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AN INTERACTING FOCK SPACE WITH PERIODIC JACOBI PARAMETER OBTAINED FROM REGULAR GRAPHS IN LARGE SCALE LIMIT

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Asymptotic spectral analysis of the adjacency matrix of a large regular graph is formulated within algebraic or quantum probability theory. We prove a quantum central limit theorem for the quantum components of the adjacency matrices of growing regular graphs under a weaker condition. A new example of growing regular graphs is constructed, for which the limit is described in terms of an interacting Fock space whose Jacobi parameter is periodic. The central limit measure is obtained from the periodic continued fraction expansion of the Cauchy transform.

1 Introduction

Let $\{\mathcal{G}_\nu = (V^{(\nu)}, E^{(\nu)})\}$ be a growing family of regular graphs (always assumed to be connected), where the growing parameter ν runs over an infinite directed set. The degree of $\mathcal{G}_\nu$ is denoted by $\kappa(\nu)$. For each graph $\mathcal{G}_\nu$ we fix an origin $x_0 \in V^{(\nu)}$ and consider the stratification induced by the natural distance function on the graph:

$$V^{(\nu)} = \bigcup_{n=0}^{\infty} V_n^{(\nu)}, \qquad V_n^{(\nu)} = \{y \in V^{(\nu)}; \partial(x_0, y) = n\}. \tag{1}$$

($V_n^{(\nu)} = \emptyset$ may occur.) The adjacency matrix A_ν of $\mathcal{G}_\nu$ admits a *quantum decomposition*:

$$A_\nu = A_\nu^+ + A_\nu^-, \tag{2}$$

which is canonically induced from the stratification (1). We are interested in asymptotic behavior of the quantum components as $\nu \to \infty$ and study it from the viewpoint of algebraic probability theory. According to the stratification (1), we define

$$\Phi_n^{(\nu)} = |V_n^{(\nu)}|^{-1/2} \sum_{x \in V_n^{(\nu)}} \delta_x, \tag{3}$$

where δ_x stands for the indicator function of the singlet $\{x\}$ and those functions form a complete orthonormal basis of $\mathcal{H}_\nu = \ell^2(V^{(\nu)})$. One can expect easily that the quantum components $A_\nu^\pm$ behave like the annihilation and creation operators on a "Fock space" spanned by the "number vectors" $\Phi_n^{(\nu)}$, where n runs over $0, 1, 2, \dots$ whenever $V_n^{(\nu)} \neq \emptyset$. In the limit as $\nu \to \infty$ this guess is realized concretely in terms of quantum central limit theorem, where the limit is described by an interacting Fock space.

To be precise we need some statistical assumptions on how the regular graphs $\mathcal{G}_\nu$ grow as $\nu \to \infty$. For $x \in V_n^{(\nu)}$ we put

$$\omega_+(x) = |\{y \in V_{n+1}^{(\nu)} \,;\, y \sim x\}|, \qquad \omega_-(x) = |\{y \in V_{n-1}^{(\nu)} \,;\, y \sim x\}|. \tag{4}$$

These are the numbers of points in the upper or lower stratum connecting with x, respectively. The average and variance of $\omega_-(x)$ over $V_n^{(\nu)}$ are defined by

$$\omega_n^{(\nu)} = |V_n^{(\nu)}|^{-1} \sum_{x \in V_n^{(\nu)}} \omega_-(x),$$

$$\sigma_n^{(\nu)2} = |V_n^{(\nu)}|^{-1} \sum_{x \in V_n^{(\nu)}} (\omega_-(x) - \omega_n^{(\nu)})^2,$$

respectively. We consider the following five conditions:

(A1) $\omega_+^{(\nu)}(x) + \omega_-^{(\nu)}(x) = \kappa(\nu)$ for all $x \in V^{(\nu)}$, in other words, there is no edge lying in a stratum;

(A2) $\lim_\nu \kappa(\nu) = \infty$;

(A3) for each $n \geq 0$ there exists a limit $\omega_n \equiv \lim_\nu \omega_n^{(\nu)} < \infty$;

(A4) $\lim_\nu \sigma_n^{(\nu)} = 0$ for all $n \geq 0$;

(A5) for each $n \geq 1$ we have

$$W_n \equiv \sup_\nu W_n^{(\nu)} < \infty, \qquad W_n^{(\nu)} = \max\{\omega_-(x) \,;\, x \in V_n^{(\nu)}\}.$$

With these notations we may claim the following

Theorem 1.1 *Let $\{\mathcal{G}_\nu = (V^{(\nu)}, E^{(\nu)})\}$ be a growing family of regular graphs satisfying conditions (A1)-(A5). Let $(\Gamma, \{\lambda_n\}, B^+, B^-)$ be the interacting Fock space associated with $\{\lambda_n\}$ given by $\lambda_n = \omega_1 \dots \omega_n$, $\lambda_0 = 1$. Then,*

$$\lim_\nu \left\langle \Phi_j^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_k^{(\nu)} \right\rangle_{\mathcal{H}_\nu} = \langle \Psi_j, B^{\epsilon_m} \dots B^{\epsilon_1} \Psi_k \rangle_\Gamma \qquad (5)$$

holds for all $j, k \geq 0$ and for any choice of $\epsilon_1, \dots, \epsilon_m \in \{\pm\}$, $m \geq 1$. Here Ψ_j, Ψ_k in the right hand side are number vectors of Γ.

The proof is deferred in Section 5. A similar result was first obtained by Hashimoto [9], where a growing family of Cayley graphs was discussed under the assumptions (A1), (A2), (A5) and

(A3″) for each n there exist constant numbers $\omega_n \geq 0$ and $C_n \geq 0$ independent of ν such that

$$|\{x \in V_n^{(\nu)}\,;\, \omega_-(x) \neq \omega_n\}| \leq C_n \kappa(\nu)^{n-1}$$

holds for all $n \geq 1$ and ν.

In fact, Hashimoto [9] proved convergence of the matrix elements (5) with respect to "coherent vectors" as well as "number vectors" under an assumption slightly stronger than (A5), and clarified Gauss-Poisson interpolation investigated in Hashimoto [8]. Later on we formulated in Hashimoto–Hora–Obata [10] and Hora–Obata [17] Hashimoto's theorem for general regular graphs and proved Theorem 1.1 under assumptions (A1), (A2), (A3″) and (A5). Note that (A3″) implies (A3) and (A4), but not conversely.

By virtue of (2) asymptotics of the adjacency matrix A_ν follows immediately from Theorem 1.1 (classical reduction). We obtain the following

Theorem 1.2 *Let $\{\mathcal{G}_\nu = (V^{(\nu)}, E^{(\nu)})\}$ and $(\Gamma, \{\lambda_n\}, B^+, B^-)$ be the same as in Theorem 1.1. Let μ be the probability measure corresponding to $(\Gamma, \{\lambda_n\}, B^+, B^-)$. Then, it holds that*

$$\lim_\nu \left\langle \Phi_0^{(\nu)}, \left(\frac{A_\nu}{\sqrt{\kappa(\nu)}}\right)^m \Phi_0^{(\nu)} \right\rangle_{\mathcal{H}_\nu} = \int_{\mathbf{R}} x^m \mu(dx), \qquad m = 0, 1, 2, \dots.$$

In particular, the moments of odd orders vanish and μ is symmetric.

It is a natural question to characterize the class of probability measures appearing as in Theorem 1.2. In Hashimoto–Hora–Obata [10] we examined some Cayley graphs with our method. The standard Gaussian measure is obtained from the lattices $\mathbf{Z}^N$ and the Wigner semi-circle law from the homogeneous trees associated with the free groups F_N. These are prototypes of classical and free central limit theorems, see e.g., Hiai–Petz [12], Voiculescu–Dykema–Nica [21]. From the Coxeter groups with the off-diagonal elements of the Coxeter matrix being ≥ 3 the Wigner semi-circle law is obtained, see Fendler [7] for a different derivation. From the symmetric group $\mathfrak{S}_N$ with Coxeter generators the standard Gaussian measure is obtained. The same occurs when $\mathfrak{S}_N$ is equipped with all the transpositions as a set of generators. On the other hand, the Wigner semi-circle law is obtained from $\mathfrak{S}_N$ equipped with the generators $\{(12), (13), \dots, (1N)\}$, see Biane [2]. No "natural" example of growing Cayley graphs is known, from which another probability measure is obtained. However, beyond Cayley graphs there are interesting examples. In Section 3 we construct a new example of growing regular graphs for which the limit is described in terms of an interacting Fock space with a periodic Jacobi parameter. Thus the Cauchy transform (also called the Stieltjes transform) of the corresponding probability measure admits a periodic continued fraction expansion. Similar probability measures are derived by Bożejko [3] from a deformation of convolution products called the r-free convolution though the range of parameter is different.

