Covering radii and shadows of binary self-dual codes

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We want to determine the image of the mapping

\[ \{ C \mid C \subset \mathbb{F}_2^n, \ C = C^\perp \} \rightarrow \mathbb{Z}[x, y] \]

defined by \( C \mapsto W_C(x, y) \), where

\[
W_C(x, y) = \sum_{c \in C} x^{n-\text{wt}(c)} y^{\text{wt}(c)},
\]

\[
\text{wt}(c) = |\{ i \mid c_i \neq 0 \}| \quad (c \in \mathbb{F}_2^n).
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If we restrict the domain to the set of doubly even codes, i.e.,

\[ \text{wt}(c) \equiv 0 \pmod{4} \quad (\forall c \in C), \]

then the image is contained in

\[ R = \mathbb{Q}[x^8 + 14x^4y^4 + y^8, W_{\text{Golay}}(x, y)] \]

Determining \( W_C(x, y) \) for a given \( C \) is computationally difficult \((|C| = 2^{n/2})\).
\[ R = \mathbb{Q}[x^8 + 14x^4y^4 + y^8, W_{\text{Golay}}(x, y)] \]

\[ W_{\text{Golay}}(x, y) = x^{24} + y^{24} + 759(x^{16}y^8 + x^8y^{16}) + 2576x^{12}y^{12}. \]

So

\[ \dim R(n) = 1 + \left\lfloor \frac{n}{24} \right\rfloor \quad (\text{if } 8 \mid n). \]
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\]

An **extremal** weight enumerator is the unique homogeneous polynomial of degree \( n \) whose coefficient of \( x^n \) is 1, and those of
\[
x^{n-4}y^4, x^{n-8}y^8, \ldots, x^{n-4\lfloor n/24 \rfloor}y^{4\lfloor n/24 \rfloor}
\]
are all zero. For example, \( W_{\text{Golay}}(x, y) \).

A code \( C \) is called **extremal** if \( W_C(x, y) \) is extremal.

Equivalently, \( C \) has minimum weight \( 4\lfloor n/24 \rfloor + 4 \), i.e.,
\[
\forall c \in C, \ \text{wt}(c) \neq 4, 8, \ldots, 4\lfloor n/24 \rfloor.
\]
Extremal doubly even self-dual codes

\[ C = C^\perp \subset \mathbb{F}_2^n, \ 8 \mid n, \]
all weights \( \equiv 0 \pmod{4} \),
minimum weight \( 4\lfloor n/24 \rfloor + 4 \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>( \geq 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n = 24k )</td>
<td>—</td>
<td>1</td>
<td>1</td>
<td>???</td>
<td>?\ldots</td>
</tr>
<tr>
<td>( n = 24k + 8 )</td>
<td>1</td>
<td>5</td>
<td>many?</td>
<td>many?</td>
<td>\ldots</td>
</tr>
<tr>
<td>( n = 24k + 16 )</td>
<td>2</td>
<td>16470</td>
<td>many?</td>
<td>many?</td>
<td>\ldots</td>
</tr>
</tbody>
</table>

- \( n = 72 \) open since Sloane (1973).
- Nonexistence for large \( n \) by Zhang (1999).
- Uniqueness for \( n = 48 \) by Houghten–Lam–Thiel–Parker (2003)
- Classification for \( n = 40 \) by Betsumiya–Harada–M. (2012)
The covering radius \( r(C') \) is defined as

\[
r(C') = \max \left\{ \min \{ \text{wt}(u) \mid u \in v + C \} \mid v + C \in \mathbb{F}_2^n/C \right\}.
\]

Computationally difficult.

Delsarte bound for extremal doubly even self-dual codes:

\[
r(C') \leq \begin{cases} 
4k & \text{if } n = 24k,
4k + 2 & \text{if } n = 24k + 8,
4k + 4 & \text{if } n = 24k + 16.
\end{cases}
\]
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\( r(C) \leq \) Delsarte bound.

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<td>4</td>
<td>16470</td>
</tr>
<tr>
<td>2</td>
<td>4 = 4</td>
<td>6 = 6</td>
<td>7, 8 \leq 8</td>
</tr>
<tr>
<td>3</td>
<td>8 = 8</td>
<td>many</td>
<td>many?</td>
</tr>
<tr>
<td>( \geq 4 )</td>
<td>?</td>
<td>many?</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
<td>( 10? = 10 )</td>
<td>( 12, 13 &lt; 14 )</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
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<td>( \ldots )</td>
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We now focus on the case \( n = 24k + 8 \).
Delsarte: \( r(C) \leq 4k + 2 \).

Suppose that a coset \( u + C \) has minimum weight \( 4k + 2 \). Let

\[
C_0 = C \cap \langle u \rangle^\perp,
\]

\[
C' = \langle C_0, u \rangle.
\]

Then \( C'' = C'^\perp \) has minimum weight \( 4k + 2 \) (not doubly even). \( S = C_0^\perp \setminus C' \) is called the shadow of \( C' \).

\[
\min(u+C') = 4k + 2 \implies \min S = 4k + 4
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For $n = 24k + 8$, the following are equivalent:

1. ∃ extremal doubly even self-dual code $C$ of length $n$ with covering radius $4k + 2$,
2. ∃ self-dual code $C'$ of length $n$ with minimum weight $4k + 2$ and its shadow has minimum weight $4k + 4$.

Bachoc–Gaborit (2004) showed: if a (not doubly even) self-dual code $C'$ of length $n$ with minimum weight $d$ and its shadow has minimum weight $s$, and

$$2d + s = \frac{n}{2} + 4,$$

then $W_{C'}(x, y)$ and $W_S(x, y)$ are uniquely determined.

$$2(4k + 2) + (4k + 4) = \frac{24k + 8}{2} + 4.$$
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$$W_{C'}(x, y) = \sum_{j=0}^{n/8} a_j (x^2 + y^2)^{n/2-4j} (x^2 y^2 (x^2 - y^2)^2)^j,$$

$$W_S(x, y) = \sum_{j=0}^{n/8} a_j (-1)^j 2^{n/2-6j} (xy)^{n/2-4j} (x^4 - y^4)^{2j},$$

the coefficients $a_j$ are uniquely determined.
For \( n = 24k + 8 \), the following are equivalent:

1. \( \exists \) extremal doubly even self-dual code \( C \) of length \( n \) with covering radius \( 4k + 2 \), (thus \( k \leq 158 \) by Zhang)
2. \( \exists \) self-dual code \( C' \) of length \( n \) with minimum weight \( 4k + 2 \) and its shadow has minimum weight \( 4k + 4 \).

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Thank you for your attention!