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ADMISSIBLE WHITE NOISE OPERATORS AND THEIR QUANTUM WHITE NOISE DERIVATIVES

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An operator on a Fock space is considered as a non-linear and non-commutative function of annihilation and creation operators at points $\{a_t, a_t^* \,;\, t \in T\}$. The derivatives with respect to a_t and a_t^*, called respectively the annihilation- and creation-derivatives, are formulated within the framework of quantum white noise theory. We prove the differentiability of an admissible white noise operator and give explicit formulae for the derivatives in terms of integral kernel operators. The qwn-derivative is a non-commutative counterpart of the Gross derivative in stochastic analysis.

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1. Introduction

In this paper we focus on an operator on the (Boson) Fock space $\Gamma(H)$ with $H = L^2(T, \nu)$, where T is a topological space equipped with a σ-finite Borel measure ν. A fundamental role is played by the annihilation and

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creation operators at a point, denoted by a_t and a_t^*, in many questions arising from quantum stochastic analysis, infinite dimensional harmonic analysis, quantum field theory and so on. The pair $\{a_t, a_t^* \,;\, t \in T\}$ is sometimes called a *quantum white noise field* on T. However, they are not well defined only within the framework of the Fock space. In literatures we find two formulations. One is to consider a_t and a_t^* as unbounded-operator-valued distribution by smearing t; the other is to formulate them as continuous operators on a certain Gelfand triple $(E) \subset \Gamma(H) \subset (E)^*$ without smearing t. The second approach is along with the classical Hida calculus and has been considerably studied under the name of *quantum white noise theory* [4, 8, 17].

We recall that every white noise operator admits a Fock expansion, i.e., $\Xi \in \mathcal{L}((E), (E)^*)$ is decomposed into an infinite sum of integral kernel operators:

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{1.1}$$

where

$$\Xi_{l,m}(\kappa_{l,m}) = \int_{T^{l+m}} \kappa_{l,m}(s_1, \cdots, s_l, t_1, \cdots, t_m) a_{s_1}^* \dots a_{s_l}^* \times a_{t_1} \cdots a_{t_m} \nu(ds_1) \dots \nu(ds_l) \nu(dt_1) \dots \nu(dt_m), \tag{1.2}$$

and $\kappa_{l,m}$ is a kernel distribution. One may accept (1.2) as a "polynomial" in a_t and a_t^*, hence (1.1) as a function of them: $\Xi = \Xi(a_t, a_t \,;\, t \in T)$. Thus we are naturally led to a kind of functional derivatives:

$$D_t^- \Xi = \frac{\delta \Xi}{\delta a_t}, \qquad D_t^+ \Xi = \frac{\delta \Xi}{\delta a_t^*}. \tag{1.3}$$

These motivated us to study the *annihilation-* and *creation-derivatives.*

The main purpose of this paper is to formulate the annihilation- and creation-derivatives (together called *qwn-derivatives*), and to study qwn-differentiability of a white noise operator. In fact, a general white noise operator is not qwn-differentiable and there is difficulty in formulating (1.3) in a direct manner. Our strategy is to establish an operator version of Gross derivative (the derivative along H) of an admissible white noise function (see e.g., [1]).

The paper is organized as follows. In Section 2 we prepare some notation in white noise theory. We introduce in Section 3 the spaces of admissible (test and generalized) functions as

$$(E) \subset \mathcal{G} \subset \Gamma(H) \subset \mathcal{G}^* \subset (E)^*$$

and discuss an admissible white noise operator, i.e., a continuous operator from $\mathcal{G}$ into $\mathcal{G}^*$. In Section 4 we give the definition of the qwn-derivatives $D_\zeta^{\pm}\Xi$ for a white noise operator Ξ. Then we prove the qwn-differentiability of an admissible white noise operator and obtain formulae for the annihilation- and creation derivatives (Theorem 4.4). This result follows from the qwn-differentiability of an admissible integral kernel operator (Theorem 4.1) and the Fock expansion of an admissible white noise operator (Theorem 4.3).

The notion equivalent to an admissible white noise function was first introduced by Lindsay–Maassen [13] and has been studied in many literatures for different purposes, e.g., Aase–Øksendal–Privault–Ubøe [1], Belavkin [2], Benth–Potthoff [3], Grothaus–Kondratiev–Streit [5], Ji [7], Lindsay–Parthasarathy [14]. We hope that our new idea of qwn-derivative opens a new direction in the study of Fock space operators. Further study and application will be discussed in [10].

2. White Noise Theory

2.1. *Underlying Gelfand Triple*

Let T be a topological space equipped with a σ-finite Borel measure ν. Let $H = L^2(T,\nu)$ be the (complex) Hilbert space of L^2-functions and the norm is denoted by $|\cdot|_0$. Let A be a selfadjoint operator (densely defined) in H satisfying the conditions (A1)–(A4) below.

(A1) $\inf \mathrm{Spec}\,(A) > 1$ and A^{-1} is of Hilbert-Schmidt type.

Then there exist a sequence

$$1 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots, \qquad \| A^{-1} \|_{\mathrm{HS}}^2 = \sum_{j=0}^{\infty} \lambda_j^{-2} < \infty,$$

and an orthonormal basis $\{e_j\}_{j=0}^{\infty}$ of H such that $Ae_j = \lambda_j e_j$. For $p \in \mathbf{R}$ we define

$$|\xi|_p^2 = |A^p\xi|_0^2 = \sum_{j=0}^{\infty} \lambda_j^{2p} |\langle \xi, e_j \rangle|^2, \qquad \xi \in H.$$

Now let $p \ge 0$. We put $E_p = \{\xi \in H\,;\, |\xi|_p < \infty\}$ and define E_{-p} to be the completion of H with respect to $|\cdot|_{-p}$. Thus we obtain a chain of Hilbert spaces $\{E_p\,;\, p \in \mathbf{R}\}$ and consider their limit spaces:

$$\mathcal{S}_A(T) = E = \operatorname*{proj\,lim}_{p\to\infty} E_p, \qquad \mathcal{S}_A^*(T) = E^* = \operatorname*{ind\,lim}_{p\to\infty} E_{-p}.$$

These are mutually dual spaces. Note also that $\mathcal{S}_A(T)$ becomes a countably Hilbert nuclear space. Identifying H with its dual space, we obtain a complex Gelfand triple:

$$E = \mathcal{S}_A(T) \subset H = L^2(T,\nu) \subset E^* = \mathcal{S}_A^*(T).$$

As usual, we understand that $\mathcal{S}_A(T)$ and $\mathcal{S}_A^*(T)$ are spaces of test functions and generalized functions (or distributions) on T, respectively.

For white noise theory $\mathcal{S}_A^*(T)$ must contain delta functions. But this is not automatic and we need further assumptions:

(A2) For each function $\xi \in \mathcal{S}_A(T)$ there exists a unique continuous function $\widetilde{\xi}$ on T such that $\xi(t) = \widetilde{\xi}(t)$ for ν-a.e. $t \in T$.

Thus $\mathcal{S}_A(T)$ is regarded as a space of continuous functions on T and we do not use the exclusive symbol $\widetilde{\xi}$. The uniqueness in (A2) is equivalent to that any continuous function on T which is zero ν-a.e. is identically zero.

(A3) For each $t \in T$ the evaluation map $\delta_t : \xi \mapsto \xi(t)$, $\xi \in \mathcal{S}_A(T)$, is a continuous linear functional, i.e., $\delta_t \in \mathcal{S}_A^*(T)$.