When condition (A1) is removed, the situation becomes full of variety. For example, the probability measures obtained in the limit are no longer symmetric. We know two examples from distance regular graphs: From a growing family of Hamming graphs the standard Gaussian measure and Poisson measures are obtained, see Hashimoto–Obata–Tabei [11]. From a growing family of Johnson graphs an exponential distribution and geometric distributions appear, see Hashimoto–Hora–Obata [10]. While, these limit distributions were first investigated by Hora [13] with a classical method. A general strategy of investigating the limit distribution has been discussed for a growing family of distance regular graphs, see Hashimoto–Hora–Obata [10]. Another type of limit procedure has been also studied by Hashimoto [8] and Hora [14,15,16].

2 Preliminaries

For the sake of the readers' convenience we assemble some basic notion and notation used in Theorems 1.1 and 1.2.

2.1 Adjacency Matrix

Let $\mathcal{G} = (V, E)$ be a regular graph of degree $1 \leq \kappa < \infty$. When $x, y \in V$ are connected by an edge, we write $x \sim y$. That $x \sim x$ never occurs. By assumption for each $x \in V$ the number $|\{y \in V\,;\, y \sim x\}| = \kappa$ is constant. In this paper, unless otherwise specified, all graphs are assumed to be connected.

Then, for any pair $x, y \in V$ there is a walk $x_0 \sim x_1 \sim \cdots \sim x_n$ such that $x = x_0$ and $y = x_n$. In that case n is called the length of the walk. The length of the shortest walk connecting x and y is denoted by $\partial(x, y)$ and is called the distance between them. By definition $\partial(x, x) = 0$. Note that $\partial(x, y) = 1$ if and only if $x \sim y$.

Let $A = (A_{xy})_{x,y \in V}$ be the adjacency matrix of $\mathcal{G}$, namely, A is a symmetric matrix defined by

$$A_{xy} = \begin{cases} 1, & x \sim y, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

The adjacency matrix is identified with a bounded operator acting on $\ell^2(V)$:

$$Af(x) = \sum_{y \sim x} f(y), \qquad x \in V, \quad f \in \ell^2(V).$$

Note that $\|A\| = \kappa$. For each $x \in V$ denote by δ_x the indicator function of the singlet $\{x\}$. Then $\{\delta_x\, ;\, x \in V\}$ forms a complete orthonormal basis of $\ell^2(V)$. Obviously,

$$A\delta_x = \sum_{y \sim x} \delta_y, \qquad x \in V. \tag{7}$$

2.2 Stratification and Quantum Decomposition

We fix a point $x_0 \in V$ as an origin of the graph. Then, the graph is stratified into a disjoint union of strata:

$$V = \bigcup_{n=0}^{\infty} V_n, \qquad V_n = \{x \in V\, ;\, \partial(x_0, x) = n\}, \tag{8}$$

where $V_n = \emptyset$ may occur. Obviously, $|V_0| = 1$, $|V_1| = \kappa$, and $|V_n| \le \kappa(\kappa - 1)^{n-1}$ for $n \ge 2$.

By the triangle inequality we see that if $x \in V_n$ and $x \sim y$, then $y \in V_{n-1} \cup V_n \cup V_{n+1}$. In this paper we avoid the case of $y \in V_n$, that is, we assume throughout the following condition:

(A1) there is no edge lying in a common stratum.

We assign to each edge $x \sim y$ of the graph $\mathcal{G} = (V, E)$ an orientation compatible with the stratification, i.e., in such a way that $x \prec y$ if $x \in V_n$ and $y \in V_{n+1}$. Then we define

$$(A^+)_{yx} = \begin{cases} A_{yx} = 1 & \text{if } y \succ x, \\ 0 & \text{otherwise,} \end{cases} \qquad (A^-)_{yx} = \begin{cases} A_{yx} = 1 & \text{if } y \prec x, \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

or equivalently,

$$A^+\delta_x = \sum_{y \succ x} \delta_y, \qquad A^-\delta_x = \sum_{y \prec x} \delta_y. \tag{10}$$

Then we come to a *quantum decomposition* of A:

$$A = A^+ + A^-, \qquad (A^+)^* = A^-. \tag{11}$$

The former relation is checked by (7) and the latter by definition (9).

2.3 Interacting Fock Space

We refer to Accardi–Bożejko [1] for more details. Let $\lambda_0 = 1, \lambda_1, \lambda_2, \cdots \geq 0$ be a sequence of nonnegative numbers and assume that if $\lambda_m = 0$ occurs for some $m \geq 1$ then $\lambda_n = 0$ for all $n \geq m$. According as $\lambda_n > 0$ for all n or $\lambda_m = 0$ occurs for some $m \geq 1$, we define a Hilbert space of infinite dimension or of finite dimension:

$$\Gamma = \sum_{n=0}^{\infty} \oplus \mathbf{C}\Psi_n, \qquad \Gamma = \sum_{n=0}^{m_0-1} \oplus \mathbf{C}\Psi_n,$$

where m_0 is the first number such that $\lambda_{m_0} = 0$, and $\{\Psi_n\}$ is an orthonormal basis. We call Ψ_n the *n-th number vector.*

The creation operator B^+ and the annihilation operator B^- are defined by

$$B^+\Psi_n = \sqrt{\frac{\lambda_{n+1}}{\lambda_n}}\, \Psi_{n+1}, \quad n \geq 0,$$

$$B^-\Psi_0 = 0, \quad B^-\Psi_n = \sqrt{\frac{\lambda_n}{\lambda_{n-1}}}\, \Psi_{n-1}, \quad n \geq 1.$$

In the case when Γ is of finite dimension we tacitly understand that $B^+\Psi_{m_0-1} = 0$. Equipped with the natural domains, $B^\pm$ become closed operators which are mutually adjoint. Then $\Gamma(\{\lambda_n\}) = (\Gamma, \{\lambda_n\}, B^+, B^-)$ is called an *interacting Fock space* associated with $\{\lambda_n\}$. By simple computation we have

$$B^+B^-\Psi_0 = 0, \quad B^+B^-\Psi_n = \frac{\lambda_n}{\lambda_{n-1}}\, \Psi_n, \quad n \geq 1, \tag{12}$$

$$B^-B^+\Psi_n = \frac{\lambda_{n+1}}{\lambda_n}\, \Psi_n, \quad n \geq 0, \tag{13}$$

$$B^{+n}\Psi_0 = \sqrt{\lambda_n}\, \Psi_n, \quad n \geq 0. \tag{14}$$

2.4 Orthogonal Polynomials

Let μ be a probability measure on $\mathbf{R}$ with finite moments of all orders, i.e.,

$$\int_{\mathbf{R}} |x|^m \mu(dx) < \infty, \qquad m = 0, 1, 2, \cdots,$$

and $\{P_n\}$ the associated orthogonal polynomials normalized in such a way that $P_n(x) = x^n + \ldots$. Then there exists uniquely a pair of sequences $\alpha_1, \alpha_2, \cdots \in \mathbf{R}$ and $\omega_1, \omega_2, \cdots \geq 0$ such that

$$\begin{aligned} &P_0(x) = 1, \\ &P_1(x) = x - \alpha_1, \\ &xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x), \quad n \geq 1 \end{aligned} \tag{15}$$

The pair $\{\alpha_n\}, \{\omega_n\}$ is called the *Szegö–Jacobi parameter.* When the probability measure μ is supported by a finite set of exactly m_0 points, the orthogonal polynomials $\{P_n\}$ terminate at $n = m_0 - 1$ and the Szegö–Jacobi parameter becomes a pair of finite sequences $\alpha_1, \ldots, \alpha_{m_0}$ and $\omega_1 \ldots, \omega_{m_0-1}$, where the last numbers are determined by (15) with $P_{n+1} = 0$. Note also that μ is symmetric if and only if $\alpha_n = 0$ for all $n \geq 1$.

