QUANTUM DECOMPOSITION AND QUANTUM CENTRAL LIMIT THEOREM

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On the basis of the canonical relation between an interacting Fock space and a system of orthogonal polynomials we introduce the notion of quantum decomposition of a real random variable in an algebraic probability space. To understand prototypes we review some basic examples appearing from the Boson, Fermion, and free Fock spaces. We then prove quantum central limit theorems for the quantum components of the adjacency matrices (combinatorial Laplacians) of a growing family of regular connected graphs. As a corollary, asymptotic properties of the adjacency matrix are obtained. Concrete examples include lattices, homogeneous trees, Cayley graphs of the Coxeter groups, Hamming graphs and Johnson graphs. In particular, asymptotic spectral distribution of the adjacency matrix of a Johnson graph is described by an interacting Fock space corresponding to the Meixner polynomials which are one-parameter deformation of the Laguerre polynomials.

Introduction

Motivated by the canonical relation between an interacting Fock space and a system of orthogonal polynomials established by Accardi–Bożejko[1], we have explicitly introduced the notion of *quantum decomposition* of a real random variable in an algebraic probability space (including a classical case in principle) in the series of works[10,11,12] to study the adjacency matrix (combinatorial Laplacian) of a large graph. This idea together with quantum central limit theorems provides a new method of obtaining approximation of the spectral distribution of the adjacency matrix.

The main purpose of this paper is to give a self-contained account of the idea of quantum decomposition together with concrete examples and applications. In particular, we shall unify results obtained in the previous works into a general statement applicable to a wider class of graphs. Thus, the mechanism of our method has become clearer and one may expect many potential applications in geometry, combinatorics, physics and so on.

The paper is organized as follows. In Section 1 we review shortly some notions in algebraic probability theory, interacting Fock space and its relation to orthogonal polynomials. Then we introduce the quantum decomposition of a classical random variable and some basic examples. In Section 2 we recall some notions in graph theory. We introduce a stratification of a graph by means of the natural distance function and an orientation compatible with the stratification. Then a quantum decomposition of the adjacency matrix is defined and the main question is formulated. In Section 3 we prove a quantum central limit theorem (Therem 3.1) for the quantum components under three essential conditions (A1)–(A3) and discuss lattices, homogeneous trees and the Cayley graphs of the Coxeter groups as concrete examples. In Section 4 we discuss a possibility of removing condition (A1) by illustration of distance-regular graphs. In fact, condition (A1) is replaced with (A4) for a certain class of distance-regular graphs and we obtain a quantum central limit theorem (Theorem 4.1). Concrete examples are Hamming graphs and Johnson graphs. It is noteworthy that the Meixner polynomials, which is a one-parameter deformation of the Laguerre polynomials, play a role in the study of Johnson graphs. Finally, introducing the *weak quantum decomposition* for a general distance-regular graph, we show the quantum central limit theorem for its quantum components.

1 Notation and Background

1.1 Algebraic Probability Space

Let $\mathcal{A}$ be a $*$-algebra with unit $1 = 1_{\mathcal{A}}$, i.e., a unital $*$-algebra. A linear function $\phi : \mathcal{A} \to \mathbf{C}$ is called a *state* if $\phi(a^*a) \geq 0$ for all $a \in \mathcal{A}$ and $\phi(1) = 1$. The pair $(\mathcal{A}, \phi)$ is called an *algebraic probability space*. Each $a \in \mathcal{A}$ is called an *algebraic random variable* or a *random variable* for short.

Consider two algebraic probability spaces $(\mathcal{A}, \phi)$ and $(\mathcal{B}, \psi)$. Algebraic random variables $a \in \mathcal{A}$ and $b \in \mathcal{B}$ are called *stochastically equivalent* if

$$\phi(a^{\epsilon_1} a^{\epsilon_2} \dots a^{\epsilon_m}) = \psi(b^{\epsilon_1} b^{\epsilon_2} \dots b^{\epsilon_m})$$

for any choice of $m = 1, 2, \dots$ and $\epsilon_1, \dots, \epsilon_m \in \{1, *\}$. For a real random variable $a = a^* \in \mathcal{A}$ it is sufficient to consider the moment sequence $\phi(a^m)$, $m = 0, 1, 2, \dots$. Let $(\mathcal{A}_n, \phi_n)$ be a sequence of algebraic probability spaces and $\{a_n\}$ a sequence of algebraic random variables such that $a_n \in \mathcal{A}_n$. Then an algebraic random variable b in another algebraic probability space $(\mathcal{B}, \psi)$ is called a *stochastic limit* of $\{a_n\}$ if

$$\lim_{n\to\infty} \phi_n(a_n^{\epsilon_1} a_n^{\epsilon_2} \dots a_n^{\epsilon_m}) = \psi(b^{\epsilon_1} b^{\epsilon_2} \dots b^{\epsilon_m})$$

for any choice of $m = 1, 2, \dots$ and $\epsilon_1, \dots, \epsilon_m \in \{1, *\}$. In that case we also say that $\{a_n\}$ *converges stochastically* to b.

1.2 Interacting Fock Space

We refer to Accardi–Bożejko[1] for more details. Let $\lambda_0 = 1, \lambda_1, \lambda_2, \dots \geq 0$ be a sequence of nonnegative numbers and assume that if $\lambda_m = 0$ occurs for some $m \geq 1$ then $\lambda_n = 0$ for all $n \geq m$. According as $\lambda_n > 0$ for all n or $\lambda_m = 0$ occurs for some $m \geq 1$, we define a Hilbert space of infinite dimension or of finite dimension:

$$\Gamma = \sum_{n=0}^{\infty} \oplus \mathbf{C}\Phi_n, \qquad \Gamma = \sum_{n=0}^{m_0-1} \oplus \mathbf{C}\Phi_n,$$

where m_0 is the first number such that $\lambda_{m_0} = 0$, and $\{\Phi_n\}$ is an orthonormal basis.

We then define the creation operator B^+ and the annihilation operator B^- by

$$B^+\Phi_n = \sqrt{\frac{\lambda_{n+1}}{\lambda_n}}\,\Phi_{n+1}, \quad n \geq 0,$$

$$B^-\Phi_0 = 0, \quad B^-\Phi_n = \sqrt{\frac{\lambda_n}{\lambda_{n-1}}}\,\Phi_{n-1}, \quad n \geq 1.$$

In the case when Γ is of finite dimension we tacitly understand that $B^+\Phi_{m_0-1} = 0$. Equipped with the natural domains, $B^\pm$ become closed operators which are mutually adjoint. Then $\Gamma(\{\lambda_n\}) = (\Gamma, \{\lambda_n\}, B^+, B^-)$ is called an *interacting Fock space* associated with $\{\lambda_n\}$. By simple computation we have

$$B^+B^-\Phi_0 = 0, \quad B^+B^-\Phi_n = \frac{\lambda_n}{\lambda_{n-1}}\,\Phi_n, \quad n \geq 1,$$

$$B^-B^+\Phi_n = \frac{\lambda_{n+1}}{\lambda_n}\,\Phi_n, \quad n \geq 0,$$

$$B^{+n}\Phi_0 = \sqrt{\lambda_n}\,\Phi_n, \quad n \geq 0.$$

The number operator N is defined as usual by

$$N\Phi_n = n\Phi_n, \qquad n \geq 0.$$

Proposition 1.1 *Let $\Gamma = (\Gamma, \{\lambda_n\}, B^+, B^-)$ be the (one-mode) Boson Fock space, i.e., the interacting Fock space with $\lambda_n = n!$. Then the Boson commutation relation holds: $B^-B^+ - B^+B^- = 1$. Moreover,*

$$\langle \Phi_0, (B^+ + B^-)^m \Phi_0 \rangle_\Gamma = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^m e^{-x^2/2} dx, \quad m = 0, 1, 2, \ldots,$$

where the probability distribution in the right hand side is the standard Gaussian distribution, i.e., with mean 0 and variance 1.

