Asymptotic Spectral Analysis of Random Graphs
— Towards Complex Networks —

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Exploring a quantum probabilistic approach to network science

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I. Graph Spectrum

Definition. A *graph* is a pair $G = (V, E)$, where V is the set of *vertices* and E the set of *edges*. For $x, y \in V$ we write $x \sim y$ (adjacent) if they are connected by an edge.

$\star$ In this talk we focus on *finite graphs* with $|V| = n < \infty$ and their limit as $n \to \infty$.

Definition. The *adjacency matrix* $A = (A_{ij})_{i,j \in V}$ is defined by $A_{ij} = \begin{cases} 1, & i \sim j, \\ 0, & \text{otherwise.} \end{cases}$

- The characteristic polynomial of G : $\varphi_G(x) = \det(xE - A)$
- The spectrum of G: eigenvalues of A, say, $\lambda_1, \dots, \lambda_n$
- The spectral distribution of G:

$$\mu_G(dx) = \frac{1}{n} \sum_{k=1}^{n} \delta(x - \lambda_k) dx$$

- The moment sequence:

$$M_m(\mu_G) = \int_{-\infty}^{+\infty} x^m \mu_G(dx) = \frac{1}{n} \sum_{k=1}^{n} \lambda_k^m = \frac{1}{n} \mathrm{Tr}\, A^m$$

$\star$ The graph spectrum is not enough to characterize a graph (up to isomorphisms), but contains valuable information about graph geometry (cf. algebraic graph theory).

II. Random Graphs

Let $V = \{0, 1, \dots, n-1\}$ be a fixed set of vertices ($|V| = n$).

$$\mathcal{G} = \{G\,;\, G \text{ is a graph whose vertex set is } V\}, \qquad |\mathcal{G}| = 2^{\binom{n}{2}}.$$

Given a probability measure P, $(\mathcal{G}, P)$ is called a *random graph*.

- The adjacency matrix $A = A_G$ of $G \in \mathcal{G}$ $\Longrightarrow$ random matrix
- The spectral distribution $\mu = \mu_G$ of $G \in \mathcal{G}$ $\Longrightarrow$ random distribution

Main objectives are:

- The *mean spectral distribution* of a random graph:

$$\mu = \mathbf{E}(\mu_G) = \sum_{G \in \mathcal{G}} P(\{G\}) \mu_G$$

- The moment sequence:

$$M_m(\mu) = \int_{-\infty}^{+\infty} x^m \mu(dx) = \frac{1}{n} \mathbf{E}(\mathrm{Tr}\, A^m), \qquad m = 1, 2, \dots.$$

- The asymptotic behaviour as $n \to \infty$ (with some scaling balance)

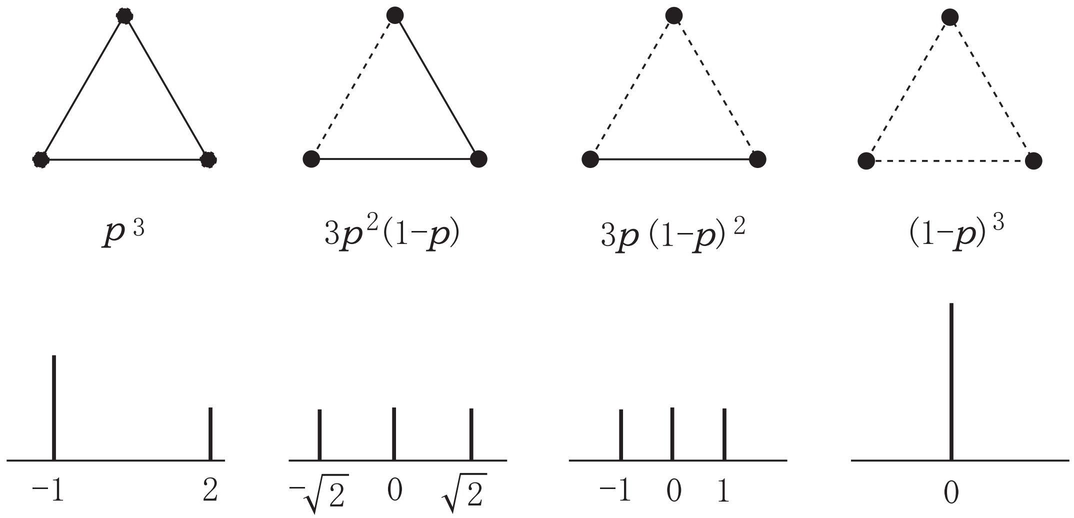
A typical example: $\mathcal{G}(n,p)$ the *Erdős–Rényi random graph*

$$V = \{0,1,2,\dots,n-1\}, \qquad 0<p<1$$

$$\mathcal{G} = \{G\,;\ \text{graph whose vertex set is } V\},$$

$$P(\{G\}) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}, \qquad |E(G)| = \#\text{edges},$$