(A4) The map $t \mapsto \delta_t \in \mathcal{S}_A^*(T)$, $t \in T$, is continuous with respect to the strong dual topology of $\mathcal{S}_A^*(T)$.

See [17] for more discussion on these assumptions.

The canonical $\mathbf{C}$-bilinear form on $E^* \times E$ is denoted by $\langle \cdot, \cdot \rangle$. In other words, we set

$$\langle x, \xi \rangle = \sum_{j=0}^{\infty} \alpha_j \beta_j \quad \text{for} \quad x = \sum_{j=0}^{\infty} \alpha_j e_j, \quad \xi = \sum_{j=0}^{\infty} \beta_j e_j, \quad \alpha_j, \beta_j \in \mathbf{C}.$$

We also write $\langle \cdot, \cdot \rangle$ for the canonical $\mathbf{C}$-bilinear form on H. Let J be the conjugate operator defined by

$$J\xi = \sum_{j=0}^{\infty} \bar{\alpha}_j e_j \qquad \text{for} \quad \xi = \sum_{j=0}^{\infty} \alpha_j e_j \in H.$$

It then follows that

$$|\xi|_0^2 = \sum_{j=0}^{\infty} |\alpha_j|^2 = \int_T |\xi(t)|^2 \, \nu(dt) = \langle J\xi, \xi \rangle .$$

The real parts of E, H, E^* are subspaces invariant under the action of J and are denoted by $E_{\mathbf{R}}$, $H_{\mathbf{R}}$ and $E_{\mathbf{R}}^*$, respectively. Then we obtain a real Gelfand triple:

$$E_{\mathbf{R}} \subset H_{\mathbf{R}} \subset E_{\mathbf{R}}^*. \tag{2.1}$$

(These are real vector spaces but not necessarily spaces of $\mathbf{R}$-valued functions.)

Remark 2.1. A prototype of our consideration is the case where $T = \mathbf{R}$ with Lebesgue measure $\nu(dt) = dt$ and

$$A = 1 + t^2 - \frac{d^2}{dt^2} = \left(t + \frac{d}{dt}\right)^* \left(t + \frac{d}{dt}\right) + 2.$$

In this case $\mathcal{S}_A(T)$ coincides with the space of rapidly decreasing functions, which is commonly denoted by $\mathcal{S}(\mathbf{R})$. Recall also that

$$e_j(t) = \left(\sqrt{\pi}\, 2^j j!\right)^{-1/2} H_j(t) e^{-t^2/2}, \qquad j = 0, 1, 2, \cdots,$$

where H_j is the Hermite polynomial of degree j, constitute an orthonormal basis of $L^2(\mathbf{R})$ and $Ae_j = (2j+2)e_j$. This prototype is suitable for stochastic processes, where $\mathbf{R}$ plays a role of the time axis. Our general framework allows to take T to be a manifold (space-time), a discrete space or even a finite set.

2.2. *Hida–Kubo–Takenaka Space*

Let E_p be the Hilbert space defined in §2.1, where $p \in \mathbf{R}$. We consider the (Boson) Fock space:

$$\Gamma(E_p) = \left\{ \phi = (f_n)_{n=0}^{\infty}\,;\, f_n \in E_p^{\widehat{\otimes} n},\ \|\phi\|_p^2 = \sum_{n=0}^{\infty} n!\, |f_n|_p^2 < \infty \right\},$$

which is essentially a direct sum of symmetric tensor powers of E_p and the weight factor $n!$ is for convention. Having obtained a chain of Fock spaces $\{\Gamma(E_p)\,;\, p \in \mathbf{R}\}$, we set

$$(E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p), \qquad (E)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p}).$$

Then we obtain a complex Gelfand triple:

$$(E) \subset \Gamma(H) \subset (E)^*, \tag{2.2}$$

which is referred to as the *Hida–Kubo–Takenaka space* [11].

By definition the topology of (E) is defined by the norms

$$\|\phi\|_p^2 = \sum_{n=0}^{\infty} n!\, |f_n|_p^2, \qquad \phi = (f_n), \qquad p \in \mathbf{R}.$$

On the other hand, for each $\Phi \in (E)^*$ there exists $p \geq 0$ such that $\Phi \in \Gamma(E_{-p})$. In this case, we have

$$\| \Phi \|_{-p}^2 \equiv \sum_{n=0}^{\infty} n! \, | F_n |_{-p}^2 < \infty, \qquad \Phi = (F_n).$$

The canonical **C**-bilinear form on $(E)^* \times (E)$ takes the form:

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \qquad \Phi = (F_n) \in (E)^*, \quad \phi = (f_n) \in (E).$$

Here we recall two important elements of $(E)^*$.

(a) *White noise.* By assumption (A3),

$$W_t = (0, \delta_t, 0, \dots), \qquad t \in T,$$

belongs to $(E)^*$ and is called a *white noise.* According as T represents time or space, the family $\{W_t \,;\, t \in T\} \subset (E)^*$ is called the *white noise process* or *white noise field* on T.

(b) *Exponential vector:* For $x \in E^*$ an *exponential vector* (or a *coherent vector*) is defined by

$$\phi_x = \left(1, x, \frac{x^{\otimes 2}}{2!}, \dots, \frac{x^{\otimes n}}{n!}, \dots \right).$$

Obviously, $\phi_x \in (E)^*$. Moreover, ϕ_ξ belongs to (E) (resp. $\Gamma(E_p)$) if and only if ξ belongs to E (resp. E_p). In particular, ϕ_0 is called the *vacuum vector.*

2.3. *White Noise Operators*

In general, a continuous operator from (E) into $(E)^*$ is called a *white noise operator.* The space of all white noise operators is denoted by $\mathcal{L}((E), (E)^*)$ and is equipped with the bounded convergence topology. It is noted that $\mathcal{L}((E), (E))$ is a subspace of $\mathcal{L}((E), (E)^*)$.

For each $t \in \mathbf{R}$ the *annihilation operator* a_t is uniquely specified by the action on exponential vectors as follows:

$$a_t \phi_\xi = \xi(t) \phi_\xi, \qquad \xi \in E.$$

It is well known that $a_t \in \mathcal{L}((E), (E))$. The *creation operator* is by definition the adjoint $a_t^* \in \mathcal{L}((E)^*, (E)^*)$. We see from (2.2) that the composition $a_{s_1}^* \dots a_{s_l}^* a_{t_1} \dots a_{t_m}$ is well defined and belongs to $\mathcal{L}((E), (E)^*)$.

Let $l, m \ge 0$ be integers and $K_{l,m} \in \mathcal{L}(E^{\otimes m}, (E^{\otimes l})^*)$. We define

$$K_{l,m}^{\circ} = S_{n+l} \circ (I_n \otimes K_{l,m}), \tag{2.3}$$

where $I_n : E^{\otimes n} \to E^{\otimes n}$ is the identity and $S_{n+l} : E^{\otimes(n+l)} \to E^{\widehat{\otimes}(n+l)}$ the symmetrizing operator. An *integral kernel operator* $\Xi_{l,m}(K_{l,m})$ is defined by the action $\phi = (f_n) \mapsto (g_n)$ given by

$$g_n = 0, \quad 0 \le n < l; \qquad g_{n+l} = \frac{(n+m)!}{n!} K_{l,m}^{\circ} f_{n+m}, \quad n \ge 0.$$

It is known that $\Xi_{l,m}(K_{l,m}) \in \mathcal{L}((E), (E)^*)$.

