A matrix approach to Yang multiplication

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A quadruple \((a, b, c, d)\) of complementary sequences of length \(n\) can be used to construct a Hadamard matrix of order \(4n\), via the Goethals-Seidel array:

\[
H = \begin{bmatrix}
A & -BR & -CR & -DR \\
BR & A & -D^\top R & C^\top R \\
CR & D^\top R & A & -B^\top R \\
DR & -C^\top R & B^\top R & A
\end{bmatrix}, \quad HH^\top = 4nI,
\]

where

\[
A, B, C, D = \text{circulant matrix with first row } a, b, c, d, \\
R = \text{back diagonal permutation matrix.}
\]
Quadruple of complementary sequences


1. Base seq. \( BS(m, n) \): length \((m, m, n, n)\)

2. Near normal seq. \( NN(n) \): a special case of \( BS(n + 1, n) \)

3. Nonperiodic complementary seq. \( NCS(n) \): length \((n, n, n, n)\)

4. Golay seq. \( GCP(n) \): \((n, n, 0, 0)\)

For \( \{0, \pm 1\} \)-sequences (ternary),

1. Normal seq. \( NS(n) \): length \((n, n, n, 0)\), weight \(2n\),

2. T-seq. \( TS(n) \): length \((n, n, n, n)\), weight \(n\)

(with some disjointness conditions).

Work done by Craigen, Doković, Kotsireas, Seberry, . . . .
Yang (1989), Theorem 4, states

\[ BS(m + 1, m) \neq \emptyset, \ BS(n + 1, n) \neq \emptyset \]
\[ \implies NCS((2m + 1)(2n + 1), 4) \neq \emptyset \]
\[ (\implies \exists H(4(2m + 1)(2n + 1))). \]

Conjecture \( BS(n + 1, n) \neq \emptyset \) for all \( n \).

In this talk: a matrix approach to prove this theorem.

Is the proof difficult?

See the next page: only 9 lines.
\[ BS(m + 1, m) \times BS(n + 1, n) \rightarrow NCS((2m + 1)(2n + 1), 4) \]

\[(a, b, c, d) \in BS(m + 1, m) \]
\[\subset \{\pm 1\}^{m+1} \times \{\pm 1\}^{m+1} \times \{\pm 1\}^m \times \{\pm 1\}^m,\]

\[(f, g, h, e) \in BS(n + 1, n) \]
\[\subset \{\pm 1\}^{n+1} \times \{\pm 1\}^{n+1} \times \{\pm 1\}^n \times \{\pm 1\}^n,\]

\[\Rightarrow\]

\[(q, r, s, t) \in NCS((2m + 1)(2n + 1), 4) \]
\[\subset (\{\pm 1\}^{(2m+1)(2n+1)})^4.\]

\[(a', b', c', d') \in (\{0, \pm 1\}^{2m+1})^4,\]
\[(f', g', h', e') \in (\{0, \pm 1\}^{2n+1})^4.\]

Our matrix approach:

\[ (Q, R, S, T) \in (\{0, \pm 1\}^{(2m+1)(2n+1)})^4. \]
Lagrange identity

Let $\mathcal{R}$ be a commutative ring with involutive automorphism $\ast$. Let $a, b, c, d, f, g, h, e \in \mathcal{R}$. Set

$$
q = af^* + cg - b^*e + dh,
$$
$$
r = bf^* + dg^* + a^*e - ch^*,
$$
$$
s = ag^* - cf - bh - d^*e,
$$
$$
t = bg - df + ah^* + c^*e.
$$

Then

$$
qq^* + rr^* + ss^* + tt^*
$$
$$
= (aa^* + bb^* + cc^* + dd^*)(ee^* + ff^* + gg^* + hh^*).
$$

We use this with

$$
\mathcal{R} = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}], \ast : x \mapsto x^{-1}, y \mapsto y^{-1}.
$$
The Hall polynomial \( f_a(x) \)

Let \( a = (a_0, \ldots, a_{n-1}) \in \mathbb{Z}^n \).