For example, $\mathcal{G}(3,p)$



The mean spectral distribution:

$$\mu_{3,p} = p^3\nu_3 + 3p^2(1-p)\nu_2 + 3p(1-p)^2\nu_1 + (1-p)^3\nu_0$$

III. Why Random Graphs?

Models of *real world networks* in our life (technological, social, biological, etc.)

Epoch-making papers:

[1] D. J. Watts and S. H. Strogatz: *Collective dynamics of 'small-world' networks*, Nature **393** (1998), 440–442. $\Longrightarrow$ small world model (short distance between vertices)

[2] A.-L. Barabási and R. Albert: *Emergence of scaling in random networks*, Science **286** (1999), 509–512. $\Longrightarrow$ scale-free model (power law of the degree distribution)

- Both models are suitable for computer simulation.
- But mathematical analysis remains open (seems very difficult).
- In particular, *spectral analysis of real world networks* is an interesting topic
- Also interesting for testing the recently developed techniques of quantum probability $\Longrightarrow$ My challenging theme

My Strategy:

1. Study Erdős–Rényi randoms (1960)
   - Original purpose was to prove existence of a graph having certain properties.
   - Referred to often as a first step toward real world networks, however, not realistic.
   - The explicit form of the asymptotic spectral distribution is still unknown.
2. Modify the WS-model and BA-model with the idea of random graphs and derive mathematically rigorous results.
3. Spectral analysis of the WS-model and BA-model and go further.

In this talk I will report preliminary consideration and some results:

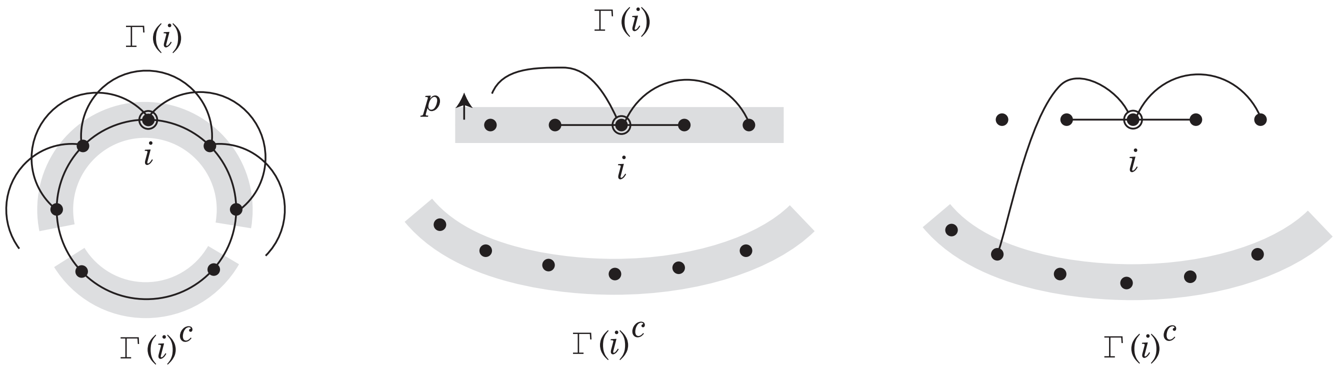
(i) Propose a model based on WS-model and ER-random graphs