Proposition 1.2 *Let $\Gamma = (\Gamma, \{\lambda_n\}, B^+, B^-)$ be the (one-mode) Fermion Fock space, i.e., the interacting Fock space with $\lambda_0 = \lambda_1 = 1$ and $\lambda_n = 0$ for $n \geq 2$. Then the Fermion commutation relation holds: $B^-B^+ + B^+B^- = 1$. Moreover,*

$$\langle \Phi_0, (B^+ + B^-)^m \Phi_0 \rangle_\Gamma = \frac{1}{2} \int_{-\infty}^{+\infty} x^m (\delta_{-1} + \delta_{+1})(dx), \quad m = 0, 1, 2, \ldots.$$

Proposition 1.3 *Let $\Gamma = (\Gamma, \{\lambda_n\}, B^+, B^-)$ be the (one-mode) free Fock space, i.e., the interacting Fock space with $\lambda_n = 1$ for all $n \geq 0$. Then the free commutation relation holds: $B^-B^+ = 1$. Moreover,*

$$\langle \Phi_0, (B^+ + B^-)^m \Phi_0 \rangle_\Gamma = \frac{1}{2\pi} \int_{-2}^{+2} x^m \sqrt{4 - x^2}\, dx, \quad m = 0, 1, 2, \ldots,$$

where the probability distribution in the right hand side is called the Wigner semi-circle law.

Remark 1.4 The above commutation relations are special cases of the so-called q-commutation relation: $B^-B^+ - qB^+B^- = 1$, where $-1 \leq q \leq 1$. This is realized by the interacting Fock space $\Gamma = (\Gamma, \{\lambda_n\}, B^+, B^-)$ with

$$\lambda_n = [n]_q! = [n]_q[n-1]_q \cdots [1]_q, \quad [n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

For more information on this topic, see e.g., Bożejko–Kümmerer–Speicher[5] and references cited therein.

1.3 Orthogonal Polynomials

Let μ be a probability measure on $\mathbf{R}$ with finite moments of all orders, i.e.,

$$\int_{\mathbf{R}} |x|^m \mu(dx) < \infty, \qquad m = 0, 1, 2, \cdots,$$

and $\{P_n\}$ the associated orthogonal polynomials normalized in such a way that $P_n(x) = x^n + \dots$. Then there exists uniquely a pair of sequences $\alpha_1, \alpha_2, \dots \in \mathbf{R}$ and $\omega_1, \omega_2, \dots \geq 0$ such that

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x - \alpha_1, \\ xP_n(x) &= P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x), \quad n \geq 1 \end{aligned} \tag{1}$$

The pair $\{\alpha_n\}, \{\omega_n\}$ is called the *Szegö–Jacobi parameter.* When the probability measure μ is supported by a finite set of exactly m_0 points, the orthogonal polynomials $\{P_n\}$ terminate at $n = m_0 - 1$ and the Szegö–Jacobi parameter becomes a pair of finite sequences $\alpha_1, \dots, \alpha_{m_0}$ and $\omega_1 \dots, \omega_{m_0-1}$, where the last numbers are determined by (1) with $P_{n+1} = 0$. Note also that μ is symmetric if and only if $\alpha_n = 0$ for all $n \geq 1$.

Theorem 1.5 (Accardi–Bożejko[1]) *Let $\{P_n\}$ be the orthogonal polynomials with respect to μ with Szegö–Jacobi parameters $\{\alpha_n\}$, $\{\omega_n\}$. Let $\Gamma(\{\lambda_n\})$ be an interacting Fock space associated with*

$$\lambda_0 = 1, \qquad \lambda_n = \omega_1\omega_2 \dots \omega_n, \quad n \geq 1. \tag{2}$$

Then there exists an isometry U from $\Gamma(\{\lambda_n\})$ into $L^2(\mathbf{R}, \mu)$ uniquely determined by

$$U\Phi_0 = P_0, \quad UB^+U^*P_n = P_{n+1}, \quad Q = U(B^+ + B^- + \alpha_{N+1})U^*,$$

where Q is the multiplication operator by x densely defined in $L^2(\mathbf{R}, \mu)$ and α_{N+1} is the operator defined by $\alpha_{N+1}\Phi_n = \alpha_{n+1}\Phi_n$.

The proof is straightforward. In fact, the isometry U is uniquely specified by $\sqrt{\lambda_n}\,\Phi_n \mapsto P_n$. A question of when U is a unitary, or equivalently when the polynomials span a dense subspace in $L^2(\mathbf{R}, \mu)$ is related to the so-called determinate moment problem, see e.g., Deift[7].

1.4 Quantum Decomposition of Position Variable

We keep the same notation as in the previous subsection. In quantum mechanics Q stands for the position variable.

Define an algebraic probability space $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is the unital $*$-algebra generated by Q and ϕ is the state defined by

$$\phi(a) = \langle P_0, aP_0 \rangle_{L^2(\mathbf{R},\mu)}, \qquad a \in \mathcal{A}.$$

Another algebraic probability space is naturally constructed from the interacting Fock space $(\Gamma, \{\lambda_m\}, B^+, B^-)$. Let $\mathcal{B}$ be the unital $*$-algebra generated by B^+, B^-, α_{N+1} and ψ the vacuum state defined by

$$\psi(b) = \langle \Phi_0, b\Phi_0 \rangle_\Gamma, \qquad b \in \mathcal{B}.$$

It then follows from Theorem 1.5 that

$$\phi(Q^m) = \psi((B^+ + B^- + \alpha_{N+1})^m), \qquad m = 0, 1, 2, \dots,$$

or equivalently

$$\int_{\mathbf{R}} x^m \mu(dx) = \langle \Phi_0, (B^+ + B^- + \alpha_{N+1})^m \Phi_0 \rangle_\Gamma, \qquad m = 0, 1, 2, \dots. \qquad (3)$$

In other words, two real random variables Q and $B^+ + B^- + \alpha_{N+1}$ are stochastically equivalent. In this sense we have

$$Q = B^+ + B^- + \alpha_{N+1},$$

which is a prototype of the *quantum decomposition.* Note that the quantum components are no longer commuting each other.