Define the Hall polynomial \( f_a(x) \in \mathbb{Z}[x] \) of \( a \) by

\[
f_a(x) = \sum_{i=0}^{n-1} a_i x^i.
\]

It is more convenient to use

\[
\psi_a(x) = x^{1-n} f_a(x^2).
\]

Example: \( a = (a_0, a_1, a_2, a_3), \ b = (b_0, b_1, b_2) \)

\[
f_a(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3,
\]

\[
\psi_a(x) = a_0 x^{-3} + a_1 x^{-1} + a_2 x^1 + a_3 x^3,
\]

\[
f_b(x) = b_0 + b_1 x + b_2 x^2,
\]

\[
\psi_b(x) = b_0 x^{-2} + b_1 x^0 + b_2 x^2.
\]
Complementary sequences

Define
\[ * : \mathbb{Z}[x^{\pm 1}] \to \mathbb{Z}[x^{\pm 1}], \quad x \mapsto x^{-1}. \]

A \( k \)-tuple \((a_1, \ldots, a_k)\) of sequences with all entries in \( \{\pm 1\} \) is said to be complementary if

\[ \sum_{i=1}^{k} f_{a_i}(x) f^*_a(x) \in \mathbb{Z}. \]

The constant term of the right-hand side is the sum of the lengths of \( a_1, \ldots, a_k \).

Example:

\begin{align*}
BS(m, n) : (a, b, c, d) & \in \{\pm 1\}^m \times \{\pm 1\}^m \times \{\pm 1\}^n \times \{\pm 1\}^n, \\
NCS(n, 4) : (q, r, s, t) & \in (\{\pm 1\}^n)^4.
\end{align*}
Yang Multiplication Theorem (C.H. Yang, 1989)

Let

\[(a, b, c, d) \in BS(m + 1, m), \quad (f, g, h, e) \in BS(n + 1, n).\]

Then \(\exists (q, r, s, t) \in NCS((2m + 1)(2n + 1)).\)

Yang’s approach:

\[f_q(x) = f_a(x^2)f_*(x^{2(2m+1)}) + xf_c(x^2)f_g(x^{2(2m+1)})\]
\[\quad - x^{2m+1}f_b^*(x^2)f_e(x^{2(2m+1)})\]
\[\quad + x^{2m+2}f_d(x^2)f_h(x^{2(2m+1)}).\]

Our matrix approach:

\[\psi_Q(x, y) = \psi_a(x)\psi_f^*(y) + \psi_c(x)\psi_g(y)\]
\[\quad - \psi_b^*(x)\psi_e(y) + \psi_d(x)\psi_h(y).\]
\( \psi_Q(x, y) \) for an \( n \times m \) matrix \( Q \)

Let \( q_0, \ldots, q_{n-1} \) denote the row vector of \( Q \):

\[
Q = \begin{bmatrix}
q_0 \\
q_1 \\
\vdots \\
q_{n-1}
\end{bmatrix}.
\]

Define

\[
\psi_Q(x, y) = \sum_{i=0}^{n-1} y^{2i+1-n} \psi_{q_i}(x) \in \mathbb{Z}[x^\pm 1, y^\pm 1].
\]

Example: Let \( Q = (q_{ij}) \) be a \( 3 \times 4 \) matrix. Then \( \psi_Q(x, y) \) is

\[
\sum \begin{bmatrix}
q_{00}x^{-3}y^{-2} & q_{01}x^{-1}y^{-2} & q_{02}x^{1}y^{-2} & q_{03}x^{3}y^{-2} \\
q_{10}x^{-3}y^{0} & q_{11}x^{-1}y^{0} & q_{12}x^{1}y^{0} & q_{13}x^{3}y^{0} \\
q_{20}x^{-3}y^{2} & q_{21}x^{-1}y^{2} & q_{22}x^{1}y^{2} & q_{23}x^{3}y^{2}
\end{bmatrix}.
\]
Lemma

For sequences \(a, b\) regarded as row vectors,

\[
\psi_{b^\top a}(x, y) = \psi_a(x)\psi_b(y). 
\]

For a matrix \(Q\), denote by \(\text{seq}(Q)\) the sequence obtained by concatenating the rows of \(Q\).

