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1. Quantum White Noise Calculus
1.1. Background

The Boson Fock space over $H = L^2(T)$ is defined by

$$\Gamma(H) = \left\{ \phi = (f_n); f_n \in H^{\hat{\otimes} n}, \|\phi\|^2 = \sum_{n=0}^{\infty} n! |f_n|^2 < \infty \right\},$$

where $T$ is a topological space equipped with a $\sigma$-finite Borel measure $dt$, $|f_n|_0$ is the usual $L^2$-norm of $H^{\hat{\otimes} n} = L^2_{\text{sym}}(T^n)$.

The annihilation and creation operator at a point $t \in T$

$$a_t : (0, \ldots, 0, \xi^{\otimes n}, 0, \ldots) \mapsto (0, \ldots, 0, n\xi(t)\xi^{\otimes (n-1)}, 0, 0, \ldots)$$

$$a_t^* : (0, \ldots, 0, \xi^{\otimes n}, 0, \ldots) \mapsto (0, \ldots, 0, 0, \xi^{\otimes n} \hat{\otimes} \delta_t, 0, \ldots)$$

A “general” Fock space operator takes the form:

$$\sum_{l,m=0}^{\infty} \int_{T^{l+m}} \kappa_{l,m}(s_1, \ldots, s_l, t_1, \ldots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m$$

Quantum field theory: e.g., Haag (1955), Berezin (1966), Krée (1988), etc.
1.2. White Noise Approach

I) Gelfand triple for $H = L^2(T)$:

$$E \subset H = L^2(T) \subset E^*, \quad E = \limproj_{p \to \infty} E_p, \quad E^* = \limind_{p \to \infty} E_{-p},$$

where $E_p$ is a dense subspace of $H$ and is a Hilbert space for itself.

II) Gelfand triple for $\Gamma(H)$ (e.g., Hida–Kubo–Takenaka space):

$$\left(E\right) \subset \Gamma(H) \subset \left(E\right)^*, \quad \left(E\right) = \limproj_{p \to \infty} \Gamma(E_p), \quad \left(E\right)^* = \limind_{p \to \infty} \Gamma(E_{-p}),$$

Notes: (1) $\Gamma(H) \cong L^2(E^*, \mu)$ (Wiener–Itô–Segal isomorphism)
(2) $\left(E\right)$ is the space of test functions and $\left(E\right)^*$ the space of distributions.

**Definition**

A continuous operator from $\left(E\right)$ into $\left(E\right)^*$ is called a *white noise operator*. Let $\mathcal{L}(\left(E\right), \left(E\right)^*)$ denote the space of white noise operators, equipped with the topology of bounded convergence.

Note: $\mathcal{L}(\left(E\right), \left(E\right)), \mathcal{L}(\left(E\right)^*, \left(E\right)^*)$ and $\mathcal{B}(\Gamma(H))$ are subspaces of $\mathcal{L}(\left(E\right), \left(E\right)^*)$. 
1.3. Integral Kernel Operators

**Theorem.**

\[ a_t \in \mathcal{L}(\mathcal{H}, \mathcal{H}) \text{ and } a_t^* \in \mathcal{L}(\mathcal{H}^*, \mathcal{H}^*) \text{ for all } t \in \mathbb{R}. \]

Moreover, both maps

\[ t \mapsto a_t \in \mathcal{L}(\mathcal{H}, \mathcal{H}) \text{ and } t \mapsto a_t^* \in \mathcal{L}(\mathcal{H}^*, \mathcal{H}^*) \]

are operator-valued rapidly decreasing functions, i.e., belongs to \( \mathcal{H} \otimes \mathcal{L}(\mathcal{H}, \mathcal{H}) \) and \( \mathcal{H} \otimes \mathcal{L}(\mathcal{H}^*, \mathcal{H}^*) \), respectively. (The pair \( \{a_t, a_t^*; t \in T\} \) is called the quantum white noise on \( T \).)