1.5 Quantum Decomposition of a Classical Random Variable

Let X be a real random variable defined on a classical probability space. The distribution μ of X is not necessarily uniquely specified by the moment sequence $\mathbf{E}(X^m)$ due to the famous determinate moment problem. If it is the case, X and Q are identified so that $X = B^+ + B^- + \alpha_{N+1}$ is also referred to as the *quantum decomposition.*

Example 1.6 Let X be a Bernoulli random variable such that $P(X = +1) = P(X = -1) = 1/2$. Then $X = B^+ + B^-$, where $B^\pm$ are the creation and annihilation operators of the Fermion Fock space defined in Proposition 1.2. In an explicit form we have

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

which is sometimes called a *quantum coin-tossing.*

Example 1.7 Let X be a real random variable obeying the standard Gaussian distribution. Then $X = B^+ + B^-$, where $B^\pm$ are the creation and annihilation operators of the Boson Fock space defined in Proposition 1.1.

Example 1.8 Let X be a real random variable obeying the Poisson measure with parameter $\lambda > 0$. Then the quantum decomposition is given by

$$X = (B^+ + \sqrt{\lambda})(B^- + \sqrt{\lambda}) = \sqrt{\lambda}(B^+ + B^-) + N + \lambda,$$

where $B^\pm$ and N are the creation, annihilation, and number operators of the Boson Fock space defined in Proposition 1.1.

Quantum decomposition of a stochastic process is a question of interest. For example, the Brownian motion is decomposed into a sum of the creation and annihilation processes defined in the Boson Fock space over $L^2(\mathbf{R})$ (hence this Boson Fock space is of infinite mode). In fact, this is a root of the quantum Itô theory initiated by Hudson–Parthasarathy[16]. Moreover, the quantum decomposition of the white noise is also performed within the framework of white noise distribution theory and is a clue to study white noise differential equations, see e.g., Chung–Ji–Obata[6].

2 Quantum Decomposition of Adjacency Matrices

2.1 Basic Notions in Graph Theory

A non-empty set V equipped with $E \subset \{\{x,y\}\,;\, x,y \in V,\, x \neq y\}$ is called a *graph* and denoted by $\mathcal{G} = (V,E)$. Elements of V and of E are called a *vertex* and an *edge*, respectively. If $\{x,y\} \in E$, we say that x and y are *adjacent* and write $x \sim y$. The *degree (or valency)* of $x \in V$ is defined by $\kappa(x) = |\{y \in V\,;\, y \sim x\}|$. A graph is called *regular* if $\kappa(x) = \kappa$ is a finite constant independent of $x \in V$.

A finite sequence $x_0, x_1, \dots, x_n \in V$ is called a *path* of length n if $x_i \sim x_{i+1}$ for all $i = 0, 1, \dots, n-1$. A graph is called *connected* if any pair of points $x, y \in V$ are connected by a path. For $x, y \in V$ the *distance* $\partial(x,y)$ is by definition the length of the shortest path connecting x and y. Obviously, $x \sim y$ if and only if $\partial(x,y) = 1$. The *diameter* of a graph is defined to be $\sup\{\partial(x,y)\,;\, x,y \in V\}$.

For a graph $\mathcal{G} = (V,E)$ the *i-th adjacency matrix* $A_i = (A_i)_{xy}$, where x, y run over V, is defined by

$$(A_i)_{xy} = \begin{cases} 1, & \partial(x,y) = i, \\ 0, & \text{otherwise}. \end{cases} \tag{4}$$

We set $A = A_1$ which is called the *adjacency matrix* for simplicity.

2.2 Stratification of a Graph and Associated Hilbert Space

From now on let $\mathcal{G} = (V,E)$ be a regular connected graph with a fixed origin $x_0 \in V$. The degree is denoted by κ. Then, the graph is stratified into a disjoint union of strata:

$$V = \bigcup_{n=0}^{\infty} V_n, \qquad V_n = \{x \in V\,;\, \partial(x_0, x) = n\}. \tag{5}$$

Obviously, $|V_0| = 1$, $|V_1| = \kappa$, and $|V_n| \le \kappa(\kappa-1)^{n-1}$ for $n \ge 2$. The next result is immediate by the triangle inequality, see Figure 1.

Lemma 2.1 *Let $x, y \in V$. If $x \in V_n$ and $x \sim y$, then $y \in V_{n-1} \cup V_n \cup V_{n+1}$.*

For $x \in V$ we denote by δ_x the indicator function of the singlet $\{x\}$. The collection $\{\delta_x\,;\, x \in V\}$ forms a complete orthonormal basis of $\ell^2(V)$. According to (5), we define a Hilbert space:

$$\Gamma(\mathcal{G}) = \sum_{n=0}^{\infty} \oplus \mathbf{C}\Phi_n, \qquad \Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} \delta_x,$$

where $\{\Phi_n\}$ becomes an orthonormal basis of $\Gamma(\mathcal{G})$.

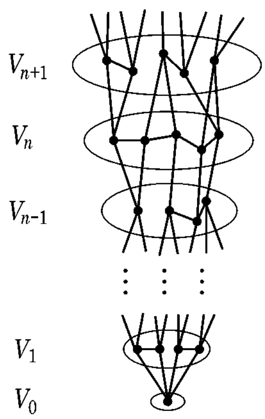

Figure 1. Stratification of $\mathcal{G} = (V, E)$ with $\kappa = 4$

2.3 Adjacency Matrix as Algebraic Random Variable

First note that A^n is well defined for all $n \ge 1$. In fact, the (x, y)-component of A^n coincides with the number of paths of length n that connect x and y. Obviously, $0 \le (A^n)_{xy} \le \kappa^n$. Hence we can define the unital $*$-algebra generated by A, which will be denoted by $\mathcal{A}$.

The adjacency matrix A acts on the Hilbert space $\ell^2(V)$ in a natural manner:

$$Af(x) = \sum_{y \in V} A_{xy} f(y) = \sum_{y \sim x} f(y), \qquad f \in \ell^2(V).$$

Note that $||A|| = \kappa$ and $\mathcal{A}$ is injectively imbedded in $\mathbf{B}(\ell^2(V))$. In general, a state ϕ on $\mathcal{A}$ is chosen by a question. In this paper we consider the *vacuum state* defined by

$$\phi(a) = \langle \delta_{x_0}, a\delta_{x_0} \rangle, \qquad a \in \mathcal{A}.$$

Thus the adjacency matrix A is regarded as a real random variable of the algebraic probability space $(\mathcal{A}, \phi)$.

2.4 Quantum Decomposition of Adjacency Matrix

By virtue of Lemma 2.1 we assign to each edge $x \sim y$ of the graph $\mathcal{G} = (V, E)$ an orientation compatible with the stratification, i.e., in such a way that $x \prec y$ if $x \in V_n$ and $y \in V_{n+1}$. For an edge $x \sim y$ lying in a stratum V_n there are two ways of assigning an orientation and, as a result, there are many ways of giving an orientation to the graph $\mathcal{G}$, see Figure 2. Then we define

$$(A^+)_{yx} = \begin{cases} A_{yx} = 1 & \text{if } y \succ x, \\ 0 & \text{otherwise,} \end{cases} \qquad (A^-)_{yx} = \begin{cases} A_{yx} = 1 & \text{if } y \prec x, \\ 0 & \text{otherwise,} \end{cases}$$

or equivalently,

$$A^+ \delta_x = \sum_{y \succ x} \delta_y, \qquad A^- \delta_x = \sum_{y \prec x} \delta_y. \tag{6}$$

As is easily verified, $(A^+)^* = A^-$ and

$$A = A^+ + A^-, \tag{7}$$

which is called a *quantum decomposition*. We keep in mind that the above quantum decomposition depends on an orientation introduced into $\mathcal{G}$.

