# Partial Unitarity Arising from Quadratic Quantum White Noise


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**Abstract**

In general, the solution to a normal-ordered white noise differential equation involving quadratic quantum white noise is a white noise operator and is not an operator acting in the $L^2$-space over the original Gaussian space where the quantum white noise is defined. The solution happens to be a unitary operator on a certain subspace of the $L^2$-space over a Gaussian space with different variance. This regularity property is referred to as *partial unitarity.*


## 1 Introduction

Given a quantum stochastic process $\{L_t\}$, we consider a normal-ordered white noise differential equation

$$\frac{d\Xi_t}{dt} = L_t \diamond \Xi_t, \qquad \Xi\big|_{t=0} = \Xi_0, \tag{1}$$

where $\diamond$ is the Wick product (or normal-ordered product). Roughly speaking, the unique solution is always found in a space of white noise operators, suitably chosen according to the coefficient $\{L_t\}$ and the initial value $\Xi_0$, see e.g., Chung–Ji–Obata [4] and Ji–Obata [6]. Let $\{a_t, a_t^*\}$ be the quantum white noise. If $L_t$ is a linear combination of $\{a_t^* a_t, a_t, a_t^*, 1\}$, the equation (1) is reduced essentially to a usual quantum stochastic differential equation for which the quantum Itô theory works well, see Parthasarathy [15]. As is well known, the higher powers of quantum white noise have rather singular nature but are well formulated in quantum white noise theory. The case when $\{L_t\}$ involves a quadratic quantum white noise $\{a_t^2, a_t^{*2}\}$ is a non-trivial step going beyond the traditional quantum Itô theory and the regularity properties of the solution are of great interest. Recall also that the quadratic quantum white noise is related to the Lévy Laplacian, see Ji–Obata–Ouerdiane [9] and Obata [14].

This paper is devoted to one of the simplest cases. We consider

$$\frac{d\Xi}{dt} = \gamma\, a_t^2 \diamond \Xi, \qquad \Xi\big|_{t=0} = \mathcal{H}_{a,b}\,, \tag{2}$$

where $\gamma, a, b \in \mathbb{C}$ are constant numbers and $\mathcal{H}_{a,b}$ a Fourier–Gauss transform. In general, the solution is merely a white noise operator. We shall prove that the solution happens to be unitary on a certain subspace of $L^2$-space over a Gaussian space whose variance is different from the one of the original space where the quantum white noise is defined. This property is called *partial unitarity.* Our result is relevant to unitarity of a (generalized) Fourier–Gauss transform investigated by Ji–Obata [7, 8]. The main results will be stated in Section 5.

There are different approaches to the quadratic quantum white noise, see e.g., Accardi–Amosov–Franz [1], Accardi–Franz–Skeide [2], Lytvynov [12], and references cited therein.

## 2 Generalized Fourier–Gauss Transforms

We adopt mostly the same notations as in [7]. Let us start with a real Gelfand triple

$$\mathcal{N} = \mathcal{S}(\mathbb{R}) \subset H = L^2(\mathbb{R}, dt) \subset \mathcal{N}^* = \mathcal{S}'(\mathbb{R}), \tag{3}$$

where $H = L^2(\mathbb{R})$ is the Hilbert space of $\mathbb{R}$-valued square-integrable functions on the real line $\mathbb{R}$ with respect to the Lebesgue measure $dt$, $\mathcal{S}(\mathbb{R})$ the space of rapidly decreasing functions and $\mathcal{S}'(\mathbb{R})$ the space of tempered distributions. The canonical bilinear form on $\mathcal{N}^* \times \mathcal{N}$ is denoted by $\langle \cdot, \cdot \rangle$, which is compatible with the inner product of $H$. By the same symbol we denote the canonical $\mathbb{C}$-bilinear form on $\mathcal{N}_{\mathbb{C}}^* \times \mathcal{N}_{\mathbb{C}}$, where the suffix means the complexification.