(ii) Combinatorial formula for the asymptotic spectral distributions

IV. Our Model

Watts-Strogatz model

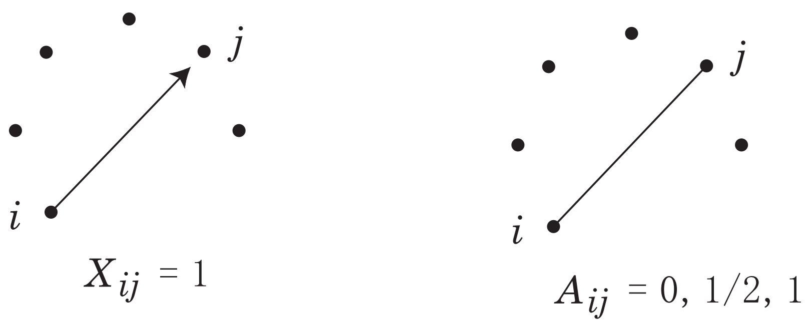
$V = \{0, 1, 2, \ldots, n-1\}$



Our model

$\{X_{ij}\,;\, i, j \in V,\, i \neq j\}$: independent random variables with values in $\{0, 1\}$

$A = (A_{ij})$ with $A_{ij} = \dfrac{1}{2}(X_{ij} + X_{ji})$: adjacency matrix of a "weighted" graph (network)



Moreover, "geometric location" is taken into account.

E.g., near vertices are connected at a large probability and remote ones at a small probability

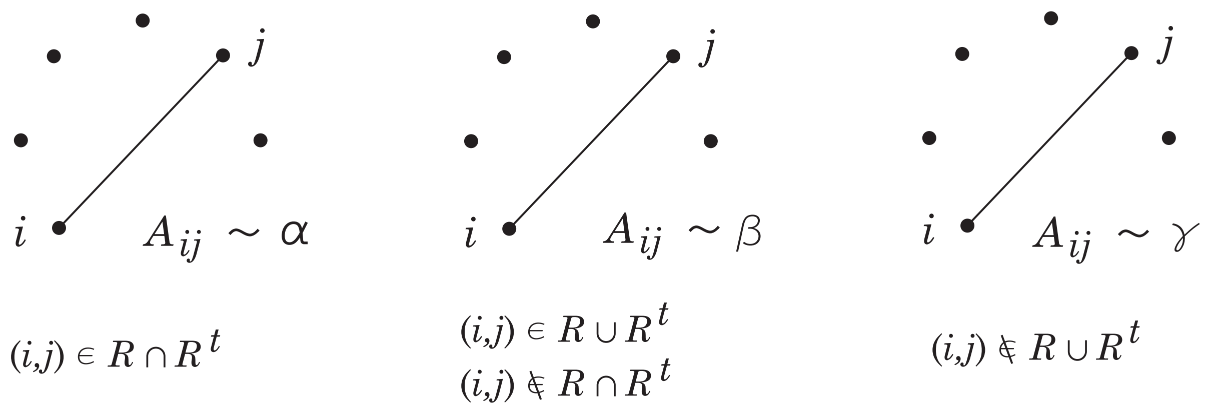
Fix a subset $R \subset \{(i,j) \in V \times V \, ; \, i \neq j\}$

Let $0 < p < 1$ and $0 < p' < 1$. Define a random variable X_{ij} by

(i) If $(i,j) \in R$, set $P(X_{ij} = 1) = p$ and $P(X_{ij} = 0) = 1 - p$.

(ii) If $(i,j) \notin R$, set $P(X_{ij} = 1) = p'$ and $P(X_{ij} = 0) = 1 - p'$.

Then $A_{ij} = \dfrac{1}{2}(X_{ij} + X_{ji})$ has 3 different distributions according to the location of i and j.



Thus, our model $\mathcal{G}(n, R; p, p')$ contains:

(i) grade of connection, i.e., $A_{ij} = 1$ (tight), $A_{ij} = 1/2$ (loose), $A_{ij} = 0$ (none).

(ii) probability of connection depends on location of vertices, $A_{ij} \sim \alpha, \beta, \gamma$.

V. Some general results on $\mathcal{G}(n, R; p, p')$

We assume "symmetry" in order that all vertices i are "statistically equivalent."

For any $i_0 \in V$ there exists a permutation σ on V such that $\sigma(i_0) = 0$ and for all $i, j \in V$, the distributions of A_{ij} and $A_{\sigma(i)\sigma(j)}$ coincide. (This is a condition on R.)