**Definition.**

Given \( \kappa_{l,m} \in (\mathcal{H} \otimes (l+m))^* \), \( l, m = 0, 1, 2, \ldots \), the integral kernel operator

\[ \Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}(\mathcal{H}, \mathcal{H}^*) \]

is defined by

\[
\langle \langle \Xi_{l,m}(\kappa_{l,m}) \varphi_\xi, \varphi_\eta \rangle \rangle = \langle \kappa_{l,m}, \eta \otimes \xi \rangle e^{\langle \xi, \eta \rangle},
\]

where \( \varphi_\xi = (1, \xi, \xi \otimes 2/2!, \cdots) \) is the exponential vector. We write

\[
\Xi_{l,m}(\kappa_{l,m}) = \int_{T^{l+m}} \kappa_{l,m}(s_1, \cdots, s_l, t_1, \cdots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m
\]
1.4. Fock Expansion

**Theorem (Obata (1993))**

Every white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ admits the infinite series expansion:

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \quad \kappa_{l,m} \in (E^\otimes (l+m))^*,$$

where the right-hand side converges in $\mathcal{L}((E), (E)^*)$. If $\Xi \in \mathcal{L}((E), (E))$, then $\kappa_{l,m} \in E^\otimes l \otimes (E^\otimes m)^*$ and the series converges in $\mathcal{L}((E), (E))$.

Berezin (1966): for bounded operators in a weak sense
Krée (1988): introduced distribution theory for operators

**Our Standpoint**

A white noise operator $\Xi$ as a function of quantum white noise:

$$\Xi = \Xi(a_s, a_t^*; s, t \in T)$$
2. Quantum White Noise Derivatives
2.1. Definition

White noise (Hida) derivative: for a white noise functional $\Phi = \Phi(\dot{B}(t); t \in T)$,

$$\frac{\delta \Phi}{\delta \dot{B}(t)} \implies \partial_t \Phi \text{ or } a_t \Phi$$

A quantum counterpart: for a white noise operator as a function of quantum white noise:

$$\Xi = \Xi(a_s, a_t^*; s, t \in T)$$

We should like to define the derivatives with respect to $a_s$ and $a_t^*$:

$$\frac{\delta \Xi}{\delta a_s} \text{ and } \frac{\delta \Xi}{\delta a_t^*}$$

Expected properties:

$$\frac{\delta}{\delta a_s} \int f(t)a_t dt = f(s)I$$

$$\frac{\delta}{\delta a_s} \int f(s, t)a_s a_t ds dt = \int f(s, t)a_t dt + \int f(t, s)a_t dt$$

$$\frac{\delta}{\delta a_t^*} \int f(s, t)a_s a_t^* ds dt = \int f(s, t)a_s ds$$
2.1. Definition

Definition (Ji–Obata (2007))

For $\Xi \in \mathcal{L}((E), (E)^*)$ and $\zeta \in E$ we define $D_\zeta^\pm \Xi \in \mathcal{L}((E), (E)^*)$ by

$$
D_\zeta^+ \Xi = [a(\zeta), \Xi], \quad D_\zeta^- \Xi = -[a^*(\zeta), \Xi].
$$

These are called the creation derivative and annihilation derivative of $\Xi$, respectively. Both together are called the quantum white noise derivatives.

Note: For $\zeta \in E$, both

$$
a(\zeta) = \Xi_{0,1}(\zeta) = \int_T \zeta(t)a_t \, dt, \quad a^*(\zeta) = \Xi_{1,0}(\zeta) = \int_T \zeta(t)a^*_t \, dt,
$$

belong to $\mathcal{L}((E), (E)) \cap \mathcal{L}((E)^*, (E)^*)$.

Some properties:

1. $(D_\zeta^+ \Xi)^* = D_\zeta^- (\Xi^*)$ and $(D_\zeta^- \Xi)^* = D_\zeta^+ (\Xi^*)$.
2. $D_\zeta^\pm$ is a continuous linear map from $\mathcal{L}((E), (E)^*)$ into itself.
3. Moreover, $(\zeta, \Xi) \mapsto D_\zeta^\pm \Xi$ is a continuous bilinear map from $E \times \mathcal{L}((E), (E)^*)$ into $\mathcal{L}((E), (E)^*)$. 