Remark 2.2 There is a slightly different idea of quantum decomposition based on a Fock space structure of $\Gamma(\mathcal{G})$. Namely, to an edge $x \sim y$ lying in a stratum V_n we do not give an orientation, we define instead a new operator A° to have $A = A^+ + A^- + A^\circ$. Obviously, these operators are more like the creation, annihilation, and number operators. This decomposition will be discussed elsewhere.

2.5 Fundamental Question

Given a "growing" family of graphs $\{\mathcal{G}_\lambda = (V^{(\lambda)}, E^{(\lambda)})\,;\, \lambda \in \Lambda\}$, where Λ is an infinite directed set, we consider the adjacency matrix A_λ of $\mathcal{G}_\lambda$ with its

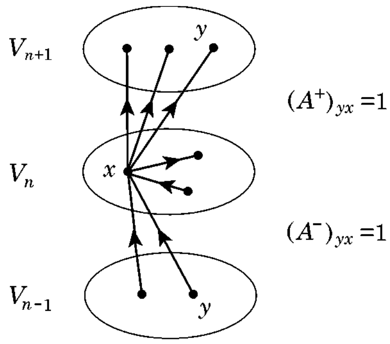

Figure 2. Quantum decomposition: $A = A^+ + A^-$

quantum decomposition $A_\lambda = A_\lambda^+ + A_\lambda^-$ as in (7). A general question is to construct an interacting Fock space $(\Gamma, \{\lambda_n\}, B^+, B^-)$ in which the limits

$$C^\pm = \lim_{\lambda \in \Lambda} \frac{A_\lambda^\pm}{Z_\lambda} \tag{8}$$

are described, where Z_λ is a normalizing constant. A rigorous statement will be naturally formulated as a quantum central limit theorem.

When $C^\pm$ admit simple expressions such as linear combinations of $B^\pm$ and the number operator N on Γ, the distribution of $C^+ + C^-$ is easily computed with the help of Theorem 1.5, see also Subsection 1.4. Note that the distribution of $C^+ + C^-$ reflects asymptotic spectral properties of A_λ.

3 Quantum Central Limit Theorem

3.1 Conditions (A1)–(A3) and Main Statement

Let $\{\mathcal{G}_\lambda = (V^{(\lambda)}, E^{(\lambda)})\,;\, \lambda \in \Lambda\}$ be a family of regular connected graphs, where Λ is an infinite directed set, with stratification:

$$V^{(\lambda)} = \bigcup_{n=0}^{\infty} V_n^{(\lambda)}.$$

The degree of $\mathcal{G}_\lambda$ is denoted by $\kappa(\lambda)$ and assume $\lim_\lambda \kappa(\lambda) = \infty$. The associated Hilbert space is denoted as usual by

$$\Gamma(\mathcal{G}_\lambda) = \sum_{n=0}^{\infty} \oplus \mathbf{C}\Phi_n^{(\lambda)}, \qquad \Phi_n^{(\lambda)} = |V_n^{(\lambda)}|^{-1/2} \sum_{x \in V_n^{(\lambda)}} \delta_x,$$

where $\{\Phi_n^{(\lambda)}\}$ forms an orthonormal basis. Keeping Lemma 2.1 in mind, for $x \in V_n^{(\lambda)}$ we put

$$\omega_+^{(\lambda)}(x) = \{y \in V_{n+1}^{(\lambda)}\,;\, y \sim x\}, \quad \omega_-^{(\lambda)}(x) = \{y \in V_{n-1}^{(\lambda)}\,;\, y \sim x\}.$$

With these notation we consider the following conditions:

(A1) there is no edge lying in a common stratum $V_n^{(\lambda)}$, i.e.,

$$|\omega_+^{(\lambda)}(x)| + |\omega_-^{(\lambda)}(x)| = \kappa(\lambda), \qquad x \in V^{(\lambda)}, \quad \lambda \in \Lambda; \tag{9}$$

(A2) for each $n = 1, 2, \ldots$ there exist an integer $\omega_n \geq 1$ and a constant $C_n \geq 0$ such that

$$|\{x \in V_n^{(\lambda)}\,;\, |\omega_-^{(\lambda)}(x)| \neq \omega_n\}| \leq C_n \kappa(\lambda)^{n-1}$$

for all $\lambda \in \Lambda$;

(A3) for any $n = 1, 2, \ldots$,

$$W_n \equiv \sup_{\lambda \in \Lambda} \sup_{x \in V_n^{(\lambda)}} |\omega_-^{(\lambda)}(x)| < \infty.$$

By (A1) there is only one orientation which is compatible with the stratification, and therefore a quantum decomposition $A_\lambda = A_\lambda^+ + A_\lambda^-$ of the adjacency matrix of $\mathcal{G}_\lambda$ is unique. Condition (A2) means that for any $\lambda \in \Lambda$ and $n = 1, 2, \ldots$, a generic $x \in V_n^{(\lambda)}$ is connected by edges with exactly ω_n vertices in the lower stratum $V_{n-1}^{(\lambda)}$. As for condition (A3), we note that $\sup_{x \in V_n^{(\lambda)}} |\omega_-^{(\lambda)}(x)| \leq \kappa(\lambda)$, which follows from (9). Hence (A3) ensures uniform boundedness. Existence of the stochastic limits (8) is now claimed in the following

Theorem 3.1 *Let $\{\mathcal{G}_\lambda = (V^{(\lambda)}, E^{(\lambda)})\,;\, \lambda \in \Lambda\}$ be a family of regular connected graphs such that $\lim_\lambda \kappa(\lambda) = \infty$. Assume conditions (A1)–(A3) are satisfied and let $(\Gamma, \{\lambda_n\}, B^+, B^-)$ be the interacting Fock space with $\lambda_n = \omega_1 \ldots \omega_n$. Then,*

$$\lim_{\lambda \in \Lambda} \left\langle \Phi_j^{(\lambda)}, \frac{A_\lambda^{\epsilon_1}}{\sqrt{\kappa(\lambda)}} \cdots \frac{A_\lambda^{\epsilon_m}}{\sqrt{\kappa(\lambda)}} \Phi_k^{(\lambda)} \right\rangle_{\ell^2(V^{(\lambda)})} = \langle \Phi_j, B^{\epsilon_1} \ldots B^{\epsilon_m} \Phi_k \rangle$$

for any choice of $j, k = 0, 1, 2, \ldots$, $m = 1, 2, \ldots$ and $\epsilon_1, \ldots, \epsilon_m \in \{\pm\}$.

