there are \(\lfloor k/4 \rfloor\) unknowns.

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Self-orthogonal designs
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**Lemma**

If $(X, \mathcal{B})$ is a self-orthogonal $3$-$(v, k, \lambda)$ design, and the dual code of its code has minimum weight $4$, then $v = 2k \equiv 0 \pmod{4}$. 
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Recall $3$-$(8, 4, 1)$ Hadamard design exists.
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**Lemma**

If $(X, B)$ is a self-orthogonal 3-$(v, k, \lambda)$ design, and the dual code of its code has minimum weight 4, then $v = 2k \equiv 0 \pmod{4}$.

Recall 3-$(8, 4, 1)$ Hadamard design exists. No self-orthogonal 3-$(12, 6, \lambda)$ design.
3-(16, 8, \lambda) design

\lambda \leq \binom{16}{8} \binom{8}{3} \binom{16}{3}^{-1} = 1287

if we don’t require self-orthogonality.

- Divisibility implies \( \lambda \equiv 0 \pmod{3} \).
3-(16, 8, \lambda) \text{ design}

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- Divisibility implies \lambda \equiv 0 \pmod{3}.
- Largest number of vectors of weight 8 in a self-orthogonal codes of length 16
  \implies \lambda \leq 18.
$3$-(16, 8, $\lambda$) design

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$\lambda = 3$: Hadamard designs.

$\lambda = 6, 9, 12, 15, 18$? disjoint union?
Theorem

The following are equivalent:

1. \( \exists \) a self-orthogonal 3-(16, 8, 3\( \mu \)) design,
2. \( \exists \) an equitable partition of the folded halved 8-cube with quotient matrix
   \[
   \begin{bmatrix}
   4(\mu - 1) & 4(8 - \mu) \\
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   \end{bmatrix}.
   \]
3. \( \mu \in \{1, 2, 3, 4, 5\} \).

In particular, there is no self-orthogonal 3-(16, 8, 18) design.
The 8-cube is the graph with vertex set \( \{0, 1\}^8 \), two vertices are adjacent whenever they differ by exactly one coordinate.
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‘halved’ = even-weight vectors
‘folded’ = identify with complement

The folded halved $8$-cube $\Gamma$ has $2^6$ vertices, and it is $28$-regular.

$$SRG(64, 28, 12, 12)$$
Let $\Gamma$ be a regular graph. An equitable partition with quotient matrix $Q$ means: the adjacency matrix $A$ is of the form

\[
A = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}, \quad A_{ij} \mathbf{1} = q_{ij} \mathbf{1}, \quad Q = (q_{ij}).
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The code $d_{16}$

The code of a self-orthogonal design is contained in a doubly even self-dual $[16, 8]$ code. There are only two such codes, $e_8 \oplus e_8$ and $d_{16}$. 
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\[ |\mathcal{B}| = 30\mu \quad (15\mu \text{ pairs}). \]
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$$|\mathcal{B}| = 30\mu \quad (15\mu \text{ pairs}).$$

$$8\mu \quad 64 = \left|\frac{1}{2}H(8, 2)\right|$$

* $7\mu \quad 35 = \left|\frac{1}{2}J(8, 4)\right|$

$$= |J_2(4, 2)| = |PG(3, 2)|$$
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Self-orthogonal 3-design

- ∃ 3-(8, 4, 1) Hadamard design
- ∄ 3-(12, 6, \(\lambda\)) design
- ∃ 3-(16, 8, 3\(\mu\)) design for \(\mu \in \{1, \ldots, 5\}\)
Self-orthogonal 3-design

- $\exists$ 3-(8, 4, 1) Hadamard design
- $\nexists$ 3-(12, 6, $\lambda$) design
- $\exists$ 3-(16, 8, 3$\mu$) design for $\mu \in \{1, \ldots, 5\}$
- $\nexists$ 3-(20, 10, $\lambda$) design
Self-orthogonal 3-design

- $\exists$ 3-$(8, 4, 1)$ Hadamard design
- $\not\exists$ 3-$(12, 6, \lambda)$ design
- $\exists$ 3-$(16, 8, 3\mu)$ design for $\mu \in \{1, \ldots, 5\}$
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These satisfy $\lfloor k/4 \rfloor + 1 \leq t = 3$. 
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These satisfy $\lfloor k/4 \rfloor + 1 \leq t = 3$.

- $\exists$ 5-(24, 12, 48) design (Uniqueness by Tonchev, 1986)
Lalaude-Labayle (2001), determined binary codes of min. wt. $k$ whose min. wt. codewords support:

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More generally, we assume

$$t = \left\lfloor \frac{k}{4} \right\rfloor + 1, \quad k = \min C,$$

but we allow

$$\mathcal{B} \subseteq \{ x \in C \mid \text{wt}(x) = k \}.$$
Theorem

∃ self-orthogonal \( t-(\nu, k, \lambda) \) design with code \( C \),

\[
t = \left\lfloor \frac{k}{4} \right\rfloor + 1, \quad k = \min C.
\]

Then

\[
2^{2t-1}t \binom{k/2}{k/2-t} \prod_{j=i}^{t-1} (k - j) \prod_{j=i}^{t-1} (\nu - j) \in \mathbb{Z}.
\]

Note: Given \( k \), there are only finitely many \( \nu \) satisfying the conclusion. Lalaude-Labayle: \( k \leq 18 \).

The only \( k > 18 \) we found which satisfies the conclusion is \( k = 24, \nu = 120, t = 7 \) (but \( \not\exists \)).
There exists self-orthogonal $t$-$(v, k, \lambda)$ design $(X, \mathcal{B})$ with code $C$, $d^\perp = \min C^\perp$,

$$t = \left\lfloor \frac{k}{4} \right\rfloor + 1, \quad \mathcal{B} \neq \{x \in C^\perp \mid \text{wt}(x) = d^\perp \}.$$

Then

$$\sum_{i=1}^{t} i(-2)^{i-1} \binom{2t - i - 1}{t - 1} \binom{d^\perp}{i} \prod_{j=i}^{t-1} \frac{v - j}{k - j} = 0.$$

**Problem** Determine all the solutions of this Diophantine equation in $(d^\perp, k, v)$. 

Thank you for your attention!

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∃ self-orthogonal $t$-$(v, k, \lambda)$ design $(X, \mathcal{B})$ with code $C$, $d_{\perp} = \min C_{\perp}$,

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