Mean degree of $\mathcal{G}(n, R; p, p')$

$$\bar{d}(\mathcal{G}(n, R; p, p')) = \frac{1}{n} \sum_{i \in V} \mathbf{E}(\deg_G(i)) = \frac{1}{n} \sum_{i \in V} \sum_{j \neq i} \mathbf{E}(A_{ij})$$
$$= \sum_{j \neq 0} \mathbf{E}(A_{0j}) = pR_2 + \frac{p + p'}{2} R_1 + p' R_0,$$

where $R_0 + R_1 + R_2 = n - 1$ and

$$R_2 = |\{j \in V\,;\, (0, j) \in R \cap R^t\}|, \quad R_0 = |\{j \in V\,;\, j \neq 0,\, (0, j) \notin R \cup R^t\}|,$$
$$R_1 = |\{j \in V\,;\, (0, j) \in R \cup R^t \text{ but } \notin R \cap R^t\}|.$$

Sparse limit (Poisson limit): $n \to \infty$ and $\bar{d} \to$ constant.

E.g., in case of $R = \{(i, i+1)\,;\, i \in V\}$, we take

$$n \to \infty, \qquad p' \to 0, \qquad np' \to \lambda \text{ (constant)}$$

Mean spectral distribution of $\mathcal{G}(n, R; p, p')$ is characterized by the moment sequence:

$$M_m = \sum_{\mathcal{L} \in \Lambda_m(\{\alpha,\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L}; n) \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e)=\alpha}} \alpha_{\kappa(e)} \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e)=\beta}} \beta_{\kappa(e)} \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e)=\gamma}} \gamma_{\kappa(e)},$$

where α_κ, β_κ, γ_κ are the κ-th moments of the distributions α, β, γ, respectively. Namely,

$$\alpha_\kappa = p^2 + \frac{2p(1-p)}{2^\kappa}, \quad \beta_\kappa = pp' + \frac{p+p'-2pp'}{2^\kappa}, \quad \gamma_\kappa = p'^2 + \frac{2p'(1-p')}{2^\kappa}.$$

$\Lambda_m(\{\alpha, \beta, \gamma\}) \ni \mathcal{L} = (\mathcal{V}, \mathcal{E}, o, \nu, \kappa)$ is a $\{\alpha, \beta, \gamma\}$-*labeled rooted graph of size* m, i.e.,

(L1) a connected graph $(\mathcal{V}, \mathcal{E})$ with $2 \le |\mathcal{V}| \le m$;

(L2) a distinguished vertex $o \in \mathcal{V}$ which is called the root;

(L3) a map $\nu : \mathcal{E} \to \{\alpha, \beta, \gamma\}$;

(L4) a map $\kappa : \mathcal{E} \to \{1, 2, \dots, m\}$ such that $\sum_{e \in \mathcal{E}} \kappa(e) = m$.

$u(\mathcal{L})$: the number of unicursal walks in $\mathcal{L}$

$t(\mathcal{L}; n)$: the number of A-admissible embeddings, i.e., injections $\varphi : \mathcal{V} \to \{0, 1, \dots, n-1\}$ such that $\varphi(o) = 0$ and for every $\{v, v'\} \in \mathcal{E}$, $\nu(\{v, v'\})$ coincides with the distribution of $A_{\varphi(v)\varphi(v')}$.

Outline of proof:

(1) By definition and symmetry assumption,

$$M_m = \frac{1}{n}\mathbf{E}(\mathrm{Tr}\, A^m) = \mathbf{E}\langle \delta_0, A^m \delta_0\rangle = \sum_{[i]\in\mathcal{W}(V,m)} \mathbf{E}(A_{0i_1}A_{i_1i_2}\cdots A_{i_{m-1}0}),$$

$$\mathcal{W}(V,m) = \{[i] : 0 \neq i_1 \neq \cdots \neq i_{m-1} \neq 0, \;\; i_k \in V\}.$$

(2) Let $G[i]$ be the underlying graph and for $e = \{j, j'\} \in E(G[i])$ we define

$$\nu(e) = \text{the distribution of } A_{jj'} = A_{j'j},$$
$$\kappa(e) = |\{0 \le s \le m-1\,;\, \{i_s, i_{s+1}\} = \{j, j'\}\}|.$$

(3) By independence of $\{A_{jj'}\,;\, 0 \le j < j' \le n-1\}$,

$$M_m = \sum_{[i]} \mathbf{E}\Bigg(\prod_{j<j'} A_{jj'}^{\kappa(j,j')}\Bigg) = \sum_{[i]}\prod_{j<j'}\mathbf{E}(A_{jj'}^{\kappa(j,j')}) = \sum_{[i]\in\mathcal{W}(V,m)}\prod_{e\in E(G[i])} M_{\kappa(e)}(\nu(e))$$

