Spherical Designs

An Introduction to Designs in Spheres and Complex Projective Spaces

Akihiro Munemasa

Graduate School of Information Sciences
Tohoku University

September 14, 2006
Why can’t we place 5 points on a sphere in a nice way, even though we can easily do the same for 4 points (tetrahedron) or for 6 points (octahedron)? We will answer this question rigorously by defining spherical design. There is no spherical 2-design in $\mathbb{R}^3$ of 4 points, or of 6 points, but not of 5 points.
Why can’t we place 5 points on a sphere in a nice way, even though we can easily do the same for 4 points (tetrahedron) or for 6 points (octahedron)? We will answer this question rigorously by defining spherical design. There is no spherical 2-design in $\mathbb{R}^3$ of 4 points, or of 6 points, but not of 5 points.
Why can’t we place 5 points on a sphere in a nice way, even though we can easily do the same for 4 points (tetrahedron) or for 6 points (octahedron)? We will answer this question rigorously by defining spherical design. There is no spherical 2-design in $\mathbb{R}^3$ of 4 points, or of 6 points, but not of 5 points.
Definition of Spherical Design

Let $d$ be a positive integer. Let $\Omega_d = \{x \in \mathbb{R}^d \mid \|x\| = 1\}$ be the unit sphere in $\mathbb{R}^d$.

A spherical $t$-design is a finite nonempty subset $X$ of $\Omega_d$ satisfying

$$\frac{1}{\text{volume}(\Omega_d)} \int_{\Omega_d} f(\xi) d\xi = \frac{1}{|X|} \sum_{x \in X} f(x)$$

for all polynomial functions $f$ of degree at most $t$. 
Definition

Let $d$ be a positive integer. Let $\Omega_d = \{ x \in \mathbb{R}^d \mid \|x\| = 1 \}$ be the unit sphere in $\mathbb{R}^d$.

A \emph{spherical $t$-design} is a finite nonempty subset $X$ of $\Omega_d$ satisfying

\[
\frac{1}{\text{volume}(\Omega_d)} \int_{\Omega_d} f(\xi) d\xi = \frac{1}{|X|} \sum_{x \in X} f(x)
\]

for all polynomial functions $f$ of degree at most $t$. 

Akihiro Munemasa
Tohoku University

Designs in Spheres
Definition of Spherical Design

Let $d$ be a positive integer. Let $\Omega_d = \{ \mathbf{x} \in \mathbb{R}^d \mid \| \mathbf{x} \| = 1 \}$ be the unit sphere in $\mathbb{R}^d$.

A spherical $t$-design is a finite nonempty subset $X$ of $\Omega_d$ satisfying

$$
\frac{1}{\text{volume}(\Omega_d)} \int_{\Omega_d} f(\xi) d\xi = \frac{1}{|X|} \sum_{\mathbf{x} \in X} f(\mathbf{x}) \tag{1}
$$

for all polynomial functions $f$ of degree at most $t$. 
Theorem (Mimura, 1990)

Let \( n, d \) be positive integers with \( d \geq 2 \). Then there exists a spherical 2-design of \( n \) points in \( \mathbb{R}^d \) unless \( n \leq d \) or \( n = d + 2 \) is odd.

In particular, there is no spherical 2-design of 5 points in \( \mathbb{R}^3 \).
If \( n \) or \( d \) is even, then the construction is easy.
If both \( n \) and \( d \) are odd, we will give a construction which is much simpler than Mimura’s.
Existence of Spherical 2-Designs

Theorem (Mimura, 1990)

Let $n, d$ be positive integers with $d \geq 2$. Then there exists a spherical 2-design of $n$ points in $\mathbb{R}^d$ unless $n \leq d$ or $n = d + 2$ is odd.

In particular, there is no spherical 2-design of 5 points in $\mathbb{R}^3$.

If $n$ or $d$ is even, then the construction is easy.

If both $n$ and $d$ are odd, we will give a construction which is much simpler than Mimura’s.
Existence of Spherical 2-Designs

**Theorem (Mimura, 1990)**

Let $n, d$ be positive integers with $d \geq 2$. Then there exists a spherical 2-design of $n$ points in $\mathbb{R}^d$ unless $n \leq d$ or $n = d + 2$ is odd.

In particular, there is no spherical 2-design of 5 points in $\mathbb{R}^3$.

If $n$ or $d$ is even, then the construction is easy.

If both $n$ and $d$ are odd, we will give a construction which is much simpler than Mimura’s.
Existence of Spherical 2-Designs

Theorem (Mimura, 1990)

Let $n, d$ be positive integers with $d \geq 2$. Then there exists a spherical 2-design of $n$ points in $\mathbb{R}^d$ unless $n \leq d$ or $n = d + 2$ is odd.