With $a \in \mathbb{C}$ and $\xi \in \mathcal{N}_{\mathbb{C}}$ we associate a continuous function $\phi_{a,\xi}$ on $\mathcal{N}^*$ defined by

$$\phi_{a,\xi}(x) = e^{\langle x,\xi \rangle - a\langle \xi,\xi \rangle/2}, \quad x \in \mathcal{N}^*, \tag{4}$$

which we call a *coherent vector* or an *exponential vector*. Let $\mathcal{E}$ be the linear space spanned by $\{\phi_{a,\xi} \, ; \, \xi \in \mathcal{N}_{\mathbb{C}}\}$. Due to the obvious relation

$$\phi_{a,\xi} = e^{(a'-a)\langle \xi,\xi \rangle/2} \phi_{a',\xi}, \qquad a, a' \in \mathbb{C}, \quad \xi \in \mathcal{N}_{\mathbb{C}}, \tag{5}$$

the space $\mathcal{E}$ does not depend on the choice of $a \in \mathbb{C}$. In general, two locally convex spaces $\mathcal{X}, \mathcal{Y}$ we denote by $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ the space of continuous operators equipped with the bounded convergence topology. In the next, we will use such space for $X$ and $Y$ are equal to $\mathcal{N}_{\mathbb{C}}$ or $\mathcal{W}$ or their dual spaces.

With a pair $A \in \mathcal{L}(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}}^*)$ and $B \in \mathcal{L}(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}})$ we associate an operator $\mathcal{G}(A, B)$ on $\mathcal{E}$ defined by

$$\mathcal{G}(A, B)\phi_{1,\xi} = e^{\langle A\xi,\xi \rangle/2} \phi_{1,B\xi} \, , \qquad \xi \in \mathcal{N}_{\mathbb{C}} \, .$$

The above formula is sufficient to define a linear operator on $\mathcal{E}$ since the exponential vectors $\{\phi_{a,\xi} \, ; \, \xi \in \mathcal{N}_{\mathbb{C}}\}$ are linearly independent. The operator $\mathcal{G}(A, B)$ is called a *generalized Fourier–Gauss transform.* Our definition is due to Chung–Ji [3], while an equivalent definition is given by Lee–Liu [11] in terms of an integral formula.

**Lemma 2.1** (1) $\mathcal{G}(A_1, B_1)\mathcal{G}(A_2, B_2) = \mathcal{G}(B_2^* A_1 B_2 + A_2, B_1 B_2)$.

(2) *$\mathcal{G}(A, B) = 1$ (the identity operator on $\mathcal{E}$) if and only if $A = 0$ and $B = 1$ (the identity operator on $\mathcal{N}_{\mathbb{C}}$).*

(3) *$\mathcal{G}(A, B)$ is invertible if and only if so is $B$, i.e., $B \in GL(\mathcal{N}_{\mathbb{C}})$. In that case, $\mathcal{G}(A, B)^{-1} = \mathcal{G}(-(B^{-1})^* A B^{-1}, B^{-1})$.*

The proof is immediate from definition. In particular,

$$\begin{aligned} \mathfrak{G} &\equiv \{\mathcal{G}(A, B) \, ; \, A \in \mathcal{L}(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}}^*), \; B \in GL(\mathcal{N}_{\mathbb{C}})\} \\ &\cong \mathcal{L}(\mathcal{N}_{\mathbb{C}}, \mathcal{N}_{\mathbb{C}}^*) \rtimes GL(\mathcal{N}_{\mathbb{C}}) \end{aligned}$$

becomes a group of linear automorphisms of $\mathcal{E}$. If both $A, B$ are scalar operators, say, $A = \alpha 1$ and $B = \beta 1$, we write simply $\mathcal{G}(\alpha, \beta)$ and is called a *Fourier–Gauss transform*. We have

$$\mathcal{G}(\alpha, \beta)\phi_{1,\xi} = e^{\alpha\langle\xi,\xi\rangle/2}\phi_{1,\beta\xi}\,, \qquad \xi \in \mathcal{N}_{\mathbb{C}}\,. \tag{6}$$

We naturally comes to a subgroup of $\mathfrak{G}$:

$$\mathfrak{G}_0 = \{\mathcal{G}(\alpha, \beta)\,;\, \alpha \in \mathbb{C},\ \beta \in \mathbb{C}^\times\} \cong \mathbb{C} \rtimes \mathbb{C}^\times,$$

where $\mathbb{C}^\times$ is the multiplicative group of non-zero complex numbers.