Remark 2.2. We have $\mathcal{L}(E^{\otimes m}, (E^{\otimes l})^*) \cong (E^{\otimes(l+m)})^*$ by the kernel theorem. For $K_{l,m} \in \mathcal{L}(E^{\otimes m}, (E^{\otimes l})^*)$ let $\kappa_{l,m}$ be the corresponding element, i.e.,

$$\left\langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle = \left\langle K_{l,m}\xi^{\otimes m}, \eta^{\otimes l} \right\rangle, \qquad \xi, \eta \in E.$$

We then easily understand that (1.2) in Introduction is a descriptive expression for $\Xi_{l,m}(K_{l,m})$. In many literatures the notation $\Xi_{l,m}(\kappa_{l,m})$ is used for $\Xi_{l,m}(K_{l,m})$, see e.g., [17].

For a white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ the *symbol* and the *Wick symbol* are defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle, \quad \widetilde{\Xi}(\xi, \eta) = \langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle \, e^{-\langle \xi, \eta \rangle}, \quad \xi, \eta \in E,$$

respectively. A white noise operator is uniquely specified by the symbol or by the Wick symbol. For an integral kernel operator we have

$$\Xi_{l,m}(K_{l,m})\widehat{\ }(\xi, \eta) = \langle K_{l,m}\xi^{\otimes m}, \eta^{\otimes l} \rangle e^{\langle \xi, \eta \rangle},$$
$$\Xi_{l,m}(K_{l,m})\widetilde{\ }(\xi, \eta) = \langle K_{l,m}\xi^{\otimes m}, \eta^{\otimes l} \rangle.$$

2.4. *Gaussian Realization*

Based on the real Gelfand triple (2.1) we define a Gaussian measure μ by its characteristic function:

$$\exp\left\{-\frac{1}{2} \, | \xi |_0^2 \right\} = \int_{E_{\mathbf{R}}^*} e^{i\langle x, \xi \rangle} \mu(dx), \qquad \xi \in E_{\mathbf{R}}.$$

The celebrated Wiener–Itô decomposition theorem says that $L^2(E_{\mathbf{R}}^*, \mu)$ is unitarily isomorphic to $\Gamma(H)$ through the correspondence:

$$\phi_\xi(x) = e^{\langle x, \xi \rangle - \langle \xi, \xi \rangle / 2} \quad \leftrightarrow \quad \phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \cdots\right), \qquad \xi \in E.$$

Taking (2.2) into account, we regard (E) as a subspace of $L^2(E^*, \mu)$. In this sense an element of (E) is called a *test white noise function* and, accordingly, an element of $(E)^*$ is called a *generalized white noise function*.

3. Admissible White Noise Operators

3.1. *Admissible White Noise Functions*

For $p \in \mathbf{R}$ we set

$$\| \phi \|_p^2 = \sum_{n=0}^{\infty} n! e^{2pn} |f_n|_0^2, \qquad \phi = (f_n) \in \Gamma(H).$$

For $p \geq 0$ we define $\mathcal{G}_p = \{\phi = (f_n) \in \Gamma(H) \,;\, \| \phi \|_p < \infty\}$ and $\mathcal{G}_{-p}$ to be the completion of $\Gamma(H)$ with respect to $\| \cdot \|_{-p}$. Then $\{\mathcal{G}_p \,;\, p \in \mathbf{R}\}$ form a chain of Hilbert spaces satisfying

$$\mathcal{G} = \operatorname*{proj\,lim}_{p\to\infty} \mathcal{G}_p \subset \mathcal{G}_p \subset \mathcal{G}_0 = \Gamma(H) \subset \mathcal{G}_{-p} \subset \mathcal{G}^* = \operatorname*{ind\,lim}_{p\to\infty} \mathcal{G}_{-p}.$$

Note that $\mathcal{G}$ is a countable Hilbert space but not necessarily a nuclear space ($\mathcal{G}$ is nuclear if and only if H is finite dimensional), and that $\mathcal{G}$ and $\mathcal{G}^*$ are mutually dual spaces.

Lemma 3.1. *For any pair p, q satisfying $0 \leq p \leq -q \log \| A^{-1} \|_{\mathrm{OP}}$ we have*

$$\| \phi \|_p \leq \| \phi \|_q \quad \textit{and} \quad \| \Phi \|_{-q} \leq \| \Phi \|_{-p},$$

where $\phi \in \Gamma(H)$ and $\Phi \in (E)^*$. (The norms can be ∞, which is understood in a usual way.)

The proof is immediate from the definition of the norms. Then, we have

$$(E) \subset \mathcal{G} \subset \Gamma(H) \subset \mathcal{G}^* \subset (E)^*. \tag{3.1}$$

An element in $\mathcal{G}$ (resp. $\mathcal{G}^*$) is called an *admissible* test (resp. generalized) function. The canonical $\mathbf{C}$-bilinear form on $\mathcal{G}^* \times \mathcal{G}$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$ too.

3.2. *Admissible White Noise Operators*

We note from the inclusion relations (3.1) that $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ is regarded as a subspace of $\mathcal{L}((E), (E)^*)$. A white noise operator belonging to the former space is called *admissible*. For an admissible operator we can find a pair of real numbers $p \geq q$ such that $\Xi \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$.

Proposition 3.1. *Let $K_{l,m} \in \mathcal{L}(E^{\otimes m}, (E^{\otimes l})^*)$. Then the integral kernel operator $\Xi_{l,m}(K_{l,m})$ is admissible if and only if $K_{l,m} \in \mathcal{L}(H^{\otimes m}, H^{\otimes l})$. In that case, for an arbitrary $q \in \mathbf{R}$ and $r > 0$ we have*

$$\| \Xi_{l,m}(K_{l,m})\phi \|_q \le C \, \| K_{l,m} \|_{\mathrm{OP}} \, \| \phi \|_{q+r},$$

where

$$C = e^{r/2+ql-(q+r)m}(l^l m^m)^{1/2} \left(\frac{e^{r/2}}{er} \right)^{(l+m)/2}$$

and $\| K_{l,m} \|_{\mathrm{OP}}$ stands for the Hilbert space operator norm. In particular, $\Xi_{l,m}(K_{l,m}) \in \mathcal{L}(\mathcal{G}_{q+r}, \mathcal{G}_q)$.

Proof. If $\Xi_{l,m}(K_{l,m})$ is admissible, there exist a pair of real numbers $p \ge q$ and a constant $C \ge 0$ such that

$$\| \Xi_{l,m}(K_{l,m})\phi \|_q \le C \, \| \phi \|_p, \qquad \phi \in \mathcal{G}_p.$$

Taking a particular $\phi = (0, \dots, 0, f_m, 0, \dots)$, one obtains easily

$$| K_{l,m} f_m |_0 \le \frac{C e^{pm-ql}}{\sqrt{l!m!}} \, | f_m |_0,$$

which shows that $K_{l,m} \in \mathcal{L}(H^{\otimes m}, H^{\otimes l})$.