Theorem 2.1 (Accardi–Bożejko [1]) *Let $\{P_n\}$ be the orthogonal polynomials with respect to μ with Szegö–Jacobi parameters $\{\alpha_n\}$, $\{\omega_n\}$. Let $\Gamma(\{\lambda_n\})$ be an interacting Fock space associated with*

$$\lambda_0 = 1, \qquad \lambda_n = \omega_1\omega_2\ldots\omega_n, \quad n \geq 1. \tag{16}$$

Then there exists an isometry U from $\Gamma(\{\lambda_n\})$ into $L^2(\mathbf{R}, \mu)$ uniquely determined by

$$U\Phi_0 = P_0, \quad UB^+U^*P_n = P_{n+1}, \quad Q = U(B^+ + B^- + \alpha_{N+1})U^*,$$

where Q is the multiplication operator by x densely defined in $L^2(\mathbf{R}, \mu)$ and α_{N+1} is the operator defined by $\alpha_{N+1}\Psi_n = \alpha_{n+1}\Psi_n$.

In fact, the isometry U is uniquely specified by $\sqrt{\lambda_n}\,\Psi_n \mapsto P_n$. A question of when U is a unitary, or equivalently when the polynomials span a dense subspace in $L^2(\mathbf{R}, \mu)$ is related to the so-called determinate moment problem, see e.g., [6,20,22].

By Theorem 2.1, given an interacting Fock space $(\Gamma, \{\lambda_n\}, B^+, B^-)$, there exists a probability measure μ such that

$$\langle \Psi_0, (B^+ + B^- + \alpha_{N+1})^m \Psi_0 \rangle = \int_{\mathbf{R}} x^m \mu(dx).$$

This μ is unique if the corresponding moment problem is determinate. There is a formula linking the Cauchy transform (also called the Stieltjes transfrom) of μ and the Szegö–Jacobi parameter $\{\alpha_n\}, \{\omega_n\}$:

$$G_\mu(z) = \int_{\mathbf{R}} \frac{\mu(dx)}{z-x} = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \frac{\omega_3}{z-\alpha_4} - \cdots . \tag{17}$$

If μ is supported by a bounded interval, $G_\mu(z)$ is holomorphic on $\{|z| > r\}$ for some $r > 0$ and the continued fraction converges. Conversely, if both $\{\alpha_n\}, \{\omega_n\}$ are bounded sequence, then the continued fraction converges uniformly on $\{|z| > r'\}$ for some $r' > 0$ and there exists a unique probability mesure μ such that (17) holds. In this case μ is supported by a bounded interval, see e.g., [6,22]. Unless μ is supported by a bounded interval, (17) is still useful but the situation becomes complicated, see [22].

Remark 2.2 The Cauchy transform is defined for every probability measure μ on $\mathbf{R}$ without assuming existence of moments, and becomes a holomorphic function on $\{\mathrm{Im}\,(z) \neq 0\}$. In fact, the integral in (17) converges absolutely and uniformly on every compact subset in that domain, see e.g., [5,22].

3 New Examples

3.1 Construction of Regular Graphs with Periodic Parameters

Theorem 3.1 *Let $a, b, k \ge 1$ be integers. Define $\kappa = abk$ and*

$$\omega_1 = 1, \quad \omega_2 = a, \quad \omega_3 = b, \quad \omega_4 = a, \quad \omega_5 = b, \dots . \tag{18}$$

Then there exists a κ-regular graph $\mathcal{G} = (V, E)$ which admits a stratification $V = \bigcup_{n=0}^\infty V_n$ such that $\omega_-(x) = \omega_n$ for $x \in V_n$, $n \ge 1$.

PROOF. We shall explicitly construct a κ-regular graph having the desired property.

1° Let V_0 and V_1 consist of a single point x_0 (origin) and of κ points, respectively. We draw edges connecting each point in V_1 and x_0. Then x_0 has κ edges.

2° We construct V_2 and edges connecting between V_1 and V_2. The number of points in V_2 is determined by counting such edges. Since each $x \in V_1$ must have $\kappa - 1$ edges connecting with points in V_2 and each $y \in V_2$ has a edges connecting with points in V_1 by request, we have the relation:

$$(\kappa - 1)|V_1| = a|V_2|.$$

Thus,

$$|V_2| = \frac{\kappa(\kappa - 1)}{a}, \tag{19}$$

which is an integer for $\kappa = abk$. We must prove that the points in V_1 and those in V_2 can be connected by edges in such a way that each point $y \in V_2$ has a edges and each $x \in V_1$ has $\kappa - 1$ edges. This is possible by looking at

$$|V_1| = \kappa = \frac{\kappa}{a} \times a, \qquad |V_2| = \frac{\kappa(\kappa - 1)}{a} = \frac{\kappa}{a} \times (\kappa - 1).$$

We can divide V_1 and V_2 into $\kappa/a = bk$ subsets:

$$V_1 = \bigcup_{i=1}^{bk} V_1^{(i)}, \quad V_2 = \bigcup_{i=1}^{bk} V_2^{(i)} \quad \text{with} \quad |V_1^{(i)}| = a, \quad |V_2^{(i)}| = \kappa - 1.$$

For each i, we draw edges between $V_1^{(i)}$ and $V_2^{(i)}$ in such a way that any pair $x \in V_1^{(i)}$ and $y \in V_2^{(i)}$ is connected. For distinct i, j there is no edge connecting between $V_1^{(i)}$ and $V_2^{(j)}$. In this way, each $x \in V_1$ has κ edges with $\omega_-(x) = 1$ and each $y \in V_2$ has a edges connecting with points in V_1.

3° We construct V_3 and edges connecting between V_2 and V_3. The number of points in V_3 is determined by the relation:

$$(\kappa - a)|V_2| = b|V_3|.$$

Hence, in view of (19) we have

$$|V_3| = \frac{\kappa(\kappa - 1)(\kappa - a)}{ab}.$$

Since

$$|V_2| = \frac{\kappa(\kappa - 1)}{a} = \frac{\kappa(\kappa - 1)}{ab} \times b, \qquad |V_3| = \frac{\kappa(\kappa - 1)}{ab} \times (\kappa - a),$$

a similar argument as in 2° allows us to draw edges between V_2 and V_3 in such a way that each point in V_2 has $\kappa - a$ edges and each point in V_3 has b edges. In total each point in $y \in V_2$ has κ edges with $\omega_-(y) = a$.

4° This procedure can be applied repeatedly and we obtain a κ-regular graph having the desired property, see Figure 1. In fact, the number of points in each stratum is given by

$$|V_0| = 1, \qquad |V_1| = \kappa,$$

$$|V_{2n}| = \frac{\kappa(\kappa-1)}{a}\left(\frac{(\kappa-a)(\kappa-b)}{ab}\right)^{n-1}, \qquad n \geq 1,$$

$$|V_{2n+1}| = \frac{\kappa(\kappa-1)(\kappa-a)}{ab}\left(\frac{(\kappa-a)(\kappa-b)}{ab}\right)^{n-1}, \qquad n \geq 1.$$

∎

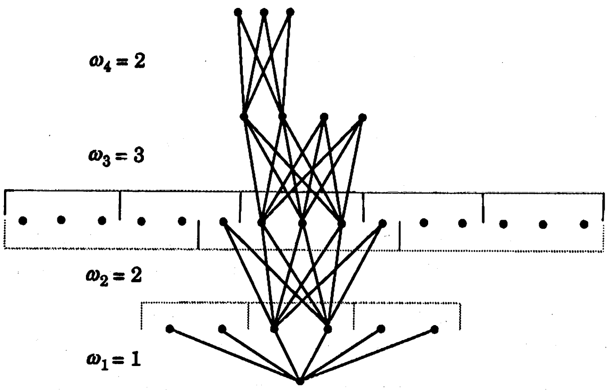

Figure 1. Construction Procedure: $a = 2$, $b = 3$, $\kappa = 6$

Remark 3.2 There are three trivial cases (i) $\kappa = 1$; (ii) $\kappa = a \geq 2$ and $b = 1$; (ii) $\kappa = b \geq 2$ and $a = 1$. Except these cases the κ-regular graph constructed in Theorem 3.1 has infinitely many strata.