For later use we define one-parameter subgroups of $\mathfrak{G}_0$. First, for $a \in \mathbb{C}$ we define

$$T_a = \mathcal{G}(a, 1) \,:\, \phi_{1,\xi} \mapsto e^{a\langle\xi,\xi\rangle/2}\phi_{1,\xi}\,, \qquad \xi \in \mathcal{N}_{\mathbb{C}}.$$

It follows immediately that

$$T_a T_{a'} = T_{a+a'}\,, \qquad T_a^{-1} = T_{-a}\,.$$

Moreover, by a straightforward computation we obtain

$$T_{a-1+2b} : \phi_{a,\xi} \mapsto e^{b\langle\xi,\xi\rangle}\phi_{1,\xi}\,, \qquad \xi \in \mathcal{N}_{\mathbb{C}}\,, \quad a, b \in \mathbb{C}. \tag{7}$$

Next, for $a, b \in \mathbb{C}$ let $\mathcal{H}_{a,b}$ be a linear operator on $\mathcal{E}$ by

$$\mathcal{H}_{a,b} : \phi_{a,\xi} \mapsto \phi_{a,b\xi}, \qquad \xi \in \mathcal{N}_{\mathbb{C}}. \tag{8}$$

Obviously,

$$\mathcal{H}_{a,b}\mathcal{H}_{a,b'} = \mathcal{H}_{a,bb'}, \qquad \mathcal{H}_{a,b}^{-1} = \mathcal{H}_{a,b^{-1}} \text{ for } b \neq 0.$$

On the other hand, by straightforward computation we obtain

$$\mathcal{H}_{a,b}\phi_{1,\xi} = e^{(a-1)(1-b^2)\langle\xi,\xi\rangle/2}\,\phi_{1,b\xi}\,,$$

which reads

$$\mathcal{H}_{a,b} = \mathcal{G}((a-1)(1-b^2), b). \tag{9}$$

# 3 Unitarity

The *Gaussian measure* with variance $a > 0$ is a probability measure $\mu_a$ on $\mathcal{N}^*$ uniquely specified by

$$e^{-a\langle\xi,\xi\rangle/2} = \int_{\mathcal{N}^*} e^{i\langle x,\xi\rangle}\mu_a(dx), \qquad \xi \in \mathcal{N}.$$

Then we have

$$\langle\langle \phi_{a,\xi}, \phi_{a,\eta}\rangle\rangle_{\mu_a} \equiv \int_{\mathcal{N}^*} \phi_{a,\xi}(x)\phi_{a,\eta}(x)\,\mu_a(dx) = e^{a\langle\xi,\eta\rangle}, \quad \xi,\eta \in \mathcal{N}_{\mathbb{C}}\,. \tag{10}$$

**Lemma 3.1** *For $a > 0$ and $|b| = 1$, the linear automorphism $\mathcal{H}_{a,b}$ of $\mathcal{E}$ extends uniquely to a unitary operator $\tilde{\mathcal{H}}_{a,b}$ on $L^2(\mathcal{N}^*, \mu_a)$.*

PROOF. Note that the inner product of $L^2(\mathcal{N}^*, \mu_a)$ is defined by $\langle\langle \bar{f}, g\rangle\rangle$. Since $\mathcal{E} \subset L^2(\mathcal{N}^*, \mu_a)$ is a dense subspace, it is sufficient to show that

$$\langle\langle \overline{\mathcal{H}_{a,b}f}, \mathcal{H}_{a,b}g\rangle\rangle_{\mu_a} = \langle\langle \overline{f}, g\rangle\rangle_{\mu_a}\,, \qquad f,g \in \mathcal{E}.$$

Verification of the above identity is straightforward from (10). ∎

Let $I \subset \mathbb{R}$ be a closed (finite or infinite) interval. We denote by $\mathcal{E}_I$ the subspace of $\mathcal{E}$ spanned by $\{\phi_{a,\xi}\,;\, \xi \in \mathcal{N}_{\mathbb{C}},\ \mathrm{supp}\,\xi \subset I\}$. By (5), $\mathcal{E}_I$ does not depend on the choice of $a \in \mathbb{C}$ either. In view of the action (6), we are ready to claim the following

**Lemma 3.2** *Each $\mathcal{G}(\alpha,\beta) \in \mathfrak{G}_0$ induces a linear automorphism of $\mathcal{E}_I$. In particular, so is $\mathcal{H}_{a,b}$ for any pair $a,b \in \mathbb{C}$ with $b \neq 0$.*