Conversely, suppose $K_{l,m} \in \mathcal{L}(H^{\otimes m}, H^{\otimes l})$. For $q \in \mathbf{R}$ and $\phi = (f_n) \in \mathcal{G}$ we have by definition

$$\| \Xi_{l,m}(K_{l,m})\phi \|_q^2 = \sum_{n=0}^{\infty} (n+l)! e^{2q(n+l)} \left(\frac{(n+m)!}{n!} \right)^2 | K_{l,m}^{\circ} f_{n+m} |_0^2. \tag{3.2}$$

Note from (2.3) that $\| K_{l,m}^{\circ} \|_{\mathrm{OP}} \le \| K_{l,m} \|_{\mathrm{OP}}$. Then for any $r > 0$, (3.2) is bounded by

$$\begin{aligned} &\le \| K_{l,m} \|_{\mathrm{OP}}^2 \sum_{n=0}^{\infty} e^{2(q+r)(n+m)} (n+m)! \, | f_{n+m} |_0^2 \\ &\qquad \times \left\{ \sup_{n \ge 0} \frac{(n+l)!}{n!} \frac{(n+m)!}{n!} e^{-2rn} \right\} e^{2ql-2(q+r)m}. \end{aligned} \tag{3.3}$$

By an elementary calculus (see e.g., [17: Section 4.1]) we have

$$\sup_{n \ge 0} \frac{(n+l)!}{n!} \frac{(n+m)!}{n!} e^{-2rn} \le e^r l^l m^m \left(\frac{e^{r/2}}{er} \right)^{l+m}. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain the desired estimate. □

3.3. *Admissible White Noise Operators with Supports*

Let $U \subset T$ be a Borel set with $\nu(U) > 0$. Then, starting from the Hilbert space $L^2(U, \nu)$ we obtain the spaces of admissible functions which are denoted by

$$\mathcal{G}(U) \subset \mathcal{G}_p(U) \subset \Gamma(L^2(U,\nu)) = \mathcal{G}_0(U) \subset \mathcal{G}_{-p}(U) \subset \mathcal{G}^*(U).$$

We identify $\mathcal{G}_p(U)$ with a closed subspace of $\mathcal{G}_p = \mathcal{G}_p(T)$ through the natural inclusion $L^2(U,\nu) \hookrightarrow L^2(T,\nu)$. An element in $\mathcal{G}_p(U)$ is called an *admissible white noise function supported by* U.

A description of the inclusion $\mathcal{G}_p(U) \hookrightarrow \mathcal{G}_p(T)$ is given in terms of tensor product decomposition. We first recall the following fact whose proof is standard and is omitted.

Lemma 3.2. *Let* $T = U_1 \cup U_2 \cup \cdots \cup U_m$ *be a partition into a disjoint union of Borel subsets up to null sets. Then, the correspondence*

$$\phi_\xi \leftrightarrow \phi_{\xi \restriction U_1} \otimes \cdots \otimes \phi_{\xi \restriction U_m}, \qquad \xi \in L^2(T,\nu),$$

gives rise to a unitary isomorphism

$$\mathcal{G}_p(T) \cong \mathcal{G}_p(U_1) \otimes \cdots \otimes \mathcal{G}_p(U_m)$$

for all $p \in \mathbf{R}$.

Now let $T = U \cup V$ be a partition, where $\nu(U) > 0$ and $\nu(V) > 0$ without loss of generality. It follows from Lemma 3.2 that

$$\mathcal{G}_p(T) \cong \mathcal{G}_p(U) \otimes \mathcal{G}_p(V) \tag{3.5}$$

and $\Phi \mapsto \Phi \otimes \phi_{0 \restriction V}$ gives the canonical inclusion $\mathcal{G}_p(U) \hookrightarrow \mathcal{G}_p(T)$.

With each continuous operator $\Xi \in \mathcal{L}(\mathcal{G}_p(U), \mathcal{G}_q(U))$, where we assume $p \geq q$ without loss of generality, we associate an admissible white noise operator $\Xi \otimes I$ according to the factorization (3.5), where I is the identity operator on $\mathcal{G}_p(V)$. Summing up, for a Borel set $U \subset T$ and a pair of real numbers $p \geq q$ we have inclusions:

$$\mathcal{L}(\mathcal{G}_p(U), \mathcal{G}_q(U)) \subset \mathcal{L}(\mathcal{G}_p(T), \mathcal{G}_q(T)) \subset \mathcal{L}((E),(E)^*).$$

An operator in $\mathcal{L}(\mathcal{G}_p(U), \mathcal{G}_q(U))$ is called an *admissible white noise operator supported by* U. Whenever no confusion occurs we use the same symbol Ξ for $\Xi \otimes I$.

The concept of an admissible white noise operator with support is useful in the study of conditional expectation and quantum martingale, see [7].

4. Quantum White Noise Derivatives

4.1. *Translation Operator*

Since each $\phi \in (E)$ is a continuous function on $E^*_{\mathbf{R}}$, for any $\zeta \in E^*_{\mathbf{R}}$ the translation $T_\zeta \phi$ defined naturally by

$$T_\zeta \phi(x) = \phi(x+\zeta), \qquad x \in E^*_{\mathbf{R}}. \tag{4.1}$$

It is known [17] that $T_\zeta \phi \in (E)$ and $T_\zeta \in \mathcal{L}((E),(E))$. However, (4.1) is not applicable to a generalized function. By the Wiener–Itô–Segal isomorphism, for $\phi = (f_n) \in (E)$ we have

$$T_\zeta \phi = \left(\sum_{m=0}^{\infty} \frac{(n+m)!}{n!m!} \zeta^{\otimes m} \widehat{\otimes}_m f_{n+m} \right)_{n=0}^{\infty}, \tag{4.2}$$

where $\widehat{\otimes}_m$ is the right m-contraction of symmetric tensor products. It is then natural to define the translation operator by extending the right hand side of (4.2). Namely, given $\zeta \in E^*$ (hereafter we allow a complex ζ) and $\Phi = (F_n) \in (E)^*$, we define $T_\zeta \Phi$ by the right hand side of (4.2) with replacing f_{n+m} by F_{n+m}, whenever well defined as an element in $(E)^*$.

Proposition 4.1. *Let $\zeta \in H$ and $\Phi \in \mathcal{G}_p$ with some $p \in \mathbf{R}$. Then, for any $q < p - \log\sqrt{2}$ it holds that $T_\zeta \Phi \in \mathcal{G}_q$ and $T_\zeta \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$.*

Proof. Let $p, q \in \mathbf{R}$. By definition,

$$\| T_\zeta \Phi \|_q^2 = \sum_{n=0}^{\infty} n! e^{2qn} \left| \sum_{m=0}^{\infty} \frac{(n+m)!}{n!m!} \zeta^{\otimes m} \widehat{\otimes}_m F_{n+m} \right|_0^2 .$$

Applying the Schwartz inequality, we have

$$\begin{aligned}
\| T_\zeta \Phi \|_q^2 &\le \sum_{n=0}^{\infty} n! e^{2qn} \left(\sum_{m=0}^{\infty} \frac{(n+m)!}{(n!m!)^2} e^{-2p(n+m)} |\zeta|_0^{2m} \right) \\
&\qquad \times \left(\sum_{m=0}^{\infty} (n+m)! e^{2p(n+m)} |F_{n+m}|_0^2 \right) \\
&\le \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} n! e^{2qn} \frac{2^{n+m}}{n!m!} e^{-2p(n+m)} |\zeta|_0^{2m} \right) \| \Phi \|_p^2 \\
&= \sum_{n=0}^{\infty} \left(2e^{-2(p-q)}\right)^n \sum_{m=0}^{\infty} \frac{1}{m!} \left(2e^{-2p} |\zeta|_0^2\right)^m \| \Phi \|_p^2 .
\end{aligned}$$