By modifying the above proof we obtain the following

Theorem 3.3 *Let $a_1, \dots, a_m, k \geq 1$ be integers. Define $\kappa = a_1 a_2 \dots a_m k$ and*

$$\omega_1 = 1, \quad \omega_2 = a_1, \quad \dots, \quad \omega_{m+1} = a_m,$$

$$\omega_{jm+i+1} = a_i, \qquad j \geq 0, \quad 1 \leq i \leq m.$$

Then there exists a κ-regular graph $\mathcal{G} = (V, E)$ which admits a stratification $V = \bigcup_{n=0}^{\infty} V_n$ such that $\omega_-(x) = \omega_n$ for $x \in V_n$, $n \geq 1$.

3.2 Limit Distribution

Let $a, b \geq 1$ be integers fixed. For each integer $k \geq 1$ let $\mathcal{G}_k$ be the abk-regular graph constructed in Theorem 3.1. As is easily verified, $\{\mathcal{G}_k\}$ fulfills conditions (A1)–(A5) and becomes an example of Theorem 1.1. Here the interacting Fock space describing the limit is determined by the parameter:

$$\lambda_0 = 1, \quad \lambda_n = \omega_1 \dots \omega_n,$$

where ω_n is a periodic sequence given in (18). We are interested in the corresponding probability measure μ.

Lemma 3.4 *The Cauchy transform of μ is given by*

$$G_\mu(z) = \frac{(2b-1)z^2 + a - b - \sqrt{z^4 - 2(a+b)z^2 + (a-b)^2}}{2z\{(b-1)z^2 + a - b + 1)\}}, \tag{20}$$

where $\mathrm{Im}\,(z) > 0$ *and* $\mathrm{Re}\,(z) > 0$.

PROOF. By general theory mentioned at the end of §2.4, the Cauchy transform of μ is given by the continued fraction:

$$G_\mu(z) = \frac{1}{z-}\,\frac{1}{z-}\,\frac{a}{z-}\,\frac{b}{z-}\,\frac{a}{z-}\,\frac{b}{z-}\cdots. \tag{21}$$

By using the periodicity, it is not hard to obtain a compact expression of $G_\mu(z)$ as in (20). ∎

By applying the Stieltjes inversion formula [5,22] we come to the following

Theorem 3.5 *Let $\chi(x)$ be the indicator function of*

$$[-\sqrt{a}-\sqrt{b}, -|\sqrt{a}-\sqrt{b}|] \cup [|\sqrt{a}-\sqrt{b}|, \sqrt{a}+\sqrt{b}].$$

and define

$$\rho_{a,b}(x) = \frac{\sqrt{2(a+b)x^2 - x^4 - (a-b)^2}}{2\pi|x|\{(b-1)x^2 + a - b + 1\}}\,\chi(x).$$

Then $\mu(dx)$ is given as follows:

(1) *If* $1 \le b \le a - 1$,

$$\mu(dx) = \left(1 - \frac{1}{a-b+1}\right)\delta_0(dx) + \rho_{a,b}(x)dx.$$

(2) *If* $b = a$ *or* $b = a + 1$,

$$\mu(dx) = \rho_{a,b}(x)dx.$$

(3) *If* $b \ge a + 2$,

$$\mu(dx) = \frac{1}{2}\left(1 - \frac{a}{(b-1-a)(b-1)}\right)(\delta_\xi + \delta_{-\xi})(dx) + \rho_{a,b}(x)dx.$$

where $\xi = \sqrt{(b-1-a)/(b-1)}$.

Remark 3.6 In [3] Bożejko introduced a one-parameter deformation of the free product called the r-free convolution, where r runs over $[0, 1]$. The Cauchy transform of the central limit measure is given by a periodic continued fraction as in (21) with $a = r$, $b = 1$.

4 Remarks on Conditions (A1)–(A5)

4.1 Statistical Quantities for Graphs

Conditions (A3) and (A4) for $n = 0, 1$ are automatically satisfied because of structure of the stratification. In fact, $\omega_0^{(\nu)} = 0$, $\omega_1^{(\nu)} = 1$ and $\sigma_0^{(\nu)} = \sigma_1^{(\nu)} = 0$ for all ν. When $V_n \neq \emptyset$ and $V_{n+1} = \emptyset$ occurs, we understand that

$$\omega_0 = 0, \quad \omega_1 = 1, \quad \omega_2 \geq 1, \quad \dots, \quad \omega_n = \kappa \geq 1, \quad \omega_{n+1} = \dots = 0. \tag{22}$$

Otherwise,

$$\omega_0 = 0, \quad \omega_1 = 1, \quad \omega_n \geq 1, \quad n \geq 2.$$

Note also that if $V_n \neq \emptyset$ and for some $n \geq 1$, we have $\omega_n \geq 1$. In fact, every $x \in V_n$ is connected with at least one point in V_{n-1}.

4.2 How Graph Grows

Roughly speaking, under conditions (A1)–(A5) the graph grows upwards by adding new points and new vertices.

Proposition 4.1 *If $\{\mathcal{G}_\nu\}$ satisfies conditions (A1), (A2) and (A3), then for each $n \geq 1$ there exists $\nu_0 = \nu_0(n)$ such that $V_n^{(\nu)} \neq \emptyset$ for all $\nu \geq \nu_0$. In particular, $\omega_n \geq 1$ for all $n \geq 1$.*

PROOF. We prove by contradiction. Suppose that there exist $n \geq 1$ and $\nu_1 < \nu_2 < \dots \to \infty$ such that $V_n^{(\nu_i)} \neq \emptyset$ and $V_{n+1}^{(\nu_i)} = \emptyset$. It follows from (A1) that $\omega_-(x) = \kappa(\nu_i)$ for all $x \in V_n^{(\nu_i)}$. Hence, taking the average over $V_n^{(\nu_i)}$, we come to $\omega_n^{(\nu_i)} = \kappa(\nu_i)$. But this is impossible by (A2) and (A3).

We show that $\omega_n \geq 1$ for all $n \geq 1$. For each $n \geq 1$ choose ν_0 as in the assertion. Then, for all $\nu \geq \nu_0$ we have $\omega_-(x) \geq 1$ for all $x \in V_n^{(\nu)}$. Hence its average satisfies $\omega_n^{(\nu)} \geq 1$ and its limit $\omega_n \geq 1$. ∎

Proposition 4.2 *If $\{\mathcal{G}_\nu\}$ satisfies (A1), (A2) and (A5), then for each $n \geq 1$ there exists $\nu_0 = \nu_0(n)$ such that every $x \in V_{n-1}^{(\nu)}$ has an edge connecting with a point in a upper stratum whenever $\nu \geq \nu_0$. In particular, $V_n^{(\nu)} \neq \emptyset$ for all $\nu \geq \nu_0$.*

PROOF. By induction on n. The assertion for $n = 1$ is clear. Assume the assertion holds up to $n-1$, where $n > 1$. By (A1) we have

$$\omega_+^{(\nu)}(x) = \kappa(\nu) - \omega_-^{(\nu)}(x), \qquad x \in V_{n-1}^{(\nu)}, \quad \nu \geq \nu_0.$$

By (A5),

$$\omega_+^{(\nu)}(x) \geq \kappa(\nu) - W_{n-1}^{(\nu)} \geq \kappa(\nu) - W_{n-1}.$$

Since W_{n-1} is independent of ν, we see by (A2) that

$$\lim_{\nu} \min\{\omega_+^{(\nu)}(x)\,;\, x \in V_{n-1}^{(\nu)}\} = \infty.$$

In particular, there exists $\nu_1 \geq \nu_0$ such that

$$\min\{\omega_+^{(\nu)}(x)\,;\, x \in V_{n-1}^{(\nu)}\} \geq 1, \qquad \nu \geq \nu_1.$$