For an interval $I$ let $1_I$ denote the indicator function. The associated multiplication operator is denoted by the same symbol. For $a > 0$ we define a linear map $E_I^a$ from $\mathcal{E}$ into $L^2(\mathcal{N}^*, \mu_a)$ by

$$E_I^a : \phi_{a,\xi} \mapsto \phi_{a,1_I\xi}, \qquad \xi \in \mathcal{N}_{\mathbb{C}}.$$

It is shown that $E_I^a$ extends to a projection on $L^2(\mathcal{N}^*, \mu_a)$, which is denoted by the same symbol. The image of this projection will be denoted by $L^2(\mu_a|\,I)$. It is noted that $\mathcal{E}_I$ is a dense subspace of $L^2(\mu_a|\,I)$.

Now we may state a generalization of Lemma 3.1, the proof of which is similar. We only need to note that $\tilde{\mathcal{H}}_{a,b}$ commutes with the projection $E_I^a$.

**Lemma 3.3** *Let $I \subset \mathbb{R}$ be a closed interval and $a,b \in \mathbb{C}$ a pair of complex numbers with $a > 0$ and $|b| = 1$. Then the linear automorphism $\mathcal{H}_{a,b} \restriction \mathcal{E}_I$ extends uniquely to a unitary operator on $L^2(\mu_a|I)$, which coincides with $\tilde{\mathcal{H}}_{a,b} \restriction L^2(\mu_a|I)$.*

# 4 White Noise Operators

We take a white noise triple

$$\mathcal{W} \subset \Gamma(H_{\mathbb{C}}) \cong L^2(\mathcal{N}^*, \mu_1) \subset \mathcal{W}^* \tag{11}$$

constructed in the standard manner [5, 6, 10, 13]. Recall that $\Gamma(H_{\mathbb{C}})$ is the Boson Fock space over $H_{\mathbb{C}}$ which is canonically identified with $L^2(\mathcal{N}^*, \mu_1)$ through the Wiener–Itô–Segal isomorphism. For instance, we may take the Hida–Kubo–Takenaka space for (11). The canonical $\mathbb{C}$-bilinear form on $\mathcal{W}^* \times \mathcal{W}$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$.

In general, a continuous operator from $\mathcal{W}$ into $\mathcal{W}^*$ is called a *white noise operator*. Since the canonical injection $\mathcal{W} \to \mathcal{W}^*$ is continuous, we have a natural inclusion $\mathcal{L}(\mathcal{W}, \mathcal{W}) \subset \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. By simple application of the famous characterization of operator symbols [6, 13] we see that every generalized Fourier–Gauss transform $\mathcal{G}(A, B)$ extends uniquely to a white noise operator in $\mathcal{L}(\mathcal{W}, \mathcal{W})$. In fact, the symbol is given by

$$\langle\langle \mathcal{G}(A, B)\phi_{1,\xi}, \phi_{1,\eta} \rangle\rangle = \exp\left\{ \frac{1}{2}\langle A\xi, \xi\rangle + \langle B\xi, \eta\rangle \right\}, \qquad \xi, \eta \in \mathcal{N}_{\mathbb{C}},$$

so the check is straightforward. The continuous extension is also called a generalized Fourier–Gauss transform and is denoted by the same symbol. Moreover, we note the following

**Proposition 4.1** *Every $\mathcal{G}(A, B) \in \mathfrak{G}$ is a topological linear automorphism of $\mathcal{W}$. In this sense $\mathfrak{G}$ is a subgroup of $GL(\mathcal{W})$.*

We say that $\{L_t\,;\, t \in \mathbb{R}\}$ is a *quantum stochastic process* if $t \mapsto L_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is continuous. Let $a_t$ and $a_t^*$ be the annihilation and creation operators at a time point $t \in \mathbb{R}$, respectively. It is known that both

$$t \mapsto a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W}), \qquad t \mapsto a_t^* \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*),$$

are $C^\infty$-maps [6]. The pair $\{a_t, a_t^*\,;\, t \in \mathbb{R}\}$ is called the *quantum white noise process*. We then see that higher powers of quantum white noise (in normal-order) are well defined white noise operators.