Given $p \in \mathbf{R}$, we choose $q \in \mathbf{R}$ such that $2e^{-2(p-q)} < 1$, i.e., $q < p - \log\sqrt{2}$. Then we come to

$$\| T_\zeta \Phi \|_q \le \frac{\exp\left(e^{-2p}|\zeta|_0^2\right)}{\sqrt{1-2e^{-2(p-q)}}} \| \Phi \|_p ,$$

which means that $T_\zeta \Phi \in \mathcal{G}_q$ and $T_\zeta \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$. □

4.2. *Gross Derivative*

Modelled after abstract Wiener space theory, we say that $\Phi \in (E)^*$ is *Gross differentiable* if for any $\zeta \in H$ the translation $T_{\epsilon\zeta}\Phi$ is defined for small $|\epsilon| < \epsilon_0$ and if

$$D_\zeta \Phi \equiv \lim_{\epsilon \to 0} \frac{T_{\epsilon\zeta}\Phi - \Phi}{\epsilon} \tag{4.3}$$

converges in $(E)^*$ with respect to the weak topology. $D_\zeta\Phi$ is called the *Gross derivative* of Φ in the direction ζ.

Proposition 4.2. *Every $\Phi = (F_n) \in \mathcal{G}^*$ is Gross differentiable and $D_\zeta \Phi = ((n+1)\zeta \widehat{\otimes}_1 F_{n+1})_{n=0}^\infty$. Moreover, D_ζ is a continuous linear operator on $\mathcal{G}^*$ equipped with the strong dual topology.*

Proof. Let $\zeta \in H$, $\Phi = (F_n) \in \mathcal{G}_p$ and set $\Psi = \left((n+1)\zeta\widehat{\otimes}_1 F_{n+1}\right)_{n=0}^\infty$. We first note that $\Psi \in \mathcal{G}_q$ for any $q < p$. In fact, by direct computation we obtain

$$\| \Psi \|_q \le e^{-p} C_{p-q} \, |\zeta|_0 \, \| \Phi \|_p , \qquad C_{p-q} = \sup_{n \ge 0} (n+1) e^{-(p-q)n}. \tag{4.4}$$

Next we show that $\Psi = D_\zeta \Phi$. It follows from (4.2) that

$$\frac{T_{\epsilon\zeta}\Phi - \Phi}{\epsilon} - \Psi = \left(\sum_{m=2}^{\infty} \frac{(n+m)!}{n!m!} \epsilon^{m-1} \zeta^{\otimes m} \widehat{\otimes}_m F_{n+m} \right)_{n=0}^{\infty}.$$

Applying a similar estimate as in the proof of Proposition 4.1, we obtain

$$\left\| \frac{T_{\epsilon\zeta}\Phi - \Phi}{\epsilon} - \Psi \right\|_q \le \frac{2\epsilon e^{-2p} \, |\zeta|_0^2 \exp(\epsilon^2 e^{-2p} \, |\zeta|_0^2)}{\sqrt{1-2e^{-2(p-q)}}} \| \Phi \|_p ,$$

where $q < p - \log\sqrt{2}$. Thus we have shown that (4.3) converges in norm and the desired assertion follows. The last assertion follows from (4.4) and general theory of locally convex spaces. □

A Gross differentiable function $\Phi \in (E)^*$ is called *pointwisely Gross differentiable* if there exists a weakly measurable function $t \mapsto \Psi_t \in (E)^*$

such that the function $t \mapsto \| \Psi_t \|_p$ belongs to $H = L^2(T, \nu)$ for some $p \in \mathbf{R}$ and

$$\langle\langle D_\zeta \Phi, \phi \rangle\rangle = \int_T \zeta(t) \langle\langle \Psi_t, \phi \rangle\rangle \, \nu(dt), \qquad \zeta \in H, \quad \phi \in (E). \tag{4.5}$$

In that case we write $\Psi_t = D_t \Phi$. Note that $D_t \Phi$ is determined for almost all $t \in T$. The pointwise Gross derivative plays a basic role in stochastic analysis and similar derivatives have been introduced by many authors in different contexts, see e.g., [6,12,15,16].

Proposition 4.3. [1: Lemma 3.10] Every $\Phi \in \mathcal{G}^*$ is pointwisely Gross differentiable and $D_t \Phi = ((n+1) F_{n+1}(t, \cdot))_{n=0}^\infty$ for $\Phi = (F_n)$. Moreover, if $\Phi = (F_n) \in \mathcal{G}_p$ and $q < p - \log \sqrt{2}$, then $D_t \Phi \in \mathcal{G}_q$ for ν-a.e. $t \in T$.

Proof. Our proof is different from the one in [1]. Consider a function $t \mapsto \Psi_t = ((n+1) F_{n+1}(t, \cdot))_{n=0}^\infty$, which is defined for almost all $t \in T$ by Fubini theorem. We note that

$$\begin{aligned}
\int_T \| \Psi_t \|_q^2 \, \nu(dt) &= \int_T \sum_{n=0}^\infty n! e^{2qn} (n+1)^2 |F_{n+1}(t, \cdot)|_0^2 \, \nu(dt) \\
&= \sum_{n=0}^\infty (n+1) e^{-2p} e^{-2(p-q)n} (n+1)! e^{2p(n+1)} \, | F_{n+1} |_0^2 \\
&\le e^{-2p} C_{p-q}^2 \| \Phi \|_p^2 < \infty.
\end{aligned} \tag{4.6}$$

Then $\Psi_t \in \mathcal{G}_q$ for almost all $t \in T$. Since $\| \Psi_t \|_{q \wedge 0} \le \| \Psi_t \|_{q \wedge 0} \le \| \Psi_t \|_q$ by Lemma 3.1, we see from (4.6) that the function $t \mapsto \| \Psi_t \|_{q \wedge 0}$ belongs to $H = L^2(T, \nu)$. Finally, (4.5) follows from Proposition 4.2 with direct computation. Thus $D_t \Phi = \Psi_t$ for almost all $t \in T$. □

Corollary 4.1. *For $\phi \in (E)$ we have $D_t \phi = a_t \phi$.*

It is shown by norm estimates that $\mathcal{G}^*$ is closed under the Wick product:

$$\Phi \diamond \Psi = \left(\sum_{k=0}^n F_k \widehat{\otimes} G_{n-k} \right)_{n=0}^\infty, \quad \Phi = (F_n), \ \Psi = (G_n).$$

Then the next result is straightforward.

