Oriented covers of the triangular graphs

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Contents of this talk

- Petersen graph
- Triangular graph $T(n)$
- Oriented cover of Triangular graph $T(n)$
- DSRG
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- Association Schemes
The Petersen graph
The Petersen graph

Definition:

\[ V = \{\{i, j\} \mid i, j \in \{1, \ldots, 5\}, \ i \neq j\}, \]
\[ E = \{\{\{i, j\}, \{k, l\}\} \mid \{i, j\} \cap \{k, l\} = \emptyset\}. \]

Properties:

- 10 vertices,
- valency 3,
- no triangle or quadrangle
- diameter 2

Characterization: Properties \(\implies\) Unique.

Generalization \(\leftarrow\) next slide.
Triangular graph $T(n)$

Petersen graph:

$$V = \{\{i, j\} \mid i, j \in \{1, \ldots, 5\}, \; i \neq j\},$$
$$E = \{\{\{i, j\}, \{k, l\}\} \mid \{i, j\} \cap \{k, l\} = \emptyset\},$$
$$\overline{E} = \{\{\{i, j\}, \{k, l\}\} \mid |\{i, j\} \cap \{k, l\}| = 1\}.$$

Triangular graph $T(n)$ ($n \geq 4$):

$$V = \{\{i, j\} \mid i, j \in \{1, \ldots, n\}, \; i \neq j\},$$
$$E = \{\{\{i, j\}, \{k, l\}\} \mid |\{i, j\} \cap \{k, l\}| = 1\}.$$

Properties:

- there are $n(n - 1)/2$ vertices,
- valency is $2(n - 2)$,
- each edge is contained in $\lambda = n - 2$ triangles,
- each pair of non-adjacent vertices has $\mu = 4$ common neighbors.

Characterization (Chang, 1959): Properties $\implies$ Unique unless $n = 8$. 
Strongly regular graphs

**Definition**

A graph $\Gamma$ is called a **strongly regular graph** (SRG) with parameters $(k, \lambda, \mu)$ if

- valency is $k$,
- each edge is contained in $\lambda$ triangles,
- each pair of non-adjacent vertices has $\mu$ common neighbors.

$T(n)$ is a SRG with parameters $(2(n - 2), n - 2, 4)$.

The Petersen graph $\overline{T(5)}$ is a SRG with parameters $(3, 0, 1)$.

The complement of a SRG is again a SRG.
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For a regular graph $\Gamma$, the following are equivalent:

- $\Gamma$ is strongly regular,
- the adjacency matrix has 3 distinct eigenvalues.

Example:

- The Petersen graph has spectrum $\{3, [-2]^4, 1^5\}$.
- $T(n)$ has spectrum $\{2(n - 2), (n - 4)^{n-1}, [-2]^{n(n-3)/2}\}$.
- $T(5)$ has spectrum $\{6, 1^4, [-2]^5\}$. 
Simultaneous diagonalization

- The Petersen graph $T(5)$ has spectrum $\{[3], [-2]^4, [1]^5\}$,
- $T(5)$ has spectrum $\{[6], [1]^4, [-2]^5\}$.

$$T(5) = I + A + J - I - A$$

Since $A$ and $J$ commute, they can be simultaneously diagonalized.

The list of eigenvalues can be tabulated in a matrix form, and it is called the eigenmatrix:

$$\begin{pmatrix} 1 & 6 & 3 \\ 1 & 1 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$
If $X$ is a finite set,

$$X \times X = R_0 \cup R_1 \cup \cdots \cup R_d \quad \text{(partition)},$$

adjacency matrices $A_0, A_1, \ldots, A_d$

satisfy

$$A_0 = I,$$

$$\sum_{i=0}^{d} A_i = J \quad \text{(all-one matrix)},$$

$$\forall i, \ A_i^\top = A_i,$$

$$\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle$$

is closed under multiplication,

then $(X, \{R_i\}_{i=0}^d)$ is called a symmetric association scheme, $\mathcal{A}$ is called its Bose-Mesner algebra.
Symmetric association schemes

For a symmetric association scheme, the Bose-Mesner algebra

$$\mathcal{A} = \langle A_0, A_1, \ldots, A_d \rangle$$

is simultaneously diagonalizable:

$$A_j \sim \begin{pmatrix} [p_{0j}]^{m_0} & & \\ & [p_{1j}]^{m_1} & \\ & & \ddots \\ & & & [p_{dj}]^{m_d} \end{pmatrix} \rightarrow P = (p_{ij})$$

eigenmatrix
Oriented cover

Triangular graph $T(n)$ ($n \geq 4$):

$$V = \{\{i, j\} \mid i, j \in \{1, \ldots, n\}, i \neq j\},$$

$$R_1 = \{\{\{i, j\}, \{k, l\}\} \mid |\{i, j\} \cap \{k, l\}| = 1\},$$

$$R_2 = \{\{\{i, j\}, \{k, l\}\} \mid \{i, j\} \cap \{k, l\} = \emptyset\},$$

Let $A_i$ be the adjacency matrix of $R_i$, for $i = 1, 2$, and set $A_0 = I$. They form a symmetric association scheme.