(4) Setting $\mathcal{L}[i] = (G[i], 0, \nu, \kappa)$,

$$\begin{aligned} M_m &= \sum_{\mathcal{L}\in\Lambda_m(\{\alpha,\beta,\gamma\})} |\{[i]\in\mathcal{W}(V,m)\,;\,\mathcal{L}[i]\cong\mathcal{L}\}| \prod_{e\in E(\mathcal{L})} M_{\kappa(e)}(\nu(e)) \\ &= \sum_{\mathcal{L}\in\Lambda_m(\{\alpha,\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L}; n) \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\alpha}} \alpha_{\kappa(e)} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\beta}} \beta_{\kappa(e)} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} \gamma_{\kappa(e)}. \end{aligned}$$

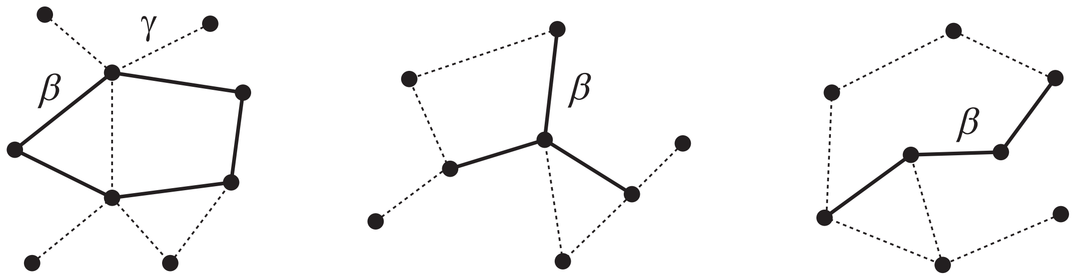
VI. Model I: $R = \{(i, i+1)\,;\, i \in V\}$

• Note that the distribution α does not appear since $R \cap R^t = \emptyset$. Hence,

$$M_m(n,p,p') = \sum_{\mathcal{L}\in\Lambda_m(\{\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L};n) \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\beta}} \beta_{\kappa(e)} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} \gamma_{\kappa(e)}.$$

• By our definition, an β-edge corresponds to an edge $\{i, i\pm 1\} \in R \cup R^t$.

• If $\mathcal{L} = (\mathcal{V}, \mathcal{E}, o, \nu, \kappa) \in \Lambda_m(\{\beta,\gamma\})$ contains a β-*cycle* or a β-*branch*, there is no A-admissible embedding (for a large n).



• The sum is taken over

$$\Lambda_m^*(\{\beta,\gamma\}) = \left\{ \mathcal{L} \in \Lambda_m(\{\beta,\gamma\})\,;\ \begin{array}{l} \text{(i) contains no } \beta\text{-cycles;} \\ \text{(ii) contains no } \beta\text{-branches} \end{array} \right\}.$$

In other words, in $\mathcal{L} \in \Lambda_m^*(\{\beta,\gamma\})$ every β-edge appears only as a linear segment (β-segment).

VII. Model I in the Sparse Limit ($n \to \infty$, $p' \to 0$, $np' \to \lambda$)

Mean degree: $\lim \bar{d}(\mathcal{G}_I(n,p,p')) = \lim\{p + (n-2)p'\} = p + \lambda$

Moments of β and γ: $\lim \beta_\kappa = \dfrac{p}{2^\kappa}\,, \qquad \lim n\gamma_\kappa = \dfrac{2\lambda}{2^\kappa}\,.$

Our target:

$$\lim M_m(n,p,p') = \lim \sum_{\mathcal{L} \in \Lambda_m^*(\{\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L};n) \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e)=\beta}} \beta_{\kappa(e)} \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e)=\gamma}} \gamma_{\kappa(e)}.$$

First a few moments:

$$\begin{aligned}
\lim M_1(n,p,p') &= 0, \\
\lim M_2(n,p,p') &= \frac{p}{2} + \frac{\lambda}{2}\,, \\
\lim M_3(n,p,p') &= 0, \\
\lim M_4(n,p,p') &= \frac{p}{8} + \frac{\lambda}{8} + \frac{p^2}{4} + p\lambda + \frac{\lambda^2}{2}\,.
\end{aligned}$$