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Remark: Pointwisely Defined QWN-Derivatives

Recall: The annihilation and creation operators

\[ a(f) = \int_T f(t) a_t \, dt, \quad a^*(f) = \int_T f(t) a^*_t \, dt. \]

It is natural to write

\[ D^+_\zeta = \int_T \zeta(t) D^+_t \, dt, \quad D^-_\zeta = \int_T \zeta(t) D^-_t \, dt. \]

In fact, this expression is useful for computation.

However, it is not straightforward to define \( D^\pm_t \) for each point \( t \in T \) because

\[ D^+_t \Xi = [a_t, \Xi] = a_t \Xi - \Xi a_t, \quad D^-_t \Xi = -[a^*_t, \Xi] = -a^*_t \Xi + \Xi a^*_t \]

are not well-defined in general.

Nevertheless, the pointwisely defined quantum white noise derivatives \( D^\pm_t \) are well formulated for admissible white noise operators (Ji–Obata, 2009, to appear).
2.2. Examples

The canonical correspondence (kernel theorem) between $S \in \mathcal{L}(E, E^*)$ and $\tau = \tau_S \in (E \otimes E)^*$ is given by $\langle \tau_S, \eta \otimes \xi \rangle = \langle S\xi, \eta \rangle$ for $\xi, \eta \in E$.

(1) The generalized Gross Laplacian associated with $S$ is defined by

$$\Delta_G(S) = \Xi_{0,2}(\tau_S) = \int_{T \times T} \tau_S(s, t)a_s a_t \, dsdt$$

Note that $\Delta_G(S) \in \mathcal{L}((E), (E))$. Then,

$$D_\zeta^+ \Delta_G(S) = 0, \quad D_\zeta^- \Delta_G(S) = a(S\zeta) + a(S^*\zeta)$$

In fact, since

$$D_t^- \Delta_G(S) = \int_T \tau_S(s, t)a_s \, ds + \int_T \tau_S(t, s)a_s \, ds$$

we have

$$D_\zeta^- \Delta_G(S) = \int_{T \times T} \tau_S(s, t)a_s \zeta(t) \, dtds + \int_{T \times T} \tau_S(t, s)a_s \zeta(t) \, dtds$$

$$= \int_T S\zeta(s)a_s \, ds + \int_T S^*\zeta(s)a_s \, ds = a(S\zeta) + a(S^*\zeta)$$
2.2. Examples

(2) The adjoint of $\Delta_G(S) \in \mathcal{L}((E)^*, (E)^*)$ is given by

$$\Delta_G^*(S) = \Xi_{2,0}(\tau_S) = \int_{T \times T} \tau_S(s, t) a_s^* a_t^* \, ds \, dt$$

The quantum white noise derivatives are given by

$$D^-_\zeta \Delta_G^*(S) = 0, \quad D^+_\zeta \Delta_G^*(S) = a^*(S\zeta) + a^*(S^*\zeta)$$

(3) The conservation operator associated with $S$ is defined by

$$\Lambda(S) = \Xi_{1,1}(\tau_S) = \int_{T \times T} \tau_S(s, t) a_s^* a_t \, ds \, dt$$

In general, $\Lambda(S) \in \mathcal{L}((E), (E)^*)$.

The quantum white noise derivatives are given by

$$D^-_\zeta \Lambda(S) = a^*(S\zeta), \quad D^+_\zeta \Lambda(S) = a(S^*\zeta).$$
2.3. Wick Products

Definition

For $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$ the Wick (or normal-ordered) product $\Xi_1 \diamond \Xi_2$ is defined by

$$(\Xi_1 \diamond \Xi_2)(\xi, \eta) = \hat{\Xi}_1(\xi, \eta)\hat{\Xi}_2(\xi, \eta)e^{-\langle \xi, \eta \rangle}, \quad \xi, \eta \in E,$$

where $\hat{\Xi}(\xi, \eta)$ is the symbol of a white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ defined by

$$\hat{\Xi}(\xi, \eta) = \langle \langle \Xi \phi_\xi, \phi_\eta \rangle \rangle, \quad \xi, \eta \in E,$$

where $\phi_\xi = (1, \xi, \cdots, \xi \otimes^n / n!, \cdots)$ is an exponential vector.