Oriented (directed) version:

$$V = \{(i, j) \mid i, j \in \{1, \ldots, n\}, i \neq j\},$$

$$R_1 = \{((i, j), (k, l)) \mid i \neq j = k \neq l \neq i\}.$$

Then $R_2, R_3, \ldots$?
V = \{ (i, j) \mid i, j \in \{1, \ldots, n\}, \ i \neq j \}.

\begin{align*}
R_1 &= \{ ((i, j), (j, i)) \mid i \neq j \}, \\
R_2 &= \{ ((i, j), (k, l)) \mid j \neq i = k \neq l \neq j \}, \\
R_3 &= \{ ((i, j), (k, l)) \mid j \neq i = l \neq k \neq j \}, \\
R_4 &= \{ ((i, j), (k, l)) \mid i \neq j = l \neq k \neq i \}, \\
R_5 &= \{ ((i, j), (k, l)) \mid i \neq j = k \neq l \neq i \}, \\
R_6 &= \{ ((i, j), (k, l)) \mid \{i, j\} \cap \{k, l\} = \emptyset \}.
\end{align*}

Let \( A_i \) be the adjacency matrix of \( R_i \), for \( i = 1, 2 \), and set \( A_0 = I \). They form an association scheme (in a broad sense), i.e., non-symmetric, non-commutative.
Non-commutative association schemes

If $X$ is a finite set,

$$X \times X = R_0 \cup R_1 \cup \cdots \cup R_d \quad \text{(partition),}$$

adjacency matrices $A_0, A_1, \ldots, A_d$

satisfy

$$A_0 = I,$$

$$\sum_{i=0}^{d} A_i = J \quad \text{(all-one matrix),}$$

$$\forall i, \exists i', A_i^\top = A_{i'},$$

then $(X, \{R_i\}_{i=0}^d)$ is called a (non-commutative) association scheme, $\mathcal{A}$ is called its Bose-Mesner algebra.
\[ V = \{(i, j) \mid i, j \in \{1, \ldots, n\}, i \neq j\}. \]

\[ R_1 = \{((i, j), (j, i)) \mid i \neq j\}, \]
\[ R_2 = \{((i, j), (k, l)) \mid j \neq i = k \neq l \neq j\}, \]
\[ R_3 = \{((i, j), (k, l)) \mid j \neq i = l \neq k \neq j\}, \]
\[ R_4 = \{((i, j), (k, l)) \mid i \neq j = l \neq k \neq i\}, \]
\[ R_5 = \{((i, j), (k, l)) \mid i \neq j = k \neq l \neq i\}, \]
\[ R_6 = \{((i, j), (k, l)) \mid \{i, j\} \cap \{k, l\} = \emptyset\}. \]

Let \( A_i \) be the adjacency matrix of \( R_i \), for \( i = 1, 2 \), and set \( A_0 = I \). They form a (non-commutative) association scheme. They cannot be simultaneously diagonalized. In fact,

\[ \langle A_0, \ldots, A_6 \rangle \cong M_1(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_1(\mathbb{C}), \]
\[ 7 = 1 + 2^2 + 1 + 1. \]
Two bases of the 7-dimensional algebra

\[ \langle A_0, \ldots, A_6 \rangle \]
\[ = \langle E_0 \rangle \oplus \langle E_1^{(1,1)}, E_1^{(1,2)}, E_1^{(2,1)}, E_1^{(2,2)} \rangle \oplus \langle E_2 \rangle \oplus \langle E_3 \rangle \]
\[ \cong M_1(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \]

Simultaneous block diagonalization:

\[ A_j \sim \begin{pmatrix} p_{0j} & & & \\ & p_{1j}^{(1,1)} & p_{1j}^{(1,2)} & \\ & p_{1j}^{(2,1)} & p_{1j}^{(2,2)} & \\ & & & p_{2j} \end{pmatrix} \]
\[ A_j = p_{0j} E_0 + \sum_{k,l} p_{1j}^{(k,l)} E_1^{(k,l)} + p_{2j} E_2 + p_{3j} E_3. \]
Eigenmatrix of the oriented cover of $T(n)$

The $j$th column of the matrix $P$ consists of the coefficients of $A_j$ when written as a linear combination of $E$'s.