Hence if $\nu \geq \nu_1$, every $x \in V_{n-1}^{(\nu)}$ possesses an edge connecting with a point in an upper stratum. In that case, obviously, $V_n^{(\nu)} \neq \emptyset$. ∎

4.3 Condition Equivalent to (A3) and (A4)

For a growing family of regular graphs $\{\mathcal{G}_\nu = (V^{(\nu)}, E^{(\nu)})\}$ consider the following condition:

(A3′) for each n there exists a constant number ω_n independent of ν such that

$$\lim_{\nu} \frac{|\{x \in V_n^{(\nu)}\,;\, \omega_-(x) = \omega_n\}|}{|V_n^{(\nu)}|} = 1. \tag{23}$$

We then come to

Proposition 4.3 *Under (A1), (A2) and (A5), we have equivalence: (A3′) $\Longleftrightarrow$ (A3), (A4).*

PROOF. ($\Longrightarrow$) Divide $V_n^{(\nu)}$ into two parts:

$$U_{\mathrm{reg}}^{(\nu)} = \{x \in V_n^{(\nu)}\,;\, \omega_-(x) = \omega_n\} \qquad U_{\mathrm{sing}}^{(\nu)} = \{x \in V_n^{(\nu)}\,;\, \omega_-(x) \neq \omega_n\},$$

where the index n is omitted for simplicity. The average of $\omega_-(x)$ is given by

$$\begin{aligned}
\omega_n^{(\nu)} &= \frac{1}{|V_n^{(\nu)}|}\sum_{x\in V_n^{(\nu)}}\omega_-(x)\\
&= \frac{1}{|V_n^{(\nu)}|}\sum_{x\in U_{\rm reg}^{(\nu)}}\omega_-(x) + \frac{1}{|V_n^{(\nu)}|}\sum_{x\in U_{\rm sing}^{(\nu)}}\omega_-(x)\\
&= \frac{|U_{\rm reg}^{(\nu)}|}{|V_n^{(\nu)}|}\omega_n + \frac{1}{|V_n^{(\nu)}|}\sum_{x\in U_{\rm sing}^{(\nu)}}\omega_-(x).
\end{aligned}$$

Since $\omega_-(x) \leq W_n$ for $x \in V_n^{(\nu)}$ by (A5), we see that

$$|\omega_n^{(\nu)} - \omega_n| \leq \left(1 - \frac{|U_{\rm reg}^{(\nu)}|}{|V_n^{(\nu)}|}\right)\omega_n + \frac{|U_{\rm sing}^{(\nu)}|}{|V_n^{(\nu)}|}W_n \leq \frac{|U_{\rm sing}^{(\nu)}|}{|V_n^{(\nu)}|}(\omega_n + W_n).$$

Applying Lemma 5.2 and (A3′), we obtain

$$\lim_\nu \frac{|U_{\rm sing}^{(\nu)}|}{|V_n^{(\nu)}|} = 0, \tag{24}$$

and hence

$$\lim_\nu \omega_n^{(\nu)} = \omega_n, \tag{25}$$

which proves (A3). We next consider the variance. By Minkowski's inequality, we obtain

$$\begin{aligned}
\sigma_n^{(\nu)} &= \left\{\frac{1}{|V_n^{(\nu)}|}\sum_{x\in V_n^{(\nu)}}(\omega_-(x) - \omega_n^{(\nu)})^2\right\}^{1/2}\\
&\leq \left\{\frac{1}{|V_n^{(\nu)}|}\sum_{x\in V_n^{(\nu)}}(\omega_-(x) - \omega_n)^2\right\}^{1/2} + \left\{\frac{1}{|V_n^{(\nu)}|}\sum_{x\in V_n^{(\nu)}}(\omega_n - \omega_n^{(\nu)})^2\right\}^{1/2},
\end{aligned}$$

where the first term is estimated by using

$$|\omega_-(x) - \omega_n| \leq \omega_-(x) + \omega_n \leq W_n + \omega_n$$

and the second term is a sum of a constant independent of x. Then

$$\sigma_n^{(\nu)} \leq \left(\frac{|U_{\rm sing}^{(\nu)}|}{|V_n^{(\nu)}|}\right)^{1/2}(W_n + \omega_n) + |\omega_n - \omega_n^{(\nu)}|.$$

Taking (24) and (25) into account, we obtain $\lim_\nu \sigma_n^{(\nu)} = 0$, which is (A4).

($\Longleftarrow$) Let $n \geq 1$ be fixed. By (A3), for any $\epsilon > 0$ there exists ν_0 such that

$$|\omega_n^{(\nu)} - \omega_n| < \epsilon, \quad \nu \geq \nu_0.$$

If $x \in V_n^{(\nu)}$ satisfies $|\omega_-(x) - \omega_n| \geq 2\epsilon$, we have

$$|\omega_-(x) - \omega_n^{(\nu)}| \geq |\omega_-(x) - \omega_n| - |\omega_n - \omega_n^{(\nu)}| \geq \epsilon.$$

Hence

$$\frac{|\{x \in V_n^{(\nu)}\,;\, |\omega_-(x) - \omega_n| \geq 2\epsilon\}|}{|V_n^{(\nu)}|} \leq \frac{|\{x \in V_n^{(\nu)}\,;\, |\omega_-(x) - \omega_n^{(\nu)}| \geq \epsilon\}|}{|V_n^{(\nu)}|}.$$

By Chebyshev's inequality and (A4) we have

$$\frac{|\{x \in V_n^{(\nu)}\,;\, |\omega_-(x) - \omega_n| \geq 2\epsilon\}|}{|V_n^{(\nu)}|} \leq \left(\frac{\sigma_n^{(\nu)}}{\epsilon}\right)^2 \to 0, \qquad \nu \to \infty. \tag{26}$$

We prove that ω_n is an integer. Suppose otherwise. Then, since $\omega_-(x)$ is always an integer, we can choose a sufficiently small $\epsilon > 0$ such that

$$V_n^{(\nu)} = \{x \in V_n^{(\nu)}\,;\, |\omega_-(x) - \omega_n| \geq 2\epsilon\}.$$

But this contradicts (26) and we conclude ω_n to be an integer. Since $\omega_(x)$ and ω_n are all integers, we may choose a sufficiently small $\epsilon > 0$ such that

$$\frac{|\{x \in V_n^{(\nu)}\,;\, \omega_-(x) \neq \omega_n\}|}{|V_n^{(\nu)}|} = \frac{|\{x \in V_n^{(\nu)}\,;\, |\omega_-(x) - \omega_n| \geq 2\epsilon\}|}{|V_n^{(\nu)}|}.$$

As is shown in (26), the right hand side tends to 0 as $\nu \to \infty$. Therefore

$$\lim_{\lambda} \frac{|\{x \in V_n^{(\nu)}\,;\, \omega_-(x) \neq \omega_n\}|}{|V_n^{(\nu)}|} = 0,$$

which proves (23). ∎

4.4 (A4) is Necessary for an Interacting Fock Space in the Limit

Lemma 4.4 *Let $\mathcal{G} = (V, E)$ be a regular graph with stratification $V = \bigcup_{n=0}^{\infty} V_n$ satisfying (A1). Let $A = A^+ + A^-$ be the quantum decomposition of the adjacency matrix. Then for $n \geq 0$ we have*

$$\langle \Phi_n, A^- A^+ \Phi_n \rangle = \|A^+ \Phi_n\|^2 = \frac{|V_{n+1}|}{|V_n|}\left(\omega_{n+1}^2 + \sigma_{n+1}^2\right), \tag{27}$$

$$\langle \Phi_n, A^+ A^- \Phi_n \rangle = \|A^- \Phi_n\|^2 = \frac{|V_{n-1}|}{|V_n|}\left((\kappa - \omega_{n-1})^2 + \sigma_{n-1}^2\right). \tag{28}$$

The proof is a direct computation and is omitted, see also (39), (40).