Some properties:

1. For any $\Xi \in \mathcal{L}((E), (E)^*)$ we have

$$a_t \diamond \Xi = \Xi \diamond a_t = \Xi a_t, \quad a^*_t \diamond \Xi = \Xi \diamond a^*_t = a^*_t \Xi.$$

2. Equipped with the Wick product, $\mathcal{L}((E), (E)^*)$ becomes a commutative algebra.
2.4. Wick Derivations

Definition

A continuous linear map \( \mathcal{D} : \mathcal{L}((E), (E)^*) \rightarrow \mathcal{L}((E), (E)^*) \) is called a Wick derivation if

\[
\mathcal{D}(\Xi_1 \diamond \Xi_2) = (\mathcal{D}\Xi_1) \diamond \Xi_2 + \Xi_1 \diamond (\mathcal{D}\Xi_2)
\]

for all \( \Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*) \).

Theorem

The creation and annihilation derivatives \( D_{\zeta}^{\pm} \) are Wick derivations for any \( \zeta \in E \).

Moreover, it is proved that a general Wick derivation \( \mathcal{D} \) is expressed in the form:

\[
\mathcal{D} = \int_T F(t) \diamond D_t^+ dt + \int_T G(t) \diamond D_t^- dt,
\]

where \( F, G \in E \otimes \mathcal{L}((E), (E)^*) \).
Proof.

In general, for $\Xi \in \mathcal{L}((E), (E)^*)$ we have

$$(D^+_\zeta \Xi)(\xi, \eta) = \langle (a(\zeta)\Xi - \Xi a(\zeta))\phi_\xi, \phi_\eta \rangle$$

$$= \langle \Xi \phi_\xi, a^*(\zeta)\phi_\eta \rangle - \langle \Xi a(\zeta)\phi_\xi, \phi_\eta \rangle$$

$$= \left. \frac{d}{dt} \right|_{t=0} \langle \Xi \phi_\xi, \phi_{\eta+t\zeta} \rangle - \langle \xi, \zeta \rangle \langle \Xi \phi_\xi, \phi_\eta \rangle$$

$$= \left. \frac{d}{dt} \right|_{t=0} \Xi(\xi, \eta + t\zeta) - \langle \xi, \zeta \rangle \Xi(\xi, \eta).$$

(1)

Then for $\Xi = \Xi_1 \odot \Xi_2$ we have

$$(D^+_\zeta \Xi)(\xi, \eta) = \left. \frac{d}{dt} \right|_{t=0} \Xi_1(\xi, t\zeta + \eta)\Xi_2(\xi, t\zeta + \eta)e^{-\langle \xi, t\zeta + \eta \rangle}$$

$$- \langle \xi, \zeta \rangle \Xi_1(\xi, \eta)\Xi_2(\xi, \eta)e^{-\langle \xi, \eta \rangle}$$

$$= \left( \left. \frac{d}{dt} \right|_{t=0} \Xi_1(\xi, t\zeta + \eta) \right) \Xi_2(\xi, \eta)e^{-\langle \xi, \eta \rangle}$$

$$+ \Xi_1(\xi, \eta) \left( \left. \frac{d}{dt} \right|_{t=0} \Xi_2(\xi, t\zeta + \eta) \right)e^{-\langle \xi, \eta \rangle}$$

$$- 2\langle \xi, \zeta \rangle \Xi_1(\xi, \eta)\Xi_2(\xi, \eta)e^{-\langle \xi, \eta \rangle}.$$

Viewing (1) once again, we obtain

$$(D^+_\zeta \Xi)(\xi, \eta) = ((D^+_\zeta \Xi_1) \odot \Xi_2)(\xi, \eta) + (\Xi_1 \odot (D^+_\zeta \Xi_2))(\xi, \eta).$$
3. Differential Equations for White Noise Operators
Given a Wick derivation $\mathcal{D}$ and a white noise operator $G \in \mathcal{L}((E), (E)^*)$, consider

$$\mathcal{D}\Xi = G \diamond \Xi \quad (2)$$

The *Wick exponential* is defined by

$$\text{wexp } Y = \sum_{n=0}^{\infty} \frac{1}{n!} Y^{\diamond n}, \quad Y \in \mathcal{L}((E), (E)^*),$$

whenever the series converges in $\mathcal{L}((E), (E)^*)$. 