Proposition 4.4. *For $\Phi, \Psi \in \mathcal{G}^*$ we have*

$$\begin{aligned}
D_\zeta (\Phi \diamond \Psi) &= (D_\zeta \Phi) \diamond \Psi + \Phi \diamond (D_\zeta \Psi), && \zeta \in H, \\
D_t (\Phi \diamond \Psi) &= (D_t \Phi) \diamond \Psi + \Phi \diamond (D_t \Psi), && \textit{for almost all } t \in T.
\end{aligned}$$

4.3. *Annihilation- and Creation-Derivatives*

Let $\Xi \in \mathcal{L}((E),(E)^*)$. It is proved that for any $\eta \in E$ there exists $\Phi_\eta \in (E)^*$ uniquely specified by

$$\langle\langle \Phi_\eta, \phi_\xi \rangle\rangle = \langle\langle \Xi^* \phi_\eta, \phi_\xi \rangle\rangle \, \langle\langle \phi_{-\eta}, \phi_\xi \rangle\rangle , \qquad \xi \in E.$$

By using the Wick product we may write

$$\Phi_\eta = (\Xi^* \phi_\eta) \diamond \phi_{-\eta}.$$

Now assume that Φ_η is Gross differentiable for all $\eta \in E$. This assumption is equivalent to that so is $\Xi^*\phi_\eta$, since $\Phi_\eta \diamond \phi_\eta = \Xi^*\phi_\eta$. If, in addition, $\langle\langle D_\zeta \Phi_\eta, \phi_\xi \rangle\rangle$ is the Wick symbol of some operator in $\mathcal{L}((E),(E)^*)$ for any $\zeta \in H$, denoted by $D_\zeta^- \Xi$, i.e.,

$$\langle\langle (D_\zeta^- \Xi)\phi_\xi, \phi_\eta \rangle\rangle e^{-\langle \xi, \eta \rangle} = \langle\langle D_\zeta \Phi_\eta, \phi_\xi \rangle\rangle, \quad \xi, \eta \in E, \tag{4.7}$$

then Ξ is said to be *differentiable in annihilation parts* and $D_\zeta^- \Xi$ is called the *annihilation-derivative* of Ξ with $\zeta \in H$. Similarly, the *creation-derivative* $D_\zeta^+ \Xi \in \mathcal{L}((E),(E)^*)$ is defined by

$$\langle\langle (D_\zeta^+ \Xi)\phi_\xi, \phi_\eta \rangle\rangle e^{-\langle \xi, \eta \rangle} = \langle\langle D_\zeta \Psi_\xi, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in E,$$

where

$$\Psi_\xi = (\Xi \phi_\xi) \diamond \phi_{-\xi}.$$

We say that $\Xi \in \mathcal{L}((E),(E)^*)$ is *qwn-differentiable* if $D_\zeta^\pm \Xi \in \mathcal{L}((E),(E)^*)$ exists for all $\zeta \in H$. The derivatives $D_\zeta^\pm \Xi$ are regarded as non-commutative extension of the Gross derivative.

Let us study the qwn-derivatives of an integral kernel operator. We need notation. For $K_{l+1,m} \in \mathcal{L}(H^{\widehat{\otimes} m}, H^{\widehat{\otimes}(l+1)})$ and $\zeta \in H$ we define $\zeta * K_{l+1,m} \in \mathcal{L}(H^{\widehat{\otimes} m}, H^{\widehat{\otimes} l})$ by

$$\left\langle (\zeta * K_{l+1,m})\xi^{\otimes m}, \eta^{\otimes l} \right\rangle = \left\langle K_{l+1,m}\xi^{\otimes m}, \eta^{\otimes l} \otimes \zeta \right\rangle, \quad \xi, \eta, \zeta \in H.$$

Similarly, for $K_{l,m+1} \in \mathcal{L}(H^{\widehat{\otimes}(m+1)}, H^{\widehat{\otimes} l})$ and $\zeta \in H$ we define $K_{l,m+1} * \zeta \in \mathcal{L}(H^{\widehat{\otimes} m}, H^{\widehat{\otimes} l})$ by

$$\left\langle (K_{l,m+1} * \zeta)\xi^{\otimes m}, \eta^{\otimes l} \right\rangle = \left\langle K_{l,m+1}\xi^{\otimes m} \otimes \zeta, \eta^{\otimes l} \right\rangle.$$

Theorem 4.1. *An admissible integral kernel operator is qwn-differentiable. Moreover, for any $K_{l,m} \in \mathcal{L}(H^{\widehat{\otimes} m}, H^{\widehat{\otimes} l})$ and $\zeta \in H$ we have*

$$D_\zeta^- \Xi_{l,m}(K_{l,m}) = m \Xi_{l,m-1}(K_{l,m} * \zeta), \tag{4.8}$$

$$D_\zeta^+ \Xi_{l,m}(K_{l,m}) = l \Xi_{l-1,m}(\zeta * K_{l,m}). \tag{4.9}$$

Proof. For simplicity we set $\Xi = \Xi_{l,m}(K_{l,m})$. It follows from (4.7) and Proposition 4.4 that $D_\zeta^- \Xi \in \mathcal{L}((E),(E)^*)$ is characterized by

$$\langle\langle (D_\zeta^- \Xi)\phi_\xi, \phi_\eta \rangle\rangle e^{-\langle \xi, \eta\rangle} = \langle\langle (D_\zeta \Xi^* \phi_\eta) \diamond \phi_{-\eta}, \phi_\xi \rangle\rangle + \langle\langle \Xi^* \phi_\eta \diamond (D_\zeta \phi_{-\eta}), \phi_\xi \rangle\rangle . \tag{4.10}$$

The right hand side being equal to

$$\begin{aligned} &\langle\langle D_\zeta \Xi^* \phi_\eta, \phi_\xi \rangle\rangle \, \langle\langle \phi_{-\eta}, \phi_\xi \rangle\rangle + \langle\langle \Xi^* \phi_\eta, \phi_\xi \rangle\rangle \, \langle\langle D_\zeta \phi_{-\eta}, \phi_\xi \rangle\rangle \\ &= e^{-\langle \xi, \eta\rangle} \langle\langle D_\zeta \Xi^* \phi_\eta, \phi_\xi \rangle\rangle - \langle \eta, \zeta \rangle \, e^{-\langle \xi, \eta\rangle} \langle\langle \Xi^* \phi_\eta, \phi_\xi \rangle\rangle , \end{aligned}$$

(4.10) is equivalent to

$$\langle\langle (D_\zeta^- \Xi)\phi_\xi, \phi_\eta \rangle\rangle = \langle\langle D_\zeta \Xi^* \phi_\eta, \phi_\xi \rangle\rangle - \langle \eta, \zeta \rangle \, \langle\langle \Xi \phi_\xi, \phi_\eta \rangle\rangle . \tag{4.11}$$

As is verified by direct computation, $D_\zeta \Xi^* \phi_\eta = (h_n)$ is given by

$$h_{m+n-1} = m(\zeta * K_{l,m}^*)\eta^{\otimes l} \otimes \frac{\eta^{\otimes n}}{n!} + \langle \eta, \zeta \rangle \, (K_{l,m}^* \eta^{\otimes l}) \otimes \frac{\eta^{\otimes (n-1)}}{(n-1)!},$$

where $n = 0, 1, 2, \ldots$ (the second term vanishes for $n = 0$). Then the first term of the right hand side of (4.11) becomes

$$\begin{aligned} \langle\langle D_\zeta \Xi^* \phi_\eta, \phi_\xi \rangle\rangle &= \sum_{n=0}^{\infty} \langle h_{m+n-1}, \xi^{\otimes (m+n-1)} \rangle \\ &= m\langle (\zeta * K_{l,m}^*)\eta^{\otimes l}, \xi^{\otimes (m-1)} \rangle \sum_{n=0}^{\infty} \frac{\langle \xi, \eta \rangle^n}{n!} \\ &\qquad + \langle \eta, \zeta \rangle \, \langle K_{l,m}^* \eta^{\otimes l}, \xi^{\otimes m} \rangle \sum_{n=1}^{\infty} \frac{\langle \xi, \eta \rangle^{n-1}}{(n-1)!} \\ &= m\langle (K_{l,m} * \zeta)\xi^{\otimes (m-1)}, \eta^{\otimes l} \rangle e^{\langle \xi, \eta \rangle} + \langle \eta, \zeta \rangle \, \langle K_{l,m} \xi^{\otimes m}, \eta^{\otimes l} \rangle e^{\langle \xi, \eta \rangle} \\ &= m\langle\langle \Xi_{l,m-1}(K_{l,m} * \zeta)\phi_\xi, \phi_\eta \rangle\rangle + \langle \eta, \zeta \rangle \, \langle\langle \Xi \phi_\xi, \phi_\eta \rangle\rangle \end{aligned}$$