In particular, taking the vacuum states (i.e., $j = k = 0$) we come to the following

Corollary 3.2 *The normalized quantum components $A_\lambda^\pm/\sqrt{\kappa(\lambda)}$ of the adjacency matrix of $\mathcal{G}_\lambda$ converges stochastically to the annihilation and creation operators on the interacting Fock space $(\Gamma, \{\lambda_n\}, B^+, B^-)$, where $\lambda_n = \omega_1 \dots \omega_n$.*

The classical reduction is now immediate.

Corollary 3.3 *It holds that*

$$\lim_{\lambda\in\Lambda} \left\langle \Phi_0^{(\lambda)}, \left(\frac{A_\lambda}{\sqrt{\kappa(\lambda)}}\right)^m \Phi_0^{(\lambda)} \right\rangle_{\ell^2(V^{(\lambda)})} = \int_{\mathbf{R}} x^m \mu(dx), \qquad m = 0, 1, 2, \dots,$$

where μ is a probability measure corresponding to the interacting Fock space $(\Gamma, \{\lambda_n\}, B^+, B^-)$.

3.2 Proof of Theorem 3.1

Theorem 3.1 was first proved by Hashimoto[10] for a class of Cayley graphs and the proof therein is easily adapted to our case. Here we only mention a sketch.

For simplicity we drop the suffix λ. By condition (A2) we split V_n into a disjoint union of two subsets:

$$V_n = V_n^{\mathrm{gen}} \cup V_n^{\mathrm{ex}},$$

$$V_n^{\mathrm{gen}} = \{x \in V_n \,;\, \omega_-(x) = \omega_n\}, \qquad V_n^{\mathrm{ex}} = \{x \in V_n \,;\, \omega_-(x) \neq \omega_n\}.$$

Lemma 3.4

$$|V_n| = \frac{\kappa^n}{\omega_n \dots \omega_1} + O(\kappa^{n-1}), \qquad n \geq 1. \tag{10}$$

PROOF. By counting the number of edges whose endpoints are in V_n we have

$$\begin{aligned} \kappa|V_n| &= \sum_{y\in V_{n+1}} |\omega_-(y)| + \sum_{x\in V_{n-1}} |\omega_+(z)| \\ &= \omega_{n+1}|V_{n+1}| + \sum_{y\in V_{n+1}^{\mathrm{ex}}} (|\omega_-(y)| - \omega_{n+1}) \\ &\qquad + (\kappa - \omega_{n-1})|V_{n-1}| - \sum_{x\in V_{n-1}^{\mathrm{ex}}} (|\omega_-(z)| - \omega_{n-1}). \end{aligned}$$

For $n \geq 1$ we put

$$S_n = \sum_{y\in V_{n+1}^{\mathrm{ex}}} (|\omega_-(y)| - \omega_{n+1}) - \sum_{x\in V_{n-1}^{\mathrm{ex}}} (|\omega_-(z)| - \omega_{n-1}),$$

where $\omega_0 = 0$. Applying two inequalities: $|V_n^{\mathrm{ex}}| \le C_n \kappa^{n-1}$ and $|\omega_-(x)| \le W_n$ for all $x \in V_n$, which follow respectively from (A2) and (A3), we come to

$$|S_n| \le W_{n+1} C_{n+1} \kappa^n + W_{n-1} C_{n-1} \kappa^{n-2} = O(\kappa^n). \tag{11}$$

Then, by repeated application of (11) we obtain (10) with no difficulty. ∎

Lemma 3.5

$$\frac{A^+}{\sqrt{\kappa}} \Phi_n = \sqrt{\omega_{n+1}}\, \Phi_{n+1} + O(\kappa^{-1/2}), \qquad n = 0, 1, 2, \dots, \tag{12}$$

$$\frac{A^-}{\sqrt{\kappa}} \Phi_0 = 0, \quad \frac{A^-}{\sqrt{\kappa}} \Phi_n = \sqrt{\omega_n}\, \Phi_{n-1} + O(\kappa^{-1}), \quad n = 1, 2, \dots. \tag{13}$$

PROOF. By definition we have

$$\begin{aligned} |V_n|^{1/2} A^+ \Phi_n &= \sum_{x \in V_n} A^+ \delta_x = \sum_{y \in V_{n+1}} \omega_-(y) \delta_y \\ &= \omega_{n+1} |V_{n+1}|^{1/2} \Phi_{n+1} + \sum_{y \in V_{n+1}^{\mathrm{ex}}} (\omega_-(y) - \omega_{n+1}) \delta_y. \end{aligned}$$

Then, (12) follows by a direct computation with the help of Lemma 3.4. The proof of (13) is similar. ∎

Consider an interacting Fock space $(\Gamma, \{\lambda_n\}, B^+, B^-)$ with $\lambda_n = \omega_1 \dots \omega_n$. Then, at a formal level we see immediately from (12) and (13) that

$$\lim_{\lambda \in \Lambda} \frac{A_\lambda^\pm}{\sqrt{\kappa(\lambda)}} = B^\pm. \tag{14}$$

In fact, the proof of Theorem 3.1 is a direct computation using Lemma 3.5.

3.3 Cayley Graphs

Consider a discrete group G with the identity e and a set of generators $\Sigma \subset G$ satisfying (i) $\sigma \in \Sigma \Rightarrow \sigma^{-1} \in \Sigma$, i.e., $\Sigma^{-1} = \Sigma$; and (ii) $e \notin \Sigma$. Then G becomes a graph, where a pair $x, y \in G$ satisfying $yx^{-1} \in \Sigma$ constitutes an edge. This is called a *Cayley graph* and denoted by (G, Σ). A Cayley graph is regular with degree $\kappa = |\Sigma|$. We consider e as the origin of the Cayley graph and introduce a stratification as usual.

Example 3.6 (Lattice) The additive group $\mathbf{Z}^N$ furnished with the standard generators $g_{\pm 1} = (\pm 1, 0, \dots, 0), \dots, g_{\pm N} = (0, \dots, 0, \pm 1)$, is the N-dimensional lattice. Conditions (A1)–(A3) are easily verified with $\kappa(N) = 2N$, $\omega_n = n$ and $W_n = n$. Hence $\lambda_n = n!$ and the limit is described by the Boson

Fock space. By Proposition 1.1 for the normalized adjacency matrix $A_N/\sqrt{2N}$ we have

$$\lim_{N\to\infty} \left\langle \Phi_0^{(N)}, \left(\frac{A_N}{\sqrt{2N}}\right)^m \Phi_0^{(N)} \right\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^m e^{-x^2/2}\, dx, \quad m = 0,1,2,\dots.$$

This reproduces the central limit theorem in the classical probability theory.

Example 3.7 (Homogeneous tree) Let F_N be the free group on N free generators $g_1, \dots, g_N$. Then the Cayley graph $(F_N, \{g_{\pm 1}, \dots, g_{\pm N}\})$, where $g_{-i} = g_i^{-1}$ for simplicity, becomes a homogeneous tree. Conditions (A1)–(A3) are easily verified with $\kappa(N) = 2N$, $\omega_n = 1$ and $W_n = 1$. Hence $\lambda_n = 1$ for all $n \ge 0$ so that the limit is described by the free Fock space. Moreover, by Proposition 1.3 for the normalized adjacency matrix $A_N/\sqrt{2N}$ we have

$$\lim_{N\to\infty} \left\langle \Phi_0^{(N)}, \left(\frac{A_N}{\sqrt{2N}}\right)^m \Phi_0^{(N)} \right\rangle = \frac{1}{2\pi} \int_{-2}^{+2} x^m \sqrt{4 - x^2}\, dx, \quad m = 0,1,2,\dots.$$

This is a prototype of the central limit theorem in the free probability theory, see e.g., Hiai–Petz[13], Voiculescu–Dykema–Nica[19].