THEOREM 1. Let $M_m = M_m(n,p,p')$ be the m-th moment of the mean spectral distribution of $\mathcal{G}_I(n,p,p')$. Then, in the sparse limit

$$\lim M_m = \begin{cases} 0, & \text{if } m \text{ is odd,} \\ \displaystyle\sum_{\mathcal{L}\in\Lambda_m^{**}(\{\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L})\, 2^{|\mathcal{E}_\gamma|+1-b(\mathcal{L})-m}\, p^{|\mathcal{E}_\beta|}(2\lambda)^{|\mathcal{E}_\gamma|}, & \text{if } m \text{ is even.} \end{cases}$$

Outline of Proof. (0) Start with

$$\lim M_m(n,p,p') = \lim \sum_{\mathcal{L}\in\Lambda_m^{*}(\{\beta,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L};n) \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\beta}} \beta_{\kappa(e)} \prod_{\substack{e\in E(\mathcal{L})\\ \nu(e)=\gamma}} \gamma_{\kappa(e)}.$$

(1) Let $\tilde{\mathcal{L}} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ be a connected graph obtained from $\mathcal{L}$ by β-*edge contraction*.

(2) Let $b(\mathcal{L})$ be the number of isolated vertices of $(\mathcal{V}, \mathcal{E}_\beta)$.

$$2^{|\tilde{\mathcal{V}}|-b(\mathcal{L})}\,(n-2m^2)^{|\tilde{\mathcal{V}}|-1} \le |t(\mathcal{L};n)| \le 2^{|\tilde{\mathcal{V}}|-b(\mathcal{L})}\,n^{|\tilde{\mathcal{V}}|-1}.$$

(3) For any connected graph $G=(V,E)$ we have $|V| \le |E|+1$. Moreover, the equality holds if and only if G is a tree.

(4) $\lim t(\mathcal{L};n) \prod \gamma_{\kappa(e)} = 0$ unless $\mathcal{L}$ belongs to

$$\Lambda_m^{**}(\{\beta,\gamma\}) = \left\{\mathcal{L}\in\Lambda_m^{*}(\{\beta,\gamma\})\,;\, \tilde{\mathcal{L}} \text{ is a tree and } |\tilde{\mathcal{E}}| = |\mathcal{E}_\gamma|\,(\Longleftrightarrow \mathcal{L} \text{ is a tree})\right\}.$$

VIII. Model II: $R = \{(i, i \pm 1)\,;\, i \in V\}$

Mean degree of $\mathcal{G}_{II}(n, p, p')$:

$$\bar{d}(\mathcal{G}_I(n, p, p')) = 2p + (n-3)p'.$$

Moment sequence of $\mathcal{G}_{II}(n, p, p')$:

$$M_m(n, p, p') = \sum_{\mathcal{L} \in \Lambda_m^*(\{\alpha,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) t(\mathcal{L}; n) \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e)=\alpha}} \alpha_{\kappa(e)} \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e)=\gamma}} \gamma_{\kappa(e)}.$$

In the sparse limit $n \longrightarrow \infty$, $p' \longrightarrow 0$, $np\prime \longrightarrow \lambda$:

$$\lim \bar{d}(\mathcal{G}_I(n, p, p')) = 2p + \lambda, \qquad \lim \alpha_\kappa = p^2 + \frac{2p(1-p)}{2^\kappa}, \qquad \lim n\gamma_\kappa = \frac{2\lambda}{2^\kappa}.$$

THEOREM 2. Let $M_m(n, p, p')$ be the m-th moment of the mean spectral distribution of $\mathcal{G}_{II}(n, p, p')$. Then, in the sparse limit

$$\lim M_m = \begin{cases} 0, & \text{if } m \text{ is odd}, \\ \displaystyle\sum_{\mathcal{L} \in \Lambda_m^{**}(\{\alpha,\gamma\})} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L})\, 2^{|\mathcal{E}_\gamma|+1-b(\mathcal{L})-m}\, (2\lambda)^{|\mathcal{E}_\gamma|} \prod_{\substack{e \in E(\mathcal{L}) \\ \nu(e)=\alpha}} ((2^{\kappa(e)} - 2)p^2 + 2p), & \\ & \text{if } m \text{ is even}. \end{cases}$$

