Complementary Ramsey numbers, graph factorizations and Ramsey graphs

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joint work with Masashi Shinohara

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For a graph $G$,

$\alpha(G)$ = independence number = $\max\{\#\text{independent set}\}$

$\omega(G)$ = clique number = $\max\{\#\text{clique}\} = \alpha(\overline{G})$.

$\omega(C_5) = \alpha(C_5) = 2$.

$\forall G$ with 6 vertices, $\omega(G) \geq 3$ or $\alpha(G) \geq 3$.

These facts can be conveniently described by the Ramsey number:

$R(3, 3) = 6$.

The smallest number of vertices required to guarantee $\alpha \geq 3$ or $\omega \geq 3$ (precise definition in the next slide).
The Ramsey number $R(m_1, m_2)$ is defined as:

\[ R(m_1, m_2) = \min \{ n \mid |V(G)| = n \implies \omega(G) \geq m_1 \text{ or } \alpha(G) \geq m_2 \} \]

A graph with $n$ vertices defines a partition of $E(K_n)$ into 2 parts, “edges” and “non-edges”.

Generalized Ramsey numbers $R(m_1, m_2, \ldots, m_k)$ can be defined if we consider partitions of $E(K_n)$ into $k$ parts, i.e., edge-colorings.
Let \([n] = \{1, 2, \ldots, n\}\), and \(E(K_n) = \binom{n}{2}\). The set of \(k\)-edge-coloring of \(K_n\) is denoted by \(C(n, k)\):

\[
C(n, k) = \{ f \mid f : E(K_n) \to [k] \}.
\]

We abbreviate

\[
\omega_i(f) = \omega([n], f^{-1}(i)), \quad \alpha_i(f) = \alpha([n], f^{-1}(i)).
\]

\[
R(m_1, \ldots, m_k) = \min\{ n \mid \forall f \in C(n, k), \exists i \in [k], \omega_i(f) \geq m_i \}
\]

\[
\bar{R}(m_1, \ldots, m_k) = \min\{ n \mid \forall f \in C(n, k), \exists i \in [k], \alpha_i(f) \geq m_i \}
\]

The last one is called the complementary Ramsey number.

\[
\bar{R}(m_1, m_2) = R(m_2, m_1) = R(m_1, m_2).
\]

So we focus on the case \(k \geq 3\). Also we assume \(m_i \geq 3\).
We submitted to our work to a conference proceedings, and received positive reviews, in 2013.

We uploaded our paper arXiv:1406.2050.

David Conlon notified to us, that the concept was introduced already by Erdős–Hajnal–Rado (1965), some results were proved by Erdős–Szemerédi (1972).

Chung–Liu (1978), “$d$-chromatic Ramsey numbers”,

$$\bar{R}(m_1, \ldots, m_k) = R^{k-1}_{k-1}(K_{m_1}, \ldots, K_{m_k}).$$

$$\bar{R}(4, 4, 4) = 10.$$ 


$$\bar{R}(m, \ldots, m) = R^{k-1}_{k-1}(K_m).$$

Xu–Shao–Su–Li (2009), “multigraph Ramsey numbers”,

$$\bar{R}(m_1, \ldots, m_k) = f^{(k-1)}(m_1, \ldots, m_k).$$

$$\bar{R}(5, 5, 5) \geq 20.$$
Given a metric space \((X, d)\) and a positive integer \(k\), classify subsets \(Y\) of \(X\) with the largest size subject to

\[
|\{d(x, y) \mid x, y \in Y, x \neq y\}| \leq k.
\]

For example, \(X = \mathbb{R}^n, k = 1 \implies\) regular simplex.

The method is by induction on \(k\).

The distance function \(d\) defines a \(k\)-edge-coloring of the complete graph on \(Y\).

If

\[
\overline{R}(m, m, \ldots, m) \leq |Y|,
\]

then \(Y\) must contain an \(m\)-subset having only \((k - 1)\) distances (so we can expect to use already obtained results for \(k - 1\)).
The vertices of the regular icosahedron is the only 12-point 3-distance set in $\mathbb{R}^3$.

- Claimed to be proven by Shinohara, arXiv:1309.2047.
- It would simplify the proof if we had $\bar{R}(5, 5, 5) = 12$, but this was not the case.
- Xu–Shao–Su–Li (2009), $\bar{R}(5, 5, 5) \geq 20$.
- What is $\bar{R}(5, 5, 5)$? It should be easier than determining $\bar{R}(5, 5) = R(5, 5)$, which is known to satisfy $43 \leq R(5, 5) \leq 48$.

In fact,

$$R(m, m) = \bar{R}(m, m) \geq \bar{R}(m, m, m) \geq \bar{R}(m, m, m, m) \cdots.$$
$\bar{R}(3, 3, 3) = 5$ by factorization

$K_4$ has a 3-edge-coloring $f$ into $2K_2$ (a 1-factorization). Then $\alpha_i(f) = 2$ for $i = 1, 2, 3$. This implies

$$\bar{R}(3, 3, 3) > 4.$$ 

The argument can be generalized to give:

**Theorem**

If $K_{mn}$ is factorable into $k$ copies of $nK_m$, then

$$\bar{R}(n + 1, \ldots, n + 1)^k = mn + 1.$$ 

Setting $m = n = 2$ and $k = 3$, we obtain $\bar{R}(3, 3, 3) = 5$. 

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Factorizations

Theorem

If $K_{mn}$ is factorable into $k$ copies of $nK_m$, then

$$\overline{R}(n+1, \ldots, n+1) = mn + 1.$$  

- Setting $m = 3$, $n = 2t + 1$, $k = 3t + 1$, the existence of a Kirkman triple system in $K_{3n}$ implies

$$\overline{R}(2t + 2, \ldots, 2t + 2) = 6t + 4.$$  

- Harborth–Möller (1999): Setting $m = n$, $k = n + 1$, if $n - 1$ MOLS of order $n$ exist, then

$$\overline{R}(n + 1, \ldots, n + 1) = n^2 + 1.$$  

The converse of the last statement also holds.
A graph $G$ is said to be a Ramsey $(s, t)$-graph if

$$\omega(G) < s \text{ and } \alpha(G) < t.$$ 

We write $G \subset H$ if $H$ is an edge-subgraph of $G$, and write $(V(G), E(G) \setminus E(H)) = G - H$.

**Theorem**

For $m_1, m_2, m_3, n \geq 2$, the following are equivalent.

(i)  $\bar{R}(m_1, m_2, m_3) \leq n$,

(ii) for any two Ramsey $(m_1, m_2)$-graphs $G$ and $H$ on the vertex set $[n]$ such that $G \supset H$, one has $\alpha(G - H) \geq m_3$. 

Chung–Liu (1978):

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and

$$
\bar{R}(k, 5, 3) = \begin{cases} 
13 & \text{if } 9 \leq k \leq 13, \\
14 & \text{if } k \geq 14.
\end{cases}
$$
We abbreviate

\[ \overline{R}(m; k) = \overline{R}(m, \ldots, m). \]

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Thank you very much for your attention!