As is mentioned in Introduction, we focus on the normal-ordered white noise differential equation:

$$\frac{d\Xi}{dt} = \gamma\, a_t^2 \diamond \Xi, \qquad \Xi\big|_{t=0} = \mathcal{H}_{a,b}, \tag{12}$$

where $\gamma, a, b \in \mathbb{C}$ are constant numbers and $\mathcal{H}_{a,b}$ is defined in (8). Recall that $\mathcal{H}_{a,b}$ is a Fourier–Gauss transform and hence, is a white noise operator. By the general theory [4, 6] there exists a unique solution to (12) in a space of white noise operators suitably chosen and is given by

$$\Xi_t = \mathcal{H}_{a,b} \diamond \exp \int_0^t \gamma\, a_s^2\, ds = \mathcal{H}_{a,b} \circ \exp \int_0^t \gamma\, a_s^2\, ds, \quad t \geq 0. \tag{13}$$

Here the Wick product $\diamond$ is replaced with the usual product (composition) of operators since the integral contains only annihilation operators.

# 5 The Main Results

**Theorem 5.1** *Let $a, b, \gamma \in \mathbb{C}$ satisfy the following conditions:*

$$|b| = 1, \qquad b \neq \pm 1, \qquad a' \equiv a + \frac{2\gamma}{1-b^2} > 0.$$

*Let $\{\Xi_t\}$ be the solution to (12), i.e., given as in (13). Then, for any $t > 0$, the white noise operator $\Xi_t$ possesses the following properties:*

(1) *$\Xi_t \upharpoonright \mathcal{E}_{[0,t]}$ extends uniquely to a unitary operator on $L^2(\mu_{a'}|\,[0,t])$.*

(2) *If $a > 0$ in addition, $\Xi_t \upharpoonright \mathcal{E}_{(-\infty,0]}$ and $\Xi_t \upharpoonright \mathcal{E}_{[t,+\infty)}$ extend uniquely to unitary operators on $L^2(\mu_a|\,(-\infty,0])$ and $L^2(\mu_a|\,[t,+\infty))$, respectively.*

As a matter of fact, it will be seen that

$$\Xi_t = \begin{cases} \mathcal{H}_{a',b} & \text{on } \mathcal{E}_{[0,t]} \\ \mathcal{H}_{a,b} & \text{on } \mathcal{E}_{(-\infty,0]} \cup \mathcal{E}_{[t,+\infty)}. \end{cases}$$

Taking into account the canonical factorizations:

$$L^2(\mathcal{N}', \mu_a) = L^2(\mu_a|(-\infty,0]) \otimes L^2(\mu_a|\,[0,t]) \otimes L^2(\mu_a|\,[t,+\infty)),$$
$$L^2(\mathcal{N}', \mu_{a'}) = L^2(\mu_{a'}|(-\infty,0]) \otimes L^2(\mu_{a'}|\,[0,t]) \otimes L^2(\mu_{a'}|\,[t,+\infty)),$$

we see that $\mathcal{E}_{(-\infty,0]} \otimes \mathcal{E}_{[0,t]} \otimes \mathcal{E}_{[t,+\infty)}$ becomes their common dense subspace. Then Theorem 5.1 says that, according to this factorization we have

$$\Xi_t = \mathcal{H}_{a,b} \otimes \mathcal{H}_{a',b} \otimes \mathcal{H}_{a,b}$$

and each factor in the right hand side extends to a unitary operator on the corresponding subspace of $L^2(\mathcal{N}', \mu_a)$ or $L^2(\mathcal{N}', \mu_{a'})$. We call this property of $\Xi_t$ the *partial unitarity*.

In fact, we prove the following more general result.

**Theorem 5.2** *Given $a, b, \gamma \in \mathbb{C}$, let $\Xi_t$ be defined as in* (13). *Assume $|b| = 1$, $b \neq \pm 1$ and choose $a', b' \in \mathbb{C}$ in such a way that*

$$\frac{1}{2}(a - a' + b')(1 - b^2) + \gamma = 0. \tag{14}$$

(1) *If $a' > 0$, the restriction $T_{b'}^{-1} \Xi_t T_{b'} \upharpoonright \mathcal{E}_{[0,t]}$ extends uniquely to a unitary operator on $L^2(\mu_{a'} | [0, t])$.*