Proposition 4.5 *Let $\{\mathcal{G}_\nu = (V^{(\nu)}, E^{(\nu)})\}$ be a growing family of regular graphs satisfying (A1)-(A3). Assume that there exists an interacting Fock space $(\Gamma, \{\lambda_n\}, B^+, B^-)$ such that*

$$\lim_\nu \left\langle \Phi_n^{(\nu)}, \frac{A_\nu^-}{\sqrt{\kappa(\nu)}} \frac{A_\nu^+}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle = \langle \Psi_n, B^- B^+ \Psi_n \rangle \tag{29}$$

$$\lim_\nu \left\langle \Phi_n^{(\nu)}, \frac{A_\nu^+}{\sqrt{\kappa(\nu)}} \frac{A_\nu^-}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle = \langle \Psi_n, B^+ B^- \Psi_n \rangle \tag{30}$$

hold for all $n \geq 0$. Then Γ is necessarily infinite dimensional and $\{\mathcal{G}_\nu\}$ fulfills condition (A4).

PROOF. In view of (12) and (13) we obtain

$$\lim_\nu \left\langle \Phi_n^{(\nu)}, \frac{A_\nu^-}{\sqrt{\kappa(\nu)}} \frac{A_\nu^+}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle = \frac{\lambda_{n+1}}{\lambda_n} \tag{31}$$

$$\lim_\nu \left\langle \Phi_n^{(\nu)}, \frac{A_\nu^+}{\sqrt{\kappa(\nu)}} \frac{A_\nu^-}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle = \frac{\lambda_n}{\lambda_{n-1}}. \tag{32}$$

On the other hand, with the help of Lemma 4.4 the left hand sides are written in terms of statistical quantities depending on ν.

We begin with (31). Since

$$\begin{aligned} \frac{\lambda_{n+1}}{\lambda_n} &= \lim_\nu \left\langle \Phi_n^{(\nu)}, \frac{A_\nu^-}{\sqrt{\kappa(\nu)}} \frac{A_\nu^+}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle \\ &= \lim_\nu \frac{|V_{n+1}^{(\nu)}|}{\kappa(\nu)|V_n^{(\nu)}|} \left(\omega_{n+1}^{(\nu)2} + \sigma_{n+1}^{(\nu)2} \right), \end{aligned} \tag{33}$$

applying Lemma 5.1 and conditions (A2), (A3), we obtain

$$\lim_\nu \frac{|V_{n+1}^{(\nu)}|}{\kappa(\nu)|V_n^{(\nu)}|} = \lim_\nu \frac{1}{\omega_{n+1}^{(\nu)}} \left(1 - \frac{\omega_n^{(\nu)}}{\kappa(\nu)} \right) = \frac{1}{\omega_{n+1}}. \tag{34}$$

Recall that $\omega_n \geq 1$ for all $n \geq 1$, see Proposition 4.1. Then (33) becomes

$$\frac{\lambda_{n+1}}{\lambda_n} = \omega_{n+1} + \frac{1}{\omega_{n+1}} \lim_\nu \sigma_{n+1}^{(\nu)2}, \qquad n \geq 0, \tag{35}$$

which guarantees also that the limit $\lim_\nu \sigma_{n+1}^{(\nu)}$ exists. Moreover, it is clear that $\lambda_n > 0$ for all $n \geq 0$. Namely, Γ is of infinite dimension.

We next consider (32). In a similar manner as above, we see from Lemma 5.1 that

$$\begin{aligned}\frac{\lambda_n}{\lambda_{n-1}} &= \lim_\nu \left\langle \Phi_n^{(\nu)}, \frac{A_\nu^+}{\sqrt{\kappa(\nu)}} \frac{A_\nu^-}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle \\ &= \lim_\nu \frac{|V_{n-1}^{(\nu)}|}{\kappa(\nu)|V_n^{(\nu)}|} \left((\kappa(\nu) - \omega_{n-1}^{(\nu)})^2 + \sigma_{n-1}^{(\nu)2} \right) \\ &= \lim_\nu \frac{\kappa(\nu)|V_{n-1}^{(\nu)}|}{|V_n^{(\nu)}|} \left\{ \left(1 - \frac{\omega_{n-1}^{(\nu)}}{\kappa(\nu)}\right)^2 + \left(\frac{\sigma_{n-1}^{(\nu)}}{\kappa(\nu)}\right)^2 \right\}.\end{aligned}$$

Since both $\lim_\nu \omega_{n-1}^{(\nu)}$ and $\lim_\nu \sigma_{n-1}^{(\nu)}$ are convergent, we see from (34) that

$$\frac{\lambda_n}{\lambda_{n-1}} = \omega_n, \qquad n \geq 1. \tag{36}$$

Finally, combining (35) and (36), we obtain $\lim_\nu \sigma_n^{(\nu)} = 0$ for $n \geq 1$. ∎

5 Proof of Theorem 1.1

5.1 Estimate of Strata

Let $\mathcal{G} = (V, E)$ be a κ-regular graph with stratification $V = \bigcup_{n=0}^\infty V_n$. We assume (A1) is satisfied, namely there is no edge lying in a stratum.

Lemma 5.1 *Let $n \geq 0$ and assume $V_n \neq \emptyset$. Then,*

$$\omega_{n+1}|V_{n+1}| = \kappa|V_n| \left(1 - \frac{\omega_n}{\kappa}\right). \tag{37}$$

PROOF. Suppose first that $V_{n+1} \neq \emptyset$. The number of edges whose endpoints lying in V_n is $\kappa|V_n|$. Dividing these edges into two parts, we have

$$\begin{aligned}\kappa|V_n| &= \sum_{x \in V_n} \omega_+(x) + \sum_{x \in V_n} \omega_-(x) \\ &= \sum_{y \in V_{n+1}} \omega_-(y) + \sum_{x \in V_n} \omega_-(x) = \omega_{n+1}|V_{n+1}| + \omega_n|V_n|.\end{aligned} \tag{38}$$

This proves the assertion. (37) is valid also for $n = 0$ since we have put $\omega_0 = 0$. If the stratification terminates at finite steps, say, $V = V_0 \cup V_1 \cup \cdots \cup V_n$ and $V_{n+1} = \emptyset$, then ω_{n+1} is not defined but at a tacit understanding $\omega_{n+1}|V_{n+1}| = 0$ we have (38). ∎

Lemma 5.2 *Let $n \geq 1$ and assume that $V_n \neq \emptyset$. Then, $\omega_1 \geq 1, \ldots, \omega_n \geq 1$ and*

$$|V_n| = \frac{\kappa^n}{\omega_1 \ldots \omega_n} + O(\kappa^{n-1}),$$

where $O(\kappa^{n-1})$ is a polynomial in κ of degree $(n-1)$.

PROOF. An immediate consequence from Lemma 37. ∎

5.2 Estimate of Error Terms

In this subsection too we fix ν so that this suffix is omitted for notational simplicity.

Explicit actions of $A^{\pm}$ on the number vectors follow directly from definitions (3) and (10). We have

$$\frac{A^+}{\sqrt{\kappa}} \Phi_n = \omega_{n+1} \left(\frac{|V_{n+1}|}{\kappa |V_n|} \right)^{1/2} \Phi_{n+1} + \frac{1}{(\kappa |V_n|)^{1/2}} \sum_{y \in V_{n+1}} (\omega_-(y) - \omega_{n+1}) \delta_y, \quad n \geq 0, \tag{39}$$

$$\frac{A^-}{\sqrt{\kappa}} \Phi_n = \left(1 - \frac{\omega_{n-1}}{\kappa}\right) \left(\frac{\kappa |V_{n-1}|}{|V_n|} \right)^{1/2} \Phi_{n-1} + \frac{1}{(\kappa |V_n|)^{1/2}} \sum_{z \in V_{n-1}} (\omega_{n-1} - \omega_-(z)) \delta_z, \quad n \geq 1, \tag{40}$$

$$\frac{A^-}{\sqrt{\kappa}} \Phi_0 = 0. \tag{41}$$

In order to express the above actions in a unified manner we need some notation:

$$\gamma_n^+ = \omega_n \left(\frac{|V_n|}{\kappa |V_{n-1}|} \right)^{1/2}, \qquad n \geq 1, \tag{42}$$

$$\gamma_n^- = \left(1 - \frac{\omega_n}{\kappa}\right) \left(\frac{\kappa |V_n|}{|V_{n+1}|} \right)^{1/2}, \qquad n \geq 0, \tag{43}$$

$$S_n^+ = \frac{1}{(\kappa |V_{n-1}|)^{1/2}} \sum_{y \in V_n} (\omega_-(y) - \omega_n) \delta_y, \qquad n \geq 1, \tag{44}$$

$$S_n^- = \frac{1}{(\kappa |V_{n+1}|)^{1/2}} \sum_{z \in V_n} (\omega_n - \omega_-(z)) \delta_z, \qquad n \geq 0. \tag{45}$$