Therefore (4.11) becomes

$$\langle\langle (D_\zeta^- \Xi)\phi_\xi, \phi_\eta \rangle\rangle = m \, \langle\langle \Xi_{l,m-1}(K_{l,m} * \zeta)\phi_\xi, \phi_\eta \rangle\rangle ,$$

from which we see that Ξ admits the annihilation derivative and (4.8) holds. A similar argument can be applied to (4.9). □

4.4. *Fock Expansion of an Admissible Operator*

We assemble some general results on an admissible white noise operator to discuss its qwn-differentiability in the next subsection.

As a special case of [9] we obtain

Lemma 4.1. *Let $p, q \in \mathbf{R}$. For each $L_{l,m} \in \mathcal{L}(H^{\widehat{\otimes} m}, H^{\widehat{\otimes} l})$ there exists a unique operator $I_{l,m}(L_{l,m}) \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ such that*

$$I_{l,m}(L_{l,m})\widehat{\ }(\xi, \eta) = \langle L_{l,m}\xi^{\otimes m}, \eta^{\otimes l} \rangle, \qquad \xi, \eta \in E.$$

In this case, $\| I_{l,m}(L_{l,m}) \|_{\mathrm{OP}} \le \sqrt{l!m!} \, \| L_{l,m} \|_{\mathrm{OP}}$.

Theorem 4.2. *Let $p, q \in \mathbf{R}$. For any $\Xi \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ there exists a unique family of operators $L_{l,m} \in \mathcal{L}(H^{\widehat{\otimes} m}, H^{\widehat{\otimes} l})$, $l, m \ge 0$, such that*

$$\Xi = \sum_{l,m=0}^{\infty} I_{l,m}(L_{l,m}), \tag{4.12}$$

where the series converges weakly in the sense that

$$\langle\!\langle \Xi\phi, \psi \rangle\!\rangle = \sum_{l,m=0}^{\infty} \langle\!\langle I_{l,m}(L_{l,m})\phi, \psi \rangle\!\rangle, \qquad \phi \in \mathcal{G}_p, \quad \psi \in \mathcal{G}_{-q}.$$

The expression (4.12) is called the *chaotic expansion* of Ξ. In fact, $L_{l,m}$ is obtained by the formula:

$$L_{l,m} = \frac{1}{l!m!} I_l^* \Xi I_m, \tag{4.13}$$

where $I_m \in \mathcal{L}(H^{\widehat{\otimes} m}, \mathcal{G}_p)$ defined by $I_m F_m = (0, \dots, 0, F_m, 0, \dots)$. On the other hand, the Fock expansion of $I_{l,m}(L_{l,m})$ is easily computed:

$$I_{l,m}(L_{l,m}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Xi_{n+l,n+m}(I^{\otimes n} \otimes L_{l,m}). \tag{4.14}$$

Inserting (4.14) into the chaotic expansion (4.12), we obtain the Fock expansion:

$$\Xi = \sum_{l,m=0} \Xi_{l,m} \left(\sum_{n=0}^{l \wedge m} \frac{(-1)^n}{n!} I^{\otimes n} \otimes L_{l-n,m-n} \right). \tag{4.15}$$

Theorem 4.3. *Let Ξ be an admissible white noise operator and let*

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(K_{l,m}) \tag{4.16}$$

be the Fock expansion. Then for all $l, m \geq 0$ *we have* $K_{l,m} \in \mathcal{L}(H^{\otimes m}, H^{\otimes l})$. *Moreover, if* $\Xi \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ *for some* $p, q \in \mathbf{R}$, *then* (4.16) *converges in* $\mathcal{L}(\mathcal{G}_{q-s+r}, \mathcal{G}_{q-s})$ *for any* $r > 0$ *and* $s > 0$ *satisfying*

$$\frac{r}{2} < 2s < \frac{3r}{2}, \quad \frac{1+e^{2q}}{re^{2s-r/2}} < 1, \quad \frac{e^{-2q}(1+e^{2p})}{re^{3r/2-2s}} < 1. \tag{4.17}$$

Proof. Given $\Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$, we define $L_{l,m}$ as in Theorem 4.2. Comparing (4.15) and (4.16), we obtain

$$K_{l,m} = \sum_{n=0}^{l\wedge m} \frac{(-1)^n}{n!} I^{\otimes n} \otimes L_{l-n,m-n}, \tag{4.18}$$

from which the first assertion is obvious. We shall prove the convergence. Suppose that $\Xi \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$ with some $p, q \in \mathbf{R}$ and denote by $\|\Xi\|$ the operator norm. It follows easily from (4.13) that

$$\| L_{l,m} \|_{\mathrm{OP}} \leq \frac{e^{pm-ql}}{\sqrt{l!m!}} \|\Xi\|, \qquad l, m \geq 0.$$

Then (4.18) becomes

$$\| K_{l,m} \|_{\mathrm{OP}} \leq \sum_{n=0}^{l\wedge m} \frac{1}{n!} \frac{e^{p(m-n)-q(l-n)}}{\sqrt{(l-n)!(m-n)!}} \|\Xi\|. \tag{4.19}$$

Applying the Schwartz inequality, we see that the last quantity is bounded by

$$\begin{aligned} &\leq \frac{e^{pm-ql}}{\sqrt{l!m!}} \left(\sum_{n=0}^{l} \frac{l! e^{2qn}}{n!(l-n)!} \right)^{1/2} \left(\sum_{n=0}^{m} \frac{m! e^{-2pn}}{n!(m-n)!} \right)^{1/2} \|\Xi\| \\ &\leq \frac{e^{pm-ql}}{\sqrt{l!m!}} (1+e^{2q})^{l/2} (1+e^{-2p})^{m/2} \|\Xi\|. \end{aligned}$$

Thus, (4.19) becomes

$$\| K_{l,m} \|_{\mathrm{OP}} \leq \frac{e^{pm-ql}}{\sqrt{l!m!}} (1+e^{2q})^{l/2} (1+e^{-2p})^{m/2} \|\Xi\|. \tag{4.20}$$

Now let $r, s > 0$. Applying Proposition 3.1 and a simple inequality $n^n \leq e^n n!$, we obtain

$$\begin{aligned} &\| \Xi_{l,m}(K_{l,m})\phi \|_{q-s} \\ &\leq e^{r/2} \left(\frac{1+e^{2q}}{re^{2s-r/2}} \right)^{l/2} \left(\frac{e^{-2q}(1+e^{2p})}{re^{3r/2-2s}} \right)^{m/2} \|\Xi\| \|\phi\|_{q-s+r}. \end{aligned} \tag{4.21}$$

Hence for any $r, s > 0$ satisfying (4.17), the Fock expansion (4.16) converges in $\mathcal{L}(\mathcal{G}_{q-s+r}, \mathcal{G}_{q-s})$. □

4.5. *QWN-Derivatives of an Admissible Operator*

Theorem 4.4. *Every admissible white noise operator is qwn-differentiable. More precisely, if the Fock expansion of* $\Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ *is given as in* (4.16), *then for any* $\zeta \in H$ *we have*

$$D_\zeta^- \Xi = \sum_{l,m=0}^{\infty} m\Xi_{l,m-1}(K_{l,m} * \zeta), \qquad D_\zeta^+ \Xi = \sum_{l,m=0}^{\infty} l\Xi_{l-1,m}(\zeta * K_{l,m}),$$

where the right hand sides converges in the same manner as mentioned in Theorem 4.3. Moreover, $D_\zeta^\pm$ *is a continuous linear operator on* $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$.