Lemma

If \(Q\) has \(m\) columns, then

\[
\psi_{\text{seq}(Q)}(x) = \psi_Q(x, x^m). 
\]
Our approach

Recall that our matrix approach was:

$$\psi_Q(x, y) = \psi_a(x)\psi_f^*(y) + \psi_c(x)\psi_g(y) - \psi_b^*(x)\psi_e(y) + \psi_d(x)\psi_h(y).$$

This is achieved by defining

$$Q = f^*\mathsf{T}a + g\mathsf{T}c - e\mathsf{T}b^* + h\mathsf{T}d,$$

where $f^*$ denotes the reverse of $f$. Note $\psi_f^*(y) = \psi_f^*(y)$.

$$\psi_{seq}(Q)(x) = \psi_a(x)\psi_f^*(x^m) + \psi_c(x)\psi_g(x^m) - \psi_b^*(x)\psi_e(x^m) + \psi_d(x)\psi_h(x^m).$$
Complementary sequences

Lemma

\[ f_a(x^2) f^*_a(x^2) = \psi_a(x) \psi^*_a(x). \]

Thus

\( a_1, \ldots, a_k: \) complementary

\[ \iff \sum_{i=1}^{k} f_{a_i}(x) f^*_{a_i}(x) \in \mathbb{Z} \]

\[ \iff \sum_{i=1}^{k} \psi_{a_i}(x) \psi^*_{a_i}(x) \in \mathbb{Z}. \]
Recall the Lagrange identity

Let $a, b, c, d, f, g, h, e \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$. Set

\[
q = af^* + cg - b^*e + dh,
\]
\[
r = bf^* + dg^* + a^*e - ch^*,
\]
\[
s = ag^* - cf - bh - d^*e,
\]
\[
t = bg - df + ah^* + c^*e.
\]

Then

\[
qq^* + rr^* + ss^* + tt^*
= (aa^* + bb^* + cc^* + dd^*)(ee^* + ff^* + gg^* + hh^*).
\]
The Lagrange identity (consequence)

Let $a, b, c, d \in \mathbb{Z}^m$, $f, g, h, e \in \mathbb{Z}^n$,

\[
Q = f^*t a + g^t c - e^t b^* + h^t d, \\
R = f^*t b + g^*t d - e^t a^* - h^*t c, \\
S = g^*t a - f^t c - h^t b + e^t d^*, \\
T = g^t b - f^t d - h^*t a + e^t c^*.
\]

Then $Q, R, S, T \in \mathbb{Z}^{n \times m}$.

\[
(\psi_Q \psi_Q^* + \psi_R \psi_R^* + \psi_S \psi_S^* + \psi_T \psi_T^*)(x, y) \\
= (\psi_a \psi_a^* + \psi_b \psi_b^* + \psi_c \psi_c^* + \psi_d \psi_d^*)(x) \\
\times (\psi_e \psi_e^* + \psi_f \psi_f^* + \psi_g \psi_g^* + \psi_h \psi_h^*)(y).
\]
The Lagrange identity (consequence)

Let \( a, b, c, d \in \mathbb{Z}^m, f, g, h, e \in \mathbb{Z}^n, \)

\[
Q = f^* a + g^t c - e^t b^* + h^t d, \\
R = f^* b + g^* d - e^t a^* - h^t c, \\
S = g^* a - f^t c - h^t b + e^t d^*, \\
T = g^t b - f^t d - h^* a + e^t c^*.
\]

Then

\[
(\psi_{\text{seq}(Q)}\psi_{\text{seq}(Q)}^* + \psi_{\text{seq}(R)}\psi_{\text{seq}(R)}^* + \psi_{\text{seq}(S)}\psi_{\text{seq}(S)}^* + \psi_{\text{seq}(T)}\psi_{\text{seq}(T)}^*)(x, x^m) \\
= (\psi_a\psi_a^* + \psi_b\psi_b^* + \psi_c\psi_c^* + \psi_d\psi_d^*)(x) \\
\times (\psi_e\psi_e^* + \psi_f\psi_f^* + \psi_g\psi_g^* + \psi_h\psi_h^*)(x^m).
\]
For $a = (a_0, \ldots, a_{m-1})$, define