Then, (39) and (40) are unified as follows:

$$\frac{A^\epsilon}{\sqrt{\kappa}}\Phi_n = \gamma^\epsilon_{n+\epsilon}\Phi_{n+\epsilon} + S^\epsilon_{n+\epsilon}, \qquad \epsilon = \pm, \quad n \ge 0, \tag{46}$$

where $n+\epsilon$ stands for $n \pm 1$ according as $\epsilon = \pm$. Setting

$$\gamma^-_{-1}\Phi_{-1} = S^-_{-1} = 0,$$

we can involve (41) in (46) too.

We next consider repeated action of $A^\pm$. Suppose we are given $m \ge 1$ and $\epsilon_1, \dots, \epsilon_m \in \{\pm\}$. Then, applying (46) repeatedly, we obtain

$$\begin{aligned}\frac{A^{\epsilon_m}}{\sqrt{\kappa}}\cdots\frac{A^{\epsilon_1}}{\sqrt{\kappa}}\Phi_n &= \gamma^{\epsilon_1}_{n+\epsilon_1}\gamma^{\epsilon_2}_{n+\epsilon_1+\epsilon_2}\cdots\gamma^{\epsilon_m}_{n+\epsilon_1+\dots+\epsilon_m}\Phi_{n+\epsilon_1+\dots+\epsilon_m} \\ &\quad + \sum_{k=1}^{m}\gamma^{\epsilon_1}_{n+\epsilon_1}\cdots\gamma^{\epsilon_{k-1}}_{n+\epsilon_1+\dots+\epsilon_{k-1}}\frac{A^{\epsilon_m}}{\sqrt{\kappa}}\cdots\frac{A^{\epsilon_{k+1}}}{\sqrt{\kappa}}S^{\epsilon_k}_{n+\epsilon_1+\dots+\epsilon_k}.\end{aligned} \tag{47}$$

Here we assumed that

$$n+\epsilon_1 \ge 0, \quad n+\epsilon_1+\epsilon_2 \ge 0, \quad \dots, \quad n+\epsilon_1+\epsilon_2+\dots+\epsilon_m \ge 0. \tag{48}$$

If a negative number appears among the above, we have

$$\frac{A^{\epsilon_m}}{\sqrt{\kappa}}\cdots\frac{A^{\epsilon_1}}{\sqrt{\kappa}}\Phi_n = 0.$$

In fact, let k be the first number such that $n+\epsilon_1+\epsilon_2+\dots+\epsilon_k < 0$. Then $n+\epsilon_1+\epsilon_2+\dots+\epsilon_{k-1} = 0$ and $\epsilon_k = -$, and hence $A^{\epsilon_{k-1}}\dots A^{\epsilon_1}\Phi_n$ is a constant multiple of Φ_0 and $A^{\epsilon_k}A^{\epsilon_{k-1}}\dots A^{\epsilon_1}\Phi_n = 0$.

We must estimate of the error term of (47). For $k \ge 1$ we set

$$W_k = \max\{\omega_-(x)\,;\, x \in V_k\}. \tag{49}$$

Obviously, $W_k \le \kappa$. Then, for $n \ge 1$ and $q \ge 0$ we define $M_{n,q}$ by

$$M_{n,q} = \begin{cases}\max\{W_{k_1}W_{k_2}\dots W_{k_q}\,;\, 1 \le k_1, k_2, \dots, k_q \le n\}, & q \ge 1, \\ 1, & q = 0.\end{cases} \tag{50}$$

Lemma 5.3 *Let $\epsilon_1, \dots, \epsilon_m \in \{\pm\}$, $m \ge 1$, be given arbitrarily. Let p and q be the numbers of $+$ and $-$ in $\{\epsilon_1, \dots, \epsilon_m\}$, respectively. Then for any $n \ge 1$ with $n+p-q \ge 0$ we have*

$$\left|\left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}}\cdots\frac{A^{\epsilon_1}}{\sqrt{\kappa}}S^+_n\right\rangle\right| \le \sigma_n M_{n+p,q}\left(\frac{\kappa^{2p-m}|V_n|}{|V_{n+p-q}|}\right)^{1/2}\left(\frac{|V_n|}{\kappa|V_{n-1}|}\right)^{1/2}. \tag{51}$$

PROOF. It is sufficient to prove the assertion under (48). If otherwise, the left hand side of (51) vanishes and the assertion is trivial. Note first

$$\frac{A^{\epsilon_m}}{\sqrt{\kappa}}\cdots\frac{A^{\epsilon_1}}{\sqrt{\kappa}}S_n^+ = \frac{1}{(\kappa|V_{n-1}|)^{1/2}}\sum_{y\in V_n}(\omega_-(y)-\omega_n)\frac{A^{\epsilon_m}}{\sqrt{\kappa}}\cdots\frac{A^{\epsilon_1}}{\sqrt{\kappa}}\delta_y$$
$$= \frac{\kappa^{-m/2}}{(\kappa|V_{n-1}|)^{1/2}}\sum_{y\in V_n}(\omega_-(y)-\omega_n)A^{\epsilon_m}\cdots A^{\epsilon_1}\delta_y. \qquad (52)$$

We use a new notation. According as $\epsilon = \pm$, we set

$$y \xrightarrow{\epsilon} z = \begin{cases} y \prec z & \epsilon = + \\ y \succ z & \epsilon = -. \end{cases}$$

For $y, z \in V$ we put

$$w(y;\epsilon_1,\ldots,\epsilon_m;z)$$
$$= |\{(z_1,\ldots,z_{m-1}) \in V^{m-1}\,;\, y \xrightarrow{\epsilon_1} z_1 \xrightarrow{\epsilon_2} z_2 \cdots \xrightarrow{\epsilon_{m-1}} z_{m-1} \xrightarrow{\epsilon_m} z\}|.$$

This counts the walks from y to z along edges with directions $\epsilon_1,\ldots,\epsilon_m$. Then (52) becomes

$$\frac{A^{\epsilon_m}}{\sqrt{\kappa}}\cdots\frac{A^{\epsilon_1}}{\sqrt{\kappa}}S_n^+$$
$$= \frac{\kappa^{-m/2}}{(\kappa|V_{n-1}|)^{1/2}}\sum_{y\in V_n}\sum_{z\in V_{n+p-q}}(\omega_-(y)-\omega_n)w(y;\epsilon_1,\ldots,\epsilon_m;z)\delta_z.$$

Therefore,

$$\left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}}\cdots\frac{A^{\epsilon_1}}{\sqrt{\kappa}}S_n^+\right\rangle = \frac{1}{|V_{n+p-q}|^{1/2}}\frac{\kappa^{-m/2}}{(\kappa|V_{n-1}|)^{1/2}}$$
$$\times\sum_{y\in V_n}\sum_{z\in V_{n+p-q}}(\omega_-(y)-\omega_n)w(y;\epsilon_1,\ldots,\epsilon_m;z). \qquad (53)$$