(2) *If*

$$a'' \equiv a' - \frac{2\gamma}{1 - b^2} > 0, \tag{15}$$

*then the restrictions $T_{b'}^{-1} \Xi_t T_{b'} \upharpoonright \mathcal{E}_{(-\infty,0]}$ and $T_{b'}^{-1} \Xi_t T_{b'} \upharpoonright \mathcal{E}_{[t,+\infty)}$ extend uniquely to unitary operators on $L^2(\mu_{a''} | (-\infty, 0])$ and $L^2(\mu_{a''} | [t, +\infty))$, respectively.*

Theorem 5.1 follows immediately from Theorem 5.2 by setting $b' = 0$. The proof of Theorem 5.2 will be divided into a few steps. The *Gross Laplacian process* is defined by

$$G_t = \int_0^t a_s^2 \, ds, \qquad t \geq 0. \tag{16}$$

In fact, $t \mapsto G_t \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ is a $C^\infty$-map.

**Lemma 5.3** *For any $\gamma \in \mathbb{C}$ we have*

$$\exp \gamma G_t = \mathcal{G}(2\gamma 1_{[0,t]}, 1). \tag{17}$$

PROOF. Since $a_t \phi_{1,\xi} = \xi(t) \phi_{1,\xi}$, we have

$$(\exp \gamma G_t)\phi_{1,\xi} = \exp\left(\gamma \int_0^t \xi(s)^2 ds\right) \phi_{1,\xi} = e^{\langle \gamma 1_{[0,t]}\xi, \xi \rangle} \phi_{1,\xi} \,,$$

which implies (17). ∎

**Lemma 5.4** *Given $a, b, \gamma \in \mathbb{C}$, let $\Xi_t$ be defined as in* (13)*. For any $a', b' \in \mathbb{C}$ and $\xi \in \mathcal{N}_{\mathbb{C}}$ we have*

$$\begin{aligned} &T_{b'}^{-1}\Xi_t T_{b'}\phi_{a',\xi} \\ &= \exp\left\{\frac{1}{2}(a-a'+b')(1-b^2)\langle\xi,\xi\rangle + \gamma\int_0^t \xi(s)^2 ds\right\}\phi_{a',b\xi}\,. \end{aligned} \tag{18}$$

PROOF. Combining (9) and (17), we obtain the solution (13) written in terms of generalized Fourier–Gauss transforms:

$$\Xi_t = \mathcal{H}_{a,b} \circ \exp\gamma G_t = \mathcal{G}((a-1)(1-b^2), b)\,\mathcal{G}(2\gamma 1_{[0,t]}, 1).$$

Then, using $T_{b'} = \mathcal{G}(b', 1)$ and applying the composition rule (Lemma 2.1), we have

$$\begin{aligned} T_{b'}^{-1}\Xi_t T_{b'} &= \mathcal{G}(-b', 1)\,\mathcal{G}((a-1)(1-b^2), b)\,\mathcal{G}(2\gamma 1_{[0,t]}, 1)\,\mathcal{G}(b', 1) \\ &= \mathcal{G}(-b'b^2 + (a-1)(1-b^2), b)\,\mathcal{G}(2\gamma 1_{[0,t]} + b', 1) \\ &= \mathcal{G}(-b'b^2 + (a-1)(1-b^2) + 2\gamma 1_{[0,t]} + b', b) \\ &= \mathcal{G}((a-1+b')(1-b^2) + 2\gamma 1_{[0,t]}, b). \end{aligned}$$

We take the action on $\phi_{a',\xi} = e^{(1-a')\langle\xi,\xi\rangle/2}\phi_{1,\xi}$ to obtain

$$\begin{aligned} T_{b'}^{-1}\Xi_t T_{b'}\phi_{a',\xi} &= e^{(1-a')\langle\xi,\xi\rangle/2} e^{\langle\{(a-1+b')(1-b^2)+2\gamma 1_{[0,t]}\}\xi,\xi\rangle/2}\phi_{1,b\xi} \\ &= e^{(1-a')\langle\xi,\xi\rangle/2} e^{\langle\{(a-1+b')(1-b^2)+2\gamma 1_{[0,t]}\}\xi,\xi\rangle/2} e^{(a'-1)b^2\langle\xi,\xi\rangle/2}\phi_{a',b\xi} \\ &= e^{\langle\{(a-a'+b')(1-b^2)+2\gamma 1_{[0,t]}\}\xi,\xi\rangle/2}\phi_{a',b\xi}\,, \end{aligned}$$