$$a/0 = (a_0, 0, a_1, 0, \ldots, 0, a_{m-1}) \quad \text{(length } 2m - 1),$$

$$0/a = (0, a_0, 0, \ldots, 0, a_{m-1}, 0) \quad \text{(length } 2m + 1).$$

Lemma

$$\psi_{a/0}(x) = \psi_{0/a}(x) = \psi_a(x^2).$$
Yang’s Theorem

Let \((a, b, c, d) \in BS(m + 1, m), (f, g, h, e) \in BS(n + 1, n)\). Then there exists \((q, r, s, t) \in NCS((2n + 1)(2m + 1))\).
Construction of the matrices $Q, R, S, T$

Let $(a, b, c, d) \in BS(m + 1, m)$, $(f, g, h, e) \in BS(n + 1, n)$. Then

$$a, b \in \{\pm 1\}^{m+1}, c, d \in \{\pm 1\}^m, f, g \in \{\pm 1\}^{n+1}, h, e \in \{\pm 1\}^n.$$ 

Set

$$a' = a/0, b' = b/0, c' = 0/c, d' = 0/d, f' = f/0, g' = g/0, h' = 0/h, e' = 0/e.$$

Define $(2n + 1) \times (2m + 1)$ matrices with entries in $\{\pm 1\}$:

$$Q = f'^* a' + g'^t c' - e'^t b'^* + h'^t d',$$
$$R = f'^* b' + g'^* d' - e'^t a'^* - h'^* c',$$
$$S = g'^* a' - f'^t c' - h'^t b' + e'^t d'^*,$$
$$T = g'^t b' - f'^t d' - h'^* a' + e'^t c'^*.$$
Set \( q = \text{seq}(Q) \), \( r = \text{seq}(R) \), \( s = \text{seq}(S) \), \( t = \text{seq}(T) \). Then

\[
(\psi_q \psi_q^* + \psi_r \psi_r^* + \psi_s \psi_s^* + \psi_t \psi_t^*)(x) \\
= (\psi_Q \psi_Q^* + \psi_R \psi_R^* + \psi_S \psi_S^* + \psi_T \psi_T^*)(x, x^{2m+1}) \\
= (\psi_{a'} \psi_{a'}^* + \psi_{b'} \psi_{b'}^* + \psi_{c'} \psi_{c'}^* + \psi_{d'} \psi_{d'}^*)(x) \\
\times (\psi_{e'} \psi_{e'}^* + \psi_{f'} \psi_{f'}^* + \psi_{g'} \psi_{g'}^* + \psi_{h'} \psi_{h'}^*)(x^{2m+1}) \\
= (\psi_a \psi_a^* + \psi_b \psi_b^* + \psi_c \psi_c^* + \psi_d \psi_d^*)(x^2) \\
\times (\psi_e \psi_e^* + \psi_f \psi_f^* + \psi_g \psi_g^* + \psi_h \psi_h^*)(x^{2(2m+1)}) \\
\in \mathbb{Z}.
\]

Thus \((q, r, s, t) \in NCS((2m + 1)(2n + 1))\). This proves Yang’s theorem (see arXiv:1705.05062 for details).
Another result of Yang (1983)

**Theorem**

Let \((a, b, c, d) \in BS(m, n)\). Suppose \(f, g \in \{0, \pm 1\}^k\) and \(e \in \{0, \pm 1\}^{k-1}\) satisfy

1. \((e, f, g)\) is complementary with weight \(2k + 1\),
2. \((0|f), (e|00) \in \{0, \pm 1\}^{k+1}\) are disjoint,
3. \(g\) and \(g^*\) have the same support.

Then \(\exists (q, r, s, t) \in TS((2k + 1)(m + n))\).

**Remarks:**

1. Yang (1983) shows this only for \(k = 6\) with \(e, f, g\) given.
2. This is different from better known Yang multiplication (1989):
   \(NS(k) \neq \emptyset, BS(m, n) \neq \emptyset \implies TS((2k+1)(m+n))\).
3. Our proof is not as neat as the one presented here. The End.