IX. The Erdős–Rényi Random Graph

Definition. Let $n \geq 1$, $0 < p < 1$. Let $\mathcal{G}(n,p)$ be the set of all graphs with vertex set $V = \{0, 1, 2, \dots, n-1\}$ with probability $P(\{G\})$ defined by

$$P(\{G\}) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}, \qquad E(G) = \{\text{edges of } G\}.$$

Some statistics of $\mathcal{G}(n,p)$

- The mean spectral distribution $\mu_{n,p}$ — rather complicated!
- The mean degree: $\bar{d}(\mathcal{G}(n,p)) = \dfrac{1}{n}\displaystyle\sum_{i \in V} \mathbf{E}(\deg_G(i)) = (n-1)p$
- The mean number of edges: $\mathbf{E}(|E(G)|) = \dfrac{n}{2}\, M_2 = \dfrac{n(n-1)}{2}\, p$
- The mean number of triangles: $\mathbf{E}(|\triangle(G)|) = \dfrac{n}{6}\, M_3 = \dfrac{n(n-1)(n-2)}{6}\, p^3$

Problem: Find $\lim \mu_{n,p}$ in the sparse limit

$$n \to \infty, \quad p \to 0, \quad np \to \lambda \text{ (constant)}.$$

- Cluster coefficient: $\mathbf{E}(|\triangle(G)|)/|\{\text{all possible triangles}\}| = p^3 \to 0$ (not realistic!)

Moment sequence of $\mu_{n,p}$ (=mean spectral distribution of $\mathcal{G}(n,p)$)

$$M_m(\mu_{n,p}) = \frac{1}{n}\,\mathbf{E}(\mathrm{Tr}\,A^m) = \frac{1}{n}\sum_{i\in V}\mathbf{E}((A^m)_{ii}) = \mathbf{E}((A^m)_{00})$$
$$= \sum_{\mathcal{L}\in\Lambda_m} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L})(n-1)(n-2)\cdots(n-(|V(\mathcal{L})|-1))\,p^{|E(\mathcal{L})|},$$

where Λ_m is the collection of all labeled graphs $\mathcal{L} = (\mathcal{V},\mathcal{E},o,\kappa)$, where

(i) $(\mathcal{V},\mathcal{E})$ is a connected graph with $2 \le |\mathcal{V}| \le m$,

(ii) $o \in V$ is a distinguished vertex,

(iii) $\kappa : \mathcal{E} \to \{1,2,\dots\}$ such that $\sum_{e\mathcal{E}} \kappa(e) = m$.

First few moments of $\mu_{n,p}$

$M_1(\mu_{n,p}) = 0, \qquad M_2(\mu_{n,p}) = (n-1)p, \qquad M_3(\mu_{n,p}) = (n-1)(n-2)p^3$

$M_4(\mu_{n,p}) = (n-1)p + 2(n-1)(n-2)p^2 + (n-1)(n-2)(n-3)p^4$

$M_5(\mu_{n,p}) = 5(n-1)(n-2)p^3 + 5(n-1)(n-2)(n-3)p^4 + (n-1)(n-2)(n-3)(n-4)p^5$

THEOREM 3.

$$M_m \equiv \lim M_m(\mu_{n,p}) = \begin{cases} 0, & m \text{ is odd,} \\ \sum_{\mathcal{L}\in\Lambda_m^{**}} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L})\,\lambda^{|E(\mathcal{L})|}, & m \text{ is even,} \end{cases}$$

where Λ_m^{**} is the collection of all labeled graphs $\mathcal{L} = (\mathcal{V}, \mathcal{E}, o, \kappa)$, where

(i) $(\mathcal{V}, \mathcal{E})$ is a *tree* with $2 \le |\mathcal{V}| \le m$,

(ii) $o \in V$ is a distinguished vertex,

(iii) $\kappa : \mathcal{E} \to \{1, 2, \dots\}$ such that $\sum_{e\in\mathcal{E}} \kappa(e) = m$.

Outline of Proof: (1)

$$\begin{aligned} M_m(\mu_{n,p}) &= \sum_{\mathcal{L}\in\Lambda_m} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L})(n-1)(n-2)\cdots(n-(|V(\mathcal{L})|-1))\,p^{|E(\mathcal{L})|} \\ &\sim \sum_{\mathcal{L}\in\Lambda_m} |\mathrm{Aut}\,(\mathcal{L})|^{-1} u(\mathcal{L}) n^{|V(\mathcal{L})|-1-|E(\mathcal{L})|}(np)^{|E(\mathcal{L})|}. \qquad (*) \end{aligned}$$

(2) If $\mathcal{L}$ is not a tree, then $|V(\mathcal{L})| \le |E(\mathcal{L})|$ and (*) vanishes.