Example 3.8 (Coxeter group) We refer to Humphreys[17] for generalities. Let $\Sigma = \{g_1, g_2, \dots\}$ be a countable infinite set and consider a Coxeter matrix $m(i,j) \in \{1, 2, \cdots, \infty\}$, where i, j run over $\{1, 2, \dots\}$, such that $m(i,i) = 1$ and $m(i,j) = m(j,i) \ge 2$ for $i \ne j$. For each $N \ge 1$ let $G^{(N)}$ be the group generated by $\Sigma^{(N)} = \{g_1, g_2, \cdots, g_N\}$ subject only to the relations:

$$(g_i g_j)^{m(i,j)} = e, \qquad i, j \in \{1, 2, \dots, N\}.$$

In case of $m(i,j) = \infty$ we understand that $g_i g_j$ is of infinite order. Note that any $g_i \in \Sigma$ is of order two and hence $\kappa(N) = N$. Condition (A1) is satisfied by every Coxeter group $G^{(N)}$. Furthermore, (A2) and (A3) are satisfied if $m(i,j) \ge 3$ for any pair of distinct i, j. In that case $\omega_n = 1$ and $W_n = 2$ for all $n \ge 1$. Hence, for a growing family of Coxeter groups $(G^{(N)}, \Sigma^{(N)})$ with Coxeter matrix satisfying $m(i,j) \ge 3$ for $i \ne j$, the situation falls into the same as in Example 3.7. The distribution of $A_N/\sqrt{2N}$ was obtained by Fendler[8] with a different method. Detailed discussion is found in Hashimoto–Hora–Obata[11].

4 More on Quantum Central Limit Theorem

4.1 Motivation

In the previous section we proved a quantum central limit theorem for adjacency matrices (Theorem 3.1) under the three assumptions (A1)–(A3). However, there are many interesting graphs which do not satisfy condition (A1).

As is discussed in Subsection 2.4, a quantum decomposition of the adjacency matrix is always possible since it is induced from an orientation compatible with the stratification, where condition (A1) is not necessary. Hence our question is how to introduce an orientation good enough to obtain quantum central limit theorems for the quantum components of the adjacency matrix. We shall illustrate this problem with distance-regular graphs.

4.2 Distance-Regular Graphs

A finite connected graph $\mathcal{G} = (V, E)$ is called *distance-regular* if for any choice of $h, i, j \in \{0, 1, \dots, d\}$, d being the diameter of the graph,

$$|\{z \in V\,;\, \partial(x,z) = i, \partial(z,y) = j\}| \equiv p^h_{ij}$$

does not depend on the choice of $x, y \in V$ such that $\partial(x,y) = h$. We call $\{p^h_{ij}\}$ the *intersection numbers* of $\mathcal{G}$. For simplicity we set

$$\kappa_n = p^0_{nn}, \qquad \kappa = \kappa_1 = p^0_{11}.$$

Obviously, the distance-regular graph is regular with degree κ.

4.3 Quantum Decomposition Induced from Euler Paths

Let $\mathcal{G} = (V, E)$ be a distance-regular graph with intersection numbers $\{p^h_{ij}\}$. We fix an arbitrary $x_0 \in V$ as the origin and introduce the stratification:

$$V = \bigcup_{n=0}^{d} V_n, \qquad V_n = \{x \in V\,;\, \partial(x_0, x) = n\}.$$

Note that $|V_n| = \kappa_n$. For a quantum decomposition of the adjacency matrix A it is sufficient to specify an orientation of an edge $x \sim y$ lying in V_n.

Let X_n be a subgraph of $\mathcal{G}$, where the set of vertices is V_n and the edges are those of $\mathcal{G}$ lying in V_n. Then X_n is a regular graph with degree p^n_{1n}. We now consider another condition:

(A4) for each $n = 1, 2, \dots$ one of the two cases occurs:

(Case 1) p^n_{1n} is even;

(Case 2) p^n_{1n} is odd and X_n admits a perfect matching.

Here we recall definition: in general, a graph (V, E) is said to admit a *perfect matching* if there is a subset $M \subset E$ such that each $x \in V$ is an endpoint of just one edge of M.

When (Case 1) occurs, by Euler's unicursal theorem there is an Euler path for X_n along which each edge is given an orientation. When (Case 2) occurs, deleting M from the subgraph X_n we obtain another subgraph $\tilde{X}_n$ which is a regular graph with even degree. Then, taking an Euler path for $\tilde{X}_n$, we give an orientation to each edge of $\tilde{X}_n$. An edge of M is given an arbitrary orientation. Thus, a distance-regular graph $\mathcal{G}$ satisfying condition (A4) is given an orientation.

Let $A = A^+ + A^-$ be the quantum decomposition induced from the above orientation. Put

$$v_n = |V_n|^{1/2}\Phi_n = \sum_{x \in V_n} \delta_x,$$

where $\{\Phi_n\}$ is an orthonormal basis of $\Gamma(\mathcal{G})$. Then, by a direct computation,

(Case 1)

$$A^{\pm} v_n = p_{1n}^{n\pm1} v_{n\pm1} + \frac{p_{1n}^n}{2} v_n, \qquad n \in \{0, 1, \dots, d\}. \tag{15}$$

(Case 2) Let V_n^- (resp. V_n^+) be the set of all $x \in V_n$ which are initial (terminal) vertex of an edge of M. Then $V_n = V_n^+ \cup V_n^-$ and

$$A^{\pm} v_n = p_{1n}^{n\pm1} v_{n\pm1} + \frac{p_{1n}^n - 1}{2} v_n + \sum_{y \in V_n^{\pm}} \delta_y, \qquad n \in \{0, 1, \dots, d\}. \tag{16}$$

Hence the action of $A^{\pm}$ is defined on $\Gamma(\mathcal{G})$ when (Case 1) occurs. This is not true for (Case 2) as in the case discussed in Section 3.