Let $y \in V_n$ be fixed. Then

$$\sum_{z\in V_{n+p-q}} w(y;\epsilon_1,\ldots,\epsilon_m;z) \qquad (54)$$

coincides with the number of walks from y to a certain point along edges with directions $\epsilon_1,\ldots,\epsilon_m$. Consider an intermediate point $z \in V_k$ in such a walk. The number of edges from z with $+$ direction is given by $\kappa - \omega_-(z)$, which is bounded by κ uniformly. On the other hand, the edges with $-$ direction is given by $\omega_-(z)$ and bounded by W_k as defined in (49). Thus (54) is obtained by a product of such numbers. Remind that + direction appears p times, − direction q times, and any walk starting from $y \in V_n$ to a certain point along edges with direction $\epsilon_1, \dots, \epsilon_m$ contained in $V_0 \cup V_1 \cup \cdots \cup V_{n+p}$. Then, by using (50) we have

$$\sum_{z \in V_{n+p-q}} w(y; \epsilon_1, \dots, \epsilon_m; z) \leq \kappa^p M_{n+p,q}.$$

The right hand side is independent of $y \in V_n$. Now we come to an estimate of (53). In fact,

$$\begin{aligned}
&\left| \left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S_n^+ \right\rangle \right| \\
&\leq \frac{\kappa^p M_{n+p,q}}{|V_{n+p-q}|^{1/2}} \frac{\kappa^{-m/2}}{(\kappa |V_{n-1}|)^{1/2}} \sum_{y \in V_n} |\omega_-(y) - \omega_n| \\
&\leq \frac{\kappa^p M_{n+p,q}}{|V_{n+p-q}|^{1/2}} \frac{\kappa^{-m/2}}{(\kappa |V_{n-1}|)^{1/2}} \left(\sum_{y \in V_n} |\omega_-(y) - \omega_n|^2 \right)^{1/2} |V_n|^{1/2} \\
&= \sigma_n M_{n+p,q} \left(\frac{\kappa^{2p-m} |V_n|}{|V_{n+p-q}|} \right)^{1/2} \left(\frac{|V_n|}{\kappa |V_{n-1}|} \right)^{1/2}.
\end{aligned}$$

This proves inequality (51). ∎

Lemma 5.4 *Let $\epsilon_1, \dots, \epsilon_m \in \{\pm\}$, $m \geq 1$, be given arbitrarily. Let p and q be the numbers of + and − among $\{\epsilon_1, \dots, \epsilon_m\}$, respectively. Then for any $n \geq 0$ with $n + p - q \geq 0$,*

$$\begin{aligned}
&\left| \left\langle \Phi_{n+p-q}, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S_n^- \right\rangle \right| \\
&\qquad \leq \sigma_n M_{n+p,q} \left(\frac{\kappa^{2p-m} |V_n|}{|V_{n+p-q}|} \right)^{1/2} \left(\frac{|V_n|}{\kappa |V_{n+1}|} \right)^{1/2}. \qquad (55)
\end{aligned}$$

PROOF. By definition (45) we have $S_0^- = 0$. Hence for $n = 0$ the left hand side of (55) vanishes and the assertion obviously holds. Suppose $n \geq 1$. We note the identity:

$$S_n^- = -\left(\frac{|V_{n-1}|}{|V_{n+1}|} \right)^{1/2} S_n^+, \qquad n \geq 1,$$

which is verified by (44) and (45). Then (55) follows from (51). ∎

5.3 Proof of Theorem 1.1

When we consider a growing family of graphs $\mathcal{G}_\nu$, quantities introduced in the previous subsection depend on the growing parameter ν.

Inserting (47) into the left hand side of (5), one obtains:

$$\begin{aligned}
&\left\langle \Phi_j^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle \\
&= \gamma_{n+\epsilon_1}^{\epsilon_1} \gamma_{n+\epsilon_1+\epsilon_2}^{\epsilon_2} \cdots \gamma_{n+\epsilon_1+\cdots+\epsilon_m}^{\epsilon_m} \left\langle \Phi_j^{(\nu)}, \Phi_{n+\epsilon_1+\cdots+\epsilon_m}^{(\nu)} \right\rangle \\
&+ \sum_{k=1}^{m} \gamma_{n+\epsilon_1}^{\epsilon_1} \cdots \gamma_{n+\epsilon_1+\cdots+\epsilon_{k-1}}^{\epsilon_{k-1}} \left\langle \Phi_j^{(\nu)}, \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_{k+1}}}{\sqrt{\kappa(\nu)}} S_{n+\epsilon_1+\cdots+\epsilon_k}^{\epsilon_k} \right\rangle . \qquad (56)
\end{aligned}$$

The coefficients γ_n^ϵ depends on ν. Explicit expressions of γ_n^ϵ being given in (42) and (43), with the help of Lemma 5.1 and condition (A3) we come to

$$\lim_\nu \gamma_n^+ = \sqrt{\omega_n}, \qquad \lim_\nu \gamma_n^- = \sqrt{\omega_{n+1}}. \qquad (57)$$

Therefore, in order to prove that the second term of (56) vanishes as $\nu \to \infty$ it is sufficient to show that

$$\lim_\nu \left\langle \Phi_j^{(\nu)}, \frac{A^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A^{\epsilon_{k+1}}}{\sqrt{\kappa(\nu)}} S_{n+\epsilon_1+\cdots+\epsilon_k}^{\epsilon_k} \right\rangle = 0. \qquad (58)$$

By Lemmas 5.3 and 5.4 we have

$$\left| \left\langle \Phi_j, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S_n^+ \right\rangle \right| \le \sigma_n M_{n+p,q} \left(\frac{\kappa^{2p-m}|V_n|}{|V_{n+p-q}|} \right)^{1/2} \left(\frac{|V_n|}{\kappa |V_{n-1}|} \right)^{1/2}, \qquad (59)$$

$$\left| \left\langle \Phi_j, \frac{A^{\epsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\epsilon_1}}{\sqrt{\kappa}} S_n^- \right\rangle \right| \le \sigma_n M_{n+p,q} \left(\frac{\kappa^{2p-m}|V_n|}{|V_{n+p-q}|} \right)^{1/2} \left(\frac{|V_n|}{\kappa |V_{n+1}|} \right)^{1/2}. \qquad (60)$$

(The left hand side vanishes unless $j = n + p - q$.) The constant numbers in the right hand sides depend on ν. We examine them one by one.

By (A5) we have

$$\sup_\nu M_{n+p,q}^{(\nu)} < \infty.$$

It follows from Lemma 5.2 that

$$\frac{\kappa^{2p-m}|V_n|}{|V_{n+p-q}|} = O(\kappa^{2p-m+n-(n+p-q)}) = O(\kappa^{p+q-m}) = O(1).$$

Since $O(1)$ is uniform in ν by (A3),

$$\sup_{\nu} \frac{\kappa(\nu)^{2p-m}|V_n^{(\nu)}|}{|V_{n+p-q}^{(\nu)}|} < \infty.$$

Similarly,

$$\sup_{\nu} \frac{|V_n^{(\nu)}|}{\kappa(\nu)|V_{n-1}^{(\nu)}|} < \infty, \qquad \lim_{\nu} \frac{|V_n^{(\nu)}|}{\kappa(\nu)|V_{n+1}^{(\nu)}|} = 0.$$

Therefore, the right hand sides of (59) and (60) vanish by (A4) in the limit as $\nu \to \infty$. Consequently, only the first term of (56) contributes to the limit and we come to

$$\begin{aligned} &\lim_{\nu} \left\langle \Phi_j^{(\nu)}, \frac{A_\nu^{\epsilon_1}}{\sqrt{\kappa(\nu)}} \cdots \frac{A_\nu^{\epsilon_m}}{\sqrt{\kappa(\nu)}} \Phi_n^{(\nu)} \right\rangle \\ &\quad = \lim_{\nu} \gamma_{n+\epsilon_1}^{\epsilon_1} \gamma_{n+\epsilon_1+\epsilon_2}^{\epsilon_2} \cdots \gamma_{n+\epsilon_1+\cdots+\epsilon_m}^{\epsilon_m} \delta_{j,n+\epsilon_1+\cdots+\epsilon_m}. \end{aligned} \tag{61}$$

Using (57) we come to the final form, which is equal to

$$\langle \Phi_j, B^{\epsilon_m} \ldots B^{\epsilon_1} \Phi_n \rangle.$$

The verification is straightforward by definition of the interacting Fock space $(\Gamma, \{\lambda_n\}, B^+, B^-)$. Thus we have completed the proof.

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