Proof. Each $\Xi_{l,m}(K_{l,m})$ is admissible by Theorem 4.3 and hence qwn-differentiable by Theorem 4.1. Then, it is sufficient to show the onvergence of the right hand sides of $D_\zeta^\pm \Xi$. Suppose that $\Xi \in \mathcal{L}(\mathcal{G}_p, \mathcal{G}_q)$. Applying Proposition 3.1 and (4.20), we obtain easily

$$\begin{aligned} &\| m\Xi_{l,m-1}(K_{l,m} * \zeta)\phi \|_{q-s} \\ &\quad \le \sqrt{r} e^{q-s+5r/4} \left(\frac{1+e^{2q}}{re^{2s-r/2}} \right)^{l/2} \sqrt{m} \left(\frac{e^{-2q}(1+e^{2p})}{re^{3r/2-2s}} \right)^{m/2} \\ &\quad \times \| \Xi \| \, |\zeta|_0 \, \| \phi \|_{q-s+r} . \end{aligned}$$

This estimate is almost the same as (4.21) and the series

$$\sum_{l,m=0}^{\infty} \| m\Xi_{l,m-1}(K_{l,m} * \zeta)\phi \|_{q-s}$$

converges whenever (4.17) is satisfied. In this case we have

$$\| (D_\zeta^- \Xi)\phi \|_{q-s} \le C \| \Xi \| \, |\zeta|_0 \, \| \phi \|_{q-s+r} ,$$

with some constant $C = C(p, q, r, s)$, which proves that D_ζ^- is a continuous linear operator on $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$. The argument for D_ζ^+ is similar. □

4.6. *Pointwise QWN-Derivatives*

A qwn-differentiable operator $\Xi \in \mathcal{L}((E), (E)^*)$ is called *pointwisely qwn-differentiable* if there exists a measurable map $t \mapsto D_t^\pm \Xi \in \mathcal{L}((E), (E)^*)$ such that

$$\langle\langle (D_\zeta^\pm \Xi)\phi_\xi, \phi_\eta \rangle\rangle = \int_T \langle\langle (D_t^\pm \Xi)\phi_\xi, \phi_\eta \rangle\rangle \zeta(t)\nu(dt), \qquad \zeta \in H, \ \xi, \eta \in E.$$

The following examples support the intuitive idea (1.3) in Introduction.

Example 4.1. For $f \in E^*$ define $K_{0,1} \in \mathcal{L}(E, \mathbf{C})$ and $K_{1,0} \in \mathcal{L}(\mathbf{C}, E^*)$ by

$$K_{0,1} : \xi \mapsto \langle f, \xi \rangle , \qquad K_{1,0} : c \mapsto cf,$$

respectively. The integral kernel operators

$$A_f = \Xi_{0,1}(K_{0,1}) = \int_T f(t) a_t \, \nu(dt), \quad A_f^* = \Xi_{1,0}(K_{1,0}) = \int_T f(t) a_t^* \, \nu(dt)$$

are respectively called annihilation and creation operators associated with f. If $f \in H$, both A_f and A_f^* are pointwisely qwn-differentiable and

$$D_t^- A_f = f(t) I, \quad D_t^+ A_f = 0; \qquad D_t^- A_f^* = 0, \quad D_t^+ A_f^* = f(t) I.$$

Example 4.2. The number operator and the Gross Laplacian are defined by

$$N = \Xi_{1,1}(I) = \int_T a_t^* a_t \, \nu(dt), \quad \Delta_G = \Xi_{0,2}(I) = \int_T a_t^2 \, \nu(dt),$$

respectively. Then we have

$$D_t^- N = a_t^*, \quad D_t^+ N = a_t, \quad D_t^- \Delta_G = 2a_t, \quad D_t^+ \Delta_G = 0.$$

References

1. K. Aase, B. Øksendal, N. Privault and J. Ubøe: *White noise generalizations of the Clark–Haussmann–Ocone theorem with application to mathematical finance*, Finance Stochast. **4** (2000), 465–496.
2. V. P. Belavkin: *A quantum nonadapted Ito formula and stochastic analysis in Fock scale*, J. Funct. Anal. **102** (1991), 414–447.
3. F. E. Benth and J. Potthoff: *On the martingale property for generalized stochastic processes*, Stoch. Stoch. Rep. **58** (1996), 349–367.
4. D. M. Chung, U. C. Ji and N. Obata: *Quantum stochastic analysis via white noise operators in weighted Fock space*, Rev. Math. Phys. **14** (2002), 241–272.
5. M. Grothaus, Yu. G. Kondratiev and L. Streit: *Complex Gaussian analysis and the Bargmann–Segal space*, Methods of Funct. Anal. and Topology **3** (1997), 46–64.
6. T. Hida: "Analysis of Brownian Functionals," Carleton Math. Lect. Notes, no. 13, Carleton University, Ottawa, 1975.
7. U. C. Ji: *Stochastic integral representation theorem for quantum semimartingales*, J. Func. Anal. **201** (2003), 1–29.
8. U. C. Ji and N. Obata: *Quantum white noise calculus*, in "Non-Commutativity, Infinite-Dimensionality and Probability at the Crossroads (N. Obata, T. Matsui and A. Hora, Eds.)," pp. 143–191, World Scientific, 2002.
9. U. C. Ji and N. Obata: *A role of Bargmann-Segal spaces in characterization and expansion of operators on Fock space*, J. Math. Soc. Japan **56** (2004), in press.

10. U. C. Ji and N. Obata: *Annihilation-derivative, creation-derivative and representation of quantum martingales*, preprint, 2003.
11. I. Kubo and S. Takenaka: *Calculus on Gaussian white noise I*, Proc. Japan Acad. **56A** (1980), 376–380.
12. H.-H. Kuo: "White Noise Distribution Theory," CRC Press, 1996.
13. J. M. Lindsay and H. Maassen: *An integral kernel approach to noise*, in "Quantum Probability and Applications III (L. Accardi and W. von Waldenfels Eds.)." Lecture Notes in Math. Vol. 1303, pp. 192–208, Springer-Verlag, 1988.
14. J. M. Lindsay and K. R. Parthasarathy: *Cohomology of power sets with applications in quantum probability*, Commun. Math. Phys. **124** (1989), 337–364.
15. P. Malliavin: "Stochastic Analysis," Springer-Verlag, 1997.
16. D. Nualart: "The Malliavin Calculus and Related Topics," Springer-Verlag, New York, 1995.
17. N. Obata: "White Noise Calculus and Fock Space," Lect. Notes in Math. Vol. 1577, Springer–Verlag, 1994.