4.4 Quantum Central Limit Theorem

We consider a growing family of distance-regular graphs $\{\mathcal{G}_\lambda\,;\, \lambda \in \Lambda\}$ such that $d(\lambda) \to \infty$ and $\kappa(\lambda) \to \infty$, where $d(\lambda)$ and $\kappa(\lambda)$ are the diameter and the degree of $\mathcal{G}_\lambda$, respectively. The adjacency matrix and the intersection numbers of $\mathcal{G}_\lambda$ are denoted by A_λ and $\{p(\lambda)_{ij}^h\}$, respectively. By definition

$$\Gamma(\mathcal{G}_\lambda) = \sum_{n=0}^{d(\lambda)} \oplus \mathbf{C}\Phi_n^{(\lambda)}.$$

Theorem 4.1 *Assume that every $\mathcal{G}_\lambda$ satisfies condition (A4) and let $A_\lambda = A_\lambda^+ + A_\lambda^-$ be the adjacency matrix of $\mathcal{G}_\lambda$ with its quantum decomposition defined as in Subsection 4.3. Assume that the limits*

$$\lim_{\lambda \in \Lambda} p(\lambda)_{1n}^{n+1} = \omega_{n+1}, \qquad \lim_{\lambda \in \Lambda} \frac{p(\lambda)_{1n}^n}{2\sqrt{\kappa(\lambda)}} = \psi_n, \tag{17}$$

exist for all $n = 0, 1, 2, \ldots$, *and let* $(\Gamma, \{\lambda_n\}, B^+, B^-)$ *be the interacting Fock space, where* $\lambda_0 = 1$, $\lambda_n = \omega_1 \ldots \omega_n$ *for* $n \geq 1$. *Define*

$$C^{\pm} = B^{\pm} + \psi_N,$$

where N *is the number operator of* Γ. *Then for any choice of* $j, k = 0, 1, 2, \ldots$, $m = 1, 2, \ldots$ *and* $\epsilon_1, \ldots, \epsilon_m \in \{\pm\}$, *it holds that*

$$\lim_{\lambda \in \Lambda} \left\langle \Phi_j^{(\lambda)}, \frac{A_\lambda^{\epsilon_1}}{\sqrt{\kappa(\lambda)}} \cdots \frac{A_\lambda^{\epsilon_m}}{\sqrt{\kappa(\lambda)}} \Phi_k^{(\lambda)} \right\rangle_{\ell^2(V^{(\lambda)})} = \langle \Phi_j, C^{\epsilon_1} \ldots C^{\epsilon_m} \Phi_k \rangle_\Gamma. \quad (18)$$

This is proved by employing a similar argumant as in Hashimoto–Obata–Tabei[12], see Hashimoto–Hora–Obata[11] for details. The classical reduction is immediate.

Corollary 4.2 *Under the same assumptions as in Theorem 4.1 it holds that*

$$\lim_{\lambda \in \Lambda} \left\langle \Phi_0^{(\lambda)}, \left(\frac{A_\lambda}{\sqrt{\kappa(\lambda)}} \right)^m \Phi_0^{(\lambda)} \right\rangle_{\ell^2(V^{(\lambda)})} = \langle \Phi_0, (B^+ + B^- + 2\psi_N)^m \Phi_0 \rangle_\Gamma, \quad (19)$$

for all $m = 0, 1, 2, \ldots$.

Remark 4.3 As we shall see later in Lemma 4.6, the inner product in the left hand side of (19) coincides with $\mathrm{Tr}((A_\lambda/\sqrt{\kappa(\lambda)})^m)$, where Tr stands for the normalized trace. Therefore a probability measure μ on $\mathbf{R}$ such that

$$\int_{\mathbf{R}} x^m \mu(dx) = \langle \Phi_0, (B^+ + B^- + 2\psi_N)^m \Phi_0 \rangle_\Gamma, \qquad m = 0, 1, 2, \ldots,$$

gives an approximation of the eigenvalue distribution of $A_\lambda/\sqrt{\kappa(\lambda)}$. For some classes of distance-regular graphs including the Hamming and Johnson graphs the distributions μ were first obtained by Hora[14] with a classical method.

4.5 Hamming Graph

Let F be a finite set of $n+1$ points and $d \geq 1$ an integer. For $x = (\xi_1, \cdots, \xi_d)$ and $y = (\eta_1, \cdots, \eta_d)$ in $V = F^d$ we put

$$\partial(x, y) = |\{i \,;\, \xi_i \neq \eta_i\}|.$$

A pair $x, y \in V$ is by definition an edge if $\partial(x, y) = 1$. Then V becomes a distance-regular graph called a *Hamming graph* and is denoted by $H(d, n+1)$. As is easily verified,

$$\begin{aligned}
p_{k1}^{k+1} &= k + 1, \qquad k = 0, 1, \cdots, d-1, \\
p_{k1}^{k} &= k(n-1), \qquad k = 0, 1, \cdots, d, \\
p_{k1}^{k-1} &= n(d-k+1), \qquad k = 1, 2, \cdots, d, \\
p_{11}^{0} &= nd = \kappa.
\end{aligned}$$

Moreover, it was proved in Hashimoto–Obata–Tabei[12] that condition (A4) is satisfied by $H(d,n+1)$ for any choice of d and n.

Let $A_{(d,n+1)} = A^+_{(d,n+1)} + A^-_{(d,n+1)}$ be the adjacency matrix of $H(d,n+1)$ and its quantum decomposition. Then the conditions in Theorem 4.1 are fulfilled with $\omega_k = k$ and $\psi_k = k\sqrt{\tau}/2$, and hence the limit is described by the Boson Fock space as follows.

Theorem 4.4 *It holds that*

$$\lim_{d,n\to\infty,n/d\to\tau} \frac{A^\pm_{(d,n+1)}}{\sqrt{nd}} = B^\pm + \frac{\sqrt{\tau}}{2} N,$$

in the sense of quantum random variables (see Theorem 4.1), where $B^\pm$ and $N = B^+B^-$ are the creation, annihilation and number operators on the Boson Fock space.

By the classical reduction we have

$$\lim_{d,n\to\infty,n/d\to\tau} \left\langle \Phi_0^{(d,n+1)}, \left(\frac{A_{(d,n+1)}}{\sqrt{nd}}\right)^m \Phi_0^{(d,n+1)} \right\rangle$$
$$= \langle \Phi_0, (\sqrt{\tau}\, B^+B^- + B^+ + B^-)^m \Phi_0 \rangle, \quad m = 0,1,2,\dots. \qquad (20)$$

The unique probability distribution ν_τ whose m-th moment is given by (20) is the standard Gaussian distribution for $\tau = 0$ and by the image of the Poisson distribution of parameter $1/\tau$ under the map $x \mapsto \sqrt{\tau}x - (1/\sqrt{\tau})$ for $\tau > 0$, see Examples 1.7 and 1.8.

4.6 Johnson Graph

Let v, d be a pair of positive integers such that $d \le v$. Put $S = \{1,2,\dots,v\}$ and $V = \{x \subset S\,;\, |x| = d\}$. We say that $x, y \in V$ are adjacent if $d - |x \cap y| = 1$. Thus a graph structure is introduced in V, which is called a *Johnson graph* and denoted by $J(v,d)$. By symmetry we may assume that $2d \le v$.

The Johnson graph $J(v,d)$ is distance-regular with intersection numbers

$$\kappa = d(v-d), \quad p^n_{1n} = n(v-2n), \quad p^{n+1}_{1n} = (n+1)^2, \qquad (21)$$

where $n = 0,1,\dots,d$. It was proved by Hashimoto–Hora–Obata[11] that every Johnson graph $J(v,d)$ fulfills condition (A4).