In particular, there is no spherical 2-design of 5 points in $\mathbb{R}^3$. If $n$ or $d$ is even, then the construction is easy. If both $n$ and $d$ are odd, we will give a construction which is much simpler than Mimura’s.
Existence of Spherical 2-Designs

Theorem (Mimura, 1990)

Let $n, d$ be positive integers with $d \geq 2$. Then there exists a spherical 2-design of $n$ points in $\mathbb{R}^d$ unless $n \leq d$ or $n = d + 2$ is odd.

In particular, there is no spherical 2-design of 5 points in $\mathbb{R}^3$. If $n$ or $d$ is even, then the construction is easy. If both $n$ and $d$ are odd, we will give a construction which is much simpler than Mimura’s.
The angle set of a finite set \( X \subset \Omega_d \) is

\[
A(X) = \{(x, y) \mid x, y \in X, \ x \neq y\} \subset [-1, 1).
\]

If we regard it as a multiset, then the property of being a spherical \( t \)-design can be described in terms of the angle set.

**Theorem (Delsarte-Goethals-Seidel)**

A finite set \( X \subset \Omega_d \) is a spherical \( t \)-design if and only if

\[
\sum_{x,y \in X} P_k((x, y)) = 0 \quad \text{for } k = 1, 2, \ldots, t,
\]

where \( P_k(x) \ (k = 1, 2, \ldots) \) are Gegenbauer polynomials.
Angle Set of Spherical Design

The angle set of a finite set $X \subset \Omega_d$ is

$$A(X) = \{(x, y) \mid x, y \in X, \ x \neq y\} \subset [-1, 1).$$

If we regard it as a multiset, then the property of being a spherical $t$-design can be described in terms of the angle set.

Theorem (Delsarte-Goethals-Seidel)

A finite set $X \subset \Omega_d$ is a spherical $t$-design if and only if

$$\sum_{x, y \in X} P_k((x, y)) = 0 \quad \text{for} \ k = 1, 2, \ldots, t,$$

where $P_k(x)$ ($k = 1, 2, \ldots$) are Gegenbauer polynomials.
Angle Set of Spherical Design

The angle set of a finite set $X \subset \Omega_d$ is

$$A(X) = \{(x, y) \mid x, y \in X, \ x \neq y\} \subset [-1, 1).$$

If we regard it as a multiset, then the property of being a spherical $t$-design can be described in terms of the angle set.

**Theorem (Delsarte-Goethals-Seidel)**

A finite set $X \subset \Omega_d$ is a spherical $t$-design if and only if

$$\sum_{x, y \in X} P_k((x, y)) = 0 \quad \text{for } k = 1, 2, \ldots, t,$$

where $P_k(x)$ ($k = 1, 2, \ldots$) are Gegenbauer polynomials.
Let $\Omega_d(\mathbb{C})$ denote the set of vectors of $\mathbb{C}^d$ of unit length. The complex projective space $P^{d-1}$ is the quotient set of $\Omega_d(\mathbb{C})$, by the equivalence relation

$$x \sim y \iff x = e^{\sqrt{-1}\theta}y \text{ for some } \theta \in \mathbb{R}. $$

**Definition**

A **$t$-design** in $P^{d-1}$ is a finite nonempty subset $X$ of $P^{d-1}$ satisfying

$$\int_{P^{d-1}} f(\xi) d\xi = \frac{1}{|X|} \sum_{x \in X} f(x)$$

for all $f \in \bigoplus_{k=0}^{t} \text{Hom}(k)$, where $d\xi$ denotes the unique normalized Haar measure invariant under the unitary group $U(d, \mathbb{C})$, and $\text{Hom}(k)$ will be defined later.
Let $\Omega_d(\mathbb{C})$ denote the set of vectors of $\mathbb{C}^d$ of unit length. The complex projective space $P^{d-1}$ is the quotient set of $\Omega_d(\mathbb{C})$, by the equivalence relation

$$x \sim y \iff x = e^{\sqrt{-1}\theta}y \quad \text{for some } \theta \in \mathbb{R}.$$ 

**Definition**

A $t$-design in $P^{d-1}$ is a finite nonempty subset $X$ of $P^{d-1}$ satisfying

$$\int_{P^{d-1}} f(\xi) d\xi = \frac{1}{|X|} \sum_{x \in X} f(x) \quad (2)$$

for all $f \in \bigoplus_{k=0}^t \text{Hom}(k)$, where $d\xi$ denotes the unique normalized Haar measure invariant under the unitary group $U(d, \mathbb{C})$, and $\text{Hom}(k)$ will be defined later.
Examples of 2-designs

- $d + 1$ mutually unbiased bases in $\mathbb{C}^d$
- Symmetric informationally complete positive operator-valued measure (SIC-POVM).
Examples of 2-designs

- $d + 1$ mutually unbiased bases in $\mathbb{C}^d$
- Symmetric informationally complete positive operator-valued measure (SIC-POVM).