(3) If $\mathcal{L}$ is a tree, then $|V(\mathcal{L})| = |E(\mathcal{L})| + 1$ and we get the result.

Cf. Bauer–Golinelli (2001), Dorogovtsev–Goltsev–Mendes–Samukhin (2003)

Alternative expression of M_m

THEOREM 4. The sparse limit of the $2m$-th moment of mean spectral distribution of the Erdős–Rényi random graph $\mathcal{G}(n,p)$ is given by

$$M_{2m} = \sum_{\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)} \lambda^{|\vartheta|}$$

Outline of Proof: (1)

$$M_{2m}(\mu_{n,p}) = \mathbf{E}((A^{2m})_{00}) = \sum_{[i]\in\mathcal{W}(V,2m)} \mathbf{E}(A_{0i_1}A_{i_0i_1}\cdots A_{i_{2m-1}0}) = \sum_{[i]\in\mathcal{W}(V,2m)} p^{|E(G[i])|}.$$

(2) In the limit, we need only consider

$$[i]: 0 \equiv i_0 \neq i_1 \neq i_2 \neq \cdots \neq i_{2m-1} \neq i_{2m} \equiv 0, \qquad i_k \in V,$$

such that $G[i]$ is a tree (by Theorem 3).

(3) For a such $[i] \in \mathcal{W}(V,2m)$ we associate a partition $\vartheta = \vartheta[i]$ of $\{1,2,\dots,2m\}$. The set of such partitions is denoted by $\mathcal{P}_{\mathrm{T}}(2m)$.

(4) For $\vartheta = \vartheta[i]$ we have $|E(G[i])| = |\vartheta|$ and

$$M_{2m} = \lim \sum_{\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)} \sum_{\substack{[i]\in\mathcal{W}(V,2m) \\ \vartheta[i]=\vartheta}} p^{|\vartheta|} = \lim \sum_{\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)} (n-1)(n-2)\cdots(n-|\vartheta|)\, p^{|\vartheta|}.$$

What is $\mathcal{P}_{\mathrm{T}}(2m)$?

• Every partition in $\mathcal{P}_{\mathrm{T}}(2m)$ is obtained from a non-crossing pair partition of $\{1, 2, \dots, 2m\}$.

Given $\vartheta \in \mathcal{P}_{\mathrm{NCP}}(2m)$, (i) two or more blocks can be joined if their depth $= 1$; (ii) two or more blocks can be joined if their depth are the same and the upper blocks are already joined.

• Let $\mathcal{P}_{\mathrm{TNC}}(2m)$ be the set of all non-crossing partitions of $\{1, 2, \dots, 2m\}$ such that each block consists of even number of points. Then $\mathcal{P}_{\mathrm{TNC}}(2m) \subset \mathcal{P}_{\mathrm{T}}(2m)$

Non-crossing approximation

$$M_{2m} = \sum_{\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)} \lambda^{|\vartheta|} \geq \sum_{\vartheta \in \mathcal{P}_{\mathrm{TNC}}(2m)} \lambda^{|\vartheta|} = M_{2m}(\pi_{\lambda/2} \boxplus \pi^{\vee}_{\lambda/2})$$

where $\pi_{\lambda/2}$ be the free Poisson distribution with parameter $\lambda/2$ and $\pi^{\vee}_{\lambda/2}$ its reflection (Bożejko, the free moment–cumulant calculus)

Asymptotics for large λ

$$M_{2m} = \sum_{\vartheta \in \mathcal{P}_{\mathrm{T}}(2m)} \lambda^{|\vartheta|} = |\mathcal{P}_{\mathrm{NCP}}(2m)|\, \lambda^{m} + O(\lambda^{m-1}).$$

explaining the Erdős–Rényi random graphs behave like a tree.

Summary

1. We proposed two models of real world complex networks.
   Motivated by Erdős–Rényi random graph + Watts-Strogatz small world model.

2. We derived combinatorial formulas for the spectral distribution in the sparse limit.

3. We examined a similar question for the Erdős–Rényi random graph.

4. Analytic study of the limit distribution is left open (a challenging problem for quantum probabilistic method).

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