Consider the growing family of Johnson graphs $J(v,d)$, where $d \to \infty$ and $2d/v \to p \in (0,1]$. Condition (17) in Theorem 4.1 is satisfied with

$$\omega_{n+1} = (n+1)^2, \qquad \psi_n = \frac{n}{\sqrt{p(2-p)}}, \qquad n = 0,1,\dots,d.$$

Then for the quantum decomposition $A_{(v,d)} = A^+_{(v,d)} + A^-_{(v,d)}$ of the adjacency matrix of $J(v,d)$ we have the following

Theorem 4.5 *Let* $0 < p \le 1$. *Then*

$$\lim_{d\to\infty, 2d/v\to p} \frac{A^{\pm}_{(v,d)}}{\sqrt{d(v-d)}} = B^{\pm} + \frac{N}{\sqrt{p(2-p)}}$$

in the sense of Theorem 4.1, where $B^{\pm}$, N *are respectively the creation, annihilation, and number operators on the interacting Fock space* $\Gamma = (\Gamma, \{(n!)^2\}, B^+, B^-)$.

The classical reduction is immediate:

$$\lim_{d\to\infty, 2d/v\to p} \left\langle \Phi_0^{(v,d)}, \left(\frac{A_{(v,d)}}{\sqrt{d(v-d)}}\right)^m \Phi_0^{(v,d)} \right\rangle$$
$$= \left\langle \Phi_0, \left(B^+ + B^- + \frac{2N}{\sqrt{p(2-p)}}\right)^m \Phi_0 \right\rangle_{\Gamma}, \qquad m = 0, 1, 2, \ldots.$$

The probability measure ν_p on $\mathbf{R}$ whose m-th moment is given as above is obtained by observing the associated orthogonal polynomials.

For $p = 1$, the Laguerre polynomials $L_n(x) = x^n + \ldots$ satisfying the recurrence formula:

$$L_0(x) = 1,$$
$$L_1(x) = x - 1,$$
$$xL_n(x) = L_{n+1}(x) + (2n+1)L_n(x) + n^2 L_{n-1}(x), \qquad n \ge 1,$$

play a role. By using the fact that the Laguerre polynomials form the orthogonal polynomials with respect to the probability measure $e^{-x}dx$ on the half line $[0, \infty)$, i.e.,

$$\int_{\mathbf{R}} x^m e^{-x} dx = \langle \Phi_0, (B^+ + B^- + 2N + 1)^m \Phi_0 \rangle_{\Gamma},$$

where $\Gamma = (\Gamma, \{(n!)^2\}, B^+, B^-)$ and N is the number operator, we see that $\nu_1(dx) = e^{-(x+1)}dx$ concentrated on $[-1, \infty)$.

For $0 < p < 1$ the Meixner polynomials (see Schoutens[18] for definition) play a role. By modification the polynomilas $M_n(x) = x^n + \ldots$ defined by

$$M_0(x) = 1,$$
$$M_1(x) = x - \sqrt{\frac{p}{2-p}},$$
$$xM_n(x) = M_{n+1}(x) + \frac{2n+p}{\sqrt{p(2-p)}} M_n(x) + n^2 M_{n-1}(x), \qquad n \ge 1,$$

form the orthogonal polynomials with respect to the probability measure

$$\widetilde{\nu}_p = \sum_{k=0}^{\infty} \frac{2(1-p)}{2-p} \left(\frac{p}{2-p}\right)^k \delta_{\frac{2(1-p)k}{\sqrt{p(2-p)}}},$$

that is,

$$\int_{\mathbf{R}} x^m \widetilde{\nu}_p(dx) = \left\langle \Phi_0, \left(B^+ + B^- + \frac{2N+p}{\sqrt{p(2-p)}}\right)^m \Phi_0 \right\rangle_{\Gamma},$$

where $\Gamma = (\Gamma, \{(n!)^2\}, B^+, B^-)$. Thus, by translation of $\widetilde{\nu}_p$ we obtain

$$\nu_p = \sum_{k=0}^{\infty} \frac{2(1-p)}{2-p} \left(\frac{p}{2-p}\right)^k \delta_{\frac{-p+2(1-p)k}{\sqrt{p(2-p)}}}, \qquad 0 < p < 1.$$

4.7 Bose–Mesner Algebra and Weak Quantum Decomposition

Let $\mathcal{G} = (V, E)$ be a general distance-regular graph for which (A4) is not necessarily fulfilled. As usual let $\mathcal{A}$ be the unital $*$-algebra generated by the adjacency matrix A. This is called the *Bose-Mesner algebra* of $\mathcal{G}$. It is known (see e.g., Bannai–Ito[4]) that the adjacency matrices $1 = A_0, A = A_1, \cdots, A_d$ are linearly independent and satisfy the relation:

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k. \tag{22}$$

Hence $\mathcal{A}$ is a vector space with linear basis $1 = A_0, A = A_1, \ldots, A_d$.

The Bose–Mesner algebra $\mathcal{A}$ becomes an algebraic probability space equipped with the normalized trace Tr. The GNS-representation of $\mathcal{A}$ is realized on the Hilbert space $\mathcal{H}(\mathcal{A})$ obtained from $\mathcal{A}$ equipped with an inner product:

$$\langle a, b\rangle_{\mathcal{A}} = \mathrm{Tr}\,(a^* b), \qquad a, b \in \mathcal{A}.$$

We see from an obvious relation:

$$\langle A_i, A_j\rangle_{\mathcal{A}} = \delta_{ij}\kappa_i = \langle v_i, v_j\rangle, \qquad v_i = \sum_{x \in V_i} \delta_x \in \ell^2(V),$$

that the correspondence $A_i \leftrightarrow v_i$ yields a unitary isomorphism between $\mathcal{H}(\mathcal{A})$ and $\Gamma(\mathcal{G}) \subset \ell^2(V)$. Moreover, since

$$A_i v_j = \sum_{h=0}^{d} p_{ij}^h v_h,$$

which is easily verified by definition, we see from (22) that the above unitary isomorphism intertwines the action of the Bose–Mesner algebra $\mathcal{A}$. In particular,

Lemma 4.6 *The action of the adjacency matrix A on $\Gamma(\mathcal{G})$ is unitarily equivalent to that induced from the GNS-representation of $(\mathcal{A}, \mathrm{Tr})$ and*

$$\langle \Phi_0, A^m \Phi_0 \rangle = \mathrm{Tr}(A^m), \qquad m = 0, 1, 2, \dots .$$

It follows from (22) with a simple triangle inequality that

$$AA_n = p_{1n}^{n-1} A_{n-1} + p_{1n}^{n} A_n + p_{1n}^{n+1} A_{n+1}, \qquad n = 0, 1, \dots, d.$$

We then define

$$A^{\pm} A_n = p_{1n}^{n\pm 1} A_{n\pm 1} + \frac{p_{1n}^{n}}{2} A_n, \qquad n = 0, 1, \dots, d.$$

Obviously $(A^+)^* = A^-$ and $A = A^+ + A^-$, which is referred to as *weak quantum decomposition*. This is equivalent to adopt (15) ignoring the orientation of the graph. Thus the weak quantum decomposition does not reflect an orientation of the graph though the actions of the quantum components are well defined on $\Gamma(\mathcal{G})$.

Theorem 4.1 remains valid if the quantum decomposition is replaced with the weak one. In fact, the proof is almost similar.

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