from which (18) follows immediately. ∎

**Lemma 5.5** *Given $a, b, \gamma \in \mathbb{C}$, let $\Xi_t$ be defined as in* (13)*. Assume $b \neq \pm 1$ and choose $a', b' \in \mathbb{C}$ in such a way that*

$$\frac{1}{2}(a-a'+b')(1-b^2) + \gamma = 0. \tag{19}$$

*Then, for $t > 0$ we have*

$$T_{b'}^{-1}\Xi_t T_{b'} = \begin{cases} \mathcal{H}_{a',b} & \text{on } \mathcal{E}_{[0,t]}\,, \\ \mathcal{H}_{a'',b} & \text{on } \mathcal{E}_{(-\infty,0]} \cup \mathcal{E}_{[t,+\infty)}\,, \end{cases} \tag{20}$$

*where*

$$a'' = a' - \frac{2\gamma}{1-b^2}\,. \tag{21}$$

PROOF. Let $\xi \in \mathcal{N}_{\mathbb{C}}$ with $\operatorname{supp} \xi \subset [0,t]$. Then, by (18) and (19) we see that

$$\begin{aligned} T_{b'}^{-1}\Xi_t T_{b'}\phi_{a',\xi} &= \exp\left\{\frac{1}{2}(a-a'+b')(1-b^2)\langle\xi,\xi\rangle + \gamma\langle\xi,\xi\rangle\right\}\phi_{a',b\xi} \\ &= \phi_{a',b\xi}\,. \end{aligned}$$

Hence $T_{b'}^{-1}\Xi_t T_{b'}\phi_{a',\xi} = \mathcal{H}_{a',b}\phi_{a',\xi}$ and the first part of (20) is proved.

We next take $\xi \in \mathcal{N}_{\mathbb{C}}$ with $\operatorname{supp} \xi \subset (-\infty,0] \cup [t,+\infty)$. Again, in view of (18) and (19) we see that

$$T_{b'}^{-1}\Xi_t T_{b'}\phi_{a',\xi} = e^{-\gamma\langle\xi,\xi\rangle}\phi_{a',b\xi}\,,$$

namely,

$$e^{(1-a')\langle\xi,\xi\rangle/2}T_{b'}^{-1}\Xi_t T_{b'}\phi_{1,\xi} = e^{-\gamma\langle\xi,\xi\rangle+(1-a')b^2\langle\xi,\xi\rangle/2}\phi_{1,b\xi}\,.$$

Therefore we have

$$\begin{aligned} T_{b'}^{-1}\Xi_t T_{b'}\phi_{1,\xi} &= e^{-\gamma\langle\xi,\xi\rangle-(1-a')(1-b^2)\langle\xi,\xi\rangle/2}\phi_{1,b\xi} \\ &= \mathcal{G}(-2\gamma-(1-a')(1-b^2),b)\phi_{1,\xi} \\ &= \mathcal{G}\Big(\Big(a'-\frac{2\gamma}{1-b^2}-1\Big)(1-b^2),b\Big)\phi_{1,\xi}\,. \end{aligned}$$

Taking (9) and (21) into account, we conclude that

$$T_{b'}^{-1}\Xi_t T_{b'}\phi_{1,\xi} = \mathcal{H}_{a'',b}\phi_{1,\xi}\,,$$

which proves the second half of (20). ∎

**Remark 5.6** Lemma 5.5 becomes uninteresting when $b=\pm 1$. In fact, in that case $\gamma = 0$ so that $\Xi_t$ is reduced to a constant independent of $t$, see (13).

PROOF OF THEOREM 5.2. (1) We already know from Lemma 5.5 that

$$T_{b'}^{-1}\Xi_t T_{b'} \restriction \mathcal{E}_{[0,t]} = \mathcal{H}_{a',b} \restriction \mathcal{E}_{[0,t]}. \tag{22}$$

Noting by assumption that $|b| = 1$ and $a' > 0$, we see from Lemma 3.3 that (22) extends to a unitary operator on $L^2(\mu_{a'}|\,[0,t])$. The proof of (2) is similar. ∎

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