$$
\begin{bmatrix}
E_0 & 1 & 1 & n-2 & n-2 & n-2 & n-2 & (n-2)(n-3) \\
E_{1}^{(1,1)} & 1 & 1 & \frac{n-4}{2} & \frac{n-4}{2} & \frac{n-4}{2} & \frac{n-4}{2} & -2(n-3) \\
E_{1}^{(1,2)} & 0 & 0 & m & -m & -m & m & 0 \\
E_{1}^{(2,1)} & 0 & 0 & m & m & -m & -m & 0 \\
E_{1}^{(2,2)} & 1 & -1 & \frac{n-2}{2} & -\frac{n-2}{2} & \frac{n-2}{2} & -\frac{n-2}{2} & 0 \\
E_2 & 1 & 1 & -1 & -1 & -1 & -1 & 2 \\
E_3 & 1 & -1 & -1 & 1 & -1 & 1 & 0
\end{bmatrix}
$$

where

$$m = \frac{\sqrt{n(n-2)}}{2}.$$
Directed strongly regular graph (Duval, 1988)

\[ V = \{(i, j) \mid i, j \in \{1, \ldots, n\}, \ i \neq j\}. \]

\[ R_1 = \{((i, j), (j, i)) \mid i \neq j\}, \]
\[ R_5 = \{((i, j), (k, l)) \mid i \neq j = k \neq l \neq i\}. \]

The matrix \( A = A_1 + A_5 \) satisfies

\[ AJ = kJ, \]
\[ A^2 = tI + \lambda A + \mu(J - I - A), \]

where

\[ k = n - 1, \quad t = 1, \quad \lambda = 0, \quad \mu = 1. \]

This is very similar to the property of the adjacency matrix of a SRG:

\[ AJ = kJ, \]
\[ A^2 = kI + \lambda A + \mu(J - I - A). \]
Directed strongly regular graph

SRG:

\[ AJ = kJ, \]
\[ A^2 = kI + \lambda A + \mu(J - I - A). \]

**Definition**

Let \( \Gamma \) be a directed graph with adjacency matrix \( A \). Then \( \Gamma \) is called a **directed strongly regular graph (DSRG)** with parameters \((k, \mu, \lambda, t)\) if

\[ AJ = kJ, \]
\[ A^2 = tI + \lambda A + \mu(J - I - A). \]

Note \( 0 \leq t \leq k \), and

\[ t = k \iff \text{SRG} \]
\[ t = 0 \iff \text{tournament}. \]
Assume

\[ AJ = kJ, \]
\[ A^2 = tI + \lambda A + \mu (J - I - A), \]
\[ 0 < t < k. \]

**Theorem (Klin-M.-Muzychuk-Zieschang 2004)**

The adjacency matrix \( A \) cannot be contained in the Bose-Mesner algebra of a commutative association scheme. In particular, algebra generated by \( A \) under \( \cdot, \circ \) has dimension at least \( 6 \).
A 2-$(v, k, 1)$ design is an incidence structure $(\mathcal{P}, \mathcal{B})$, where $\mathcal{B} \subset \binom{\mathcal{P}}{k}$, and every pair $i, j \in \mathcal{P}$ is contained in $\lambda$ members of $\mathcal{B}$.

Assume $(\mathcal{P}, \mathcal{B})$ is a 2-$(v, k, 1)$ design, and set

$$\mathcal{F} = \{(i, B) \in \mathcal{P} \times \mathcal{B} \mid x \in B\}.$$ 

Example: $\mathcal{B} = \binom{\mathcal{P}}{2}$, $\mathcal{P} = \{1, \ldots, n\}$. Then

$$\mathcal{F} = \{(i, \{i, j\}) \mid i, j \in \{1, \ldots, n\}, i \neq j\}$$

which corresponds bijectively to

$$V = \{(i, j) \mid i, j \in \{1, \ldots, n\}, i \neq j\}.$$
Assume \((\mathcal{P}, \mathcal{B})\) is a 2-\((v, k, 1)\) design, and set

\[ \mathcal{F} = \{(i, B) \in \mathcal{P} \times \mathcal{B} \mid x \in B\}, \]

\[ R_1 = \{((i, B), (j, B)) \mid i \neq j\}, \]

\[ R_2 = \{((i, B), (i, B')) \mid B \neq B'\}, \]

\[ R_3 = \{((i, B), (j, B')) \mid j \neq i \in B'\}, \]

\[ R_4 = \{((i, B), (j, B')) \mid i, j \notin B \cap B' \neq \emptyset\}, \]

\[ R_5 = \{((i, B), (j, B')) \mid i \neq j \in B\}, \]

\[ R_6 = \{((i, B), (j, B')) \mid B \cap B' = \emptyset\}. \]

Let \(A_i\) be the adjacency matrix of \(R_i\), for \(i = 1, 2\), and set \(A_0 = I\). The flag algebra is (Klin-M.-Muzychuk-Zieschang, 2004):

\[ \langle A_0, \ldots, A_6 \rangle \cong M_1(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_1(\mathbb{C}). \]
The end

Problem

- Generalize our result on the calculation of the eigenmatrix for the oriented $T(n)$, to that of the flag algebra of a $2-(v, k, 1)$ design.
- Find other classes of DSRG for which eigenmatrix can be calculated. Is the eigenmatrix determined by $(k, \mu, \lambda, t)$?

Tomorrow, I will give another talk at China University of Geoscience in Beijing:

*Quasi-symmetric 2-$(56, 16, 18)$ designs constructed from the dual of the quasi-symmetric 2-$(21, 6, 4)$ design as a Hoffman coclique*

where I will describe another relationship between strongly regular graphs and 2-designs. Thank you very much for your attention.