Theorem

Assume that there exists an operator $Y \in \mathcal{L}((E), (E)^*)$ such that $\mathcal{D}Y = G$ and $\text{wexp} \ Y$ is defined in $\mathcal{L}((E), (E)^*)$. Then every solution to

$$\mathcal{D}\Xi = G \diamond \Xi$$

(3)

is of the form:

$$\Xi = (\text{wexp} \ Y) \diamond F,$$

(4)

where $F \in \mathcal{L}((E), (E)^*)$ satisfying $\mathcal{D}F = 0$.

Proof.

It is straightforward to see that (4) is a solution to (3). To prove the converse, let $\Xi$ be an arbitrary solution to (3). Set

$$F = (\text{wexp} (-Y)) \diamond \Xi.$$

Obviously, $F \in \mathcal{L}((E), (E)^*)$ and $\Xi = (\text{wexp} Y) \diamond F$. We only need to show that $\mathcal{D}F = 0$. In fact,

$$\mathcal{D}F = -\mathcal{D}Y \diamond (\text{wexp} (-Y)) \diamond \Xi + (\text{wexp} (-Y)) \diamond \mathcal{D}\Xi$$

$$= -G \diamond (\text{wexp} (-Y)) \diamond \Xi + (\text{wexp} (-Y)) \diamond G \diamond \Xi = 0.$$

This completes the proof. 

\qed
3.2. Example (1)

Let us consider the (system of) differential equations:

\[ D_\zeta^+ \Xi = 0, \quad \zeta \in E. \]  

(5)

We expect easily that \( \Xi = \Xi(a_s, a_t^*; s, t \in T) \) does not depend on the creation operators. In fact, by Fock expansion we see that the solutions to (5) are given by

\[ \Xi = \sum_{m=0}^{\infty} \Xi_{0,m}(\kappa_{0,m}). \]

In a similar manner, the solutions to

\[ D_\zeta^- \Xi = 0, \quad \zeta \in E, \]  

(6)

are given by

\[ \Xi = \sum_{l=0}^{\infty} \Xi_{l,0}(\kappa_{l,0}). \]

Consequently, a white noise operator satisfying both (5) and (6) are the scalar operators. Thus, the irreducibility of the canonical commutation relation is reproduced.
3.3. Example (2)

Let us consider the differential equation:

\[ D_\zeta \Xi = 2a(\zeta) \diamond \Xi, \quad \zeta \in E. \]  

We need to find \( Y \in \mathcal{L}((E), (E)^*) \) satisfying \( D_\zeta Y = 2a(\zeta) \).

In fact, \( Y = \Delta_G \) is a solution.

Moreover, it is easily verified that \( \text{wexp} \Delta_G \) is defined in \( \mathcal{L}((E), (E)) \).

Then, a general solution to (7) is of the form:

\[ \Xi = (\text{wexp} \Delta_G) \diamond F, \]  

where \( D_\zeta F = 0 \) for all \( \zeta \in E \).
Now we consider the differential equation:

\[
\begin{cases}
D^+_{\zeta} \Xi = 0, \\
D^-_{\zeta} \Xi = 2a(\zeta) \diamond \Xi, & \zeta \in E.
\end{cases}
\]  

(9)

By Example (2) the solution is of the form:

\[ \Xi = (\text{wexp } \Delta_G) \diamond F, \quad D^-_{\zeta} F = 0 \text{ for all } \zeta \in E. \]

We need only to find additional conditions for $F$ satisfying $D^+_{\zeta} \Xi = 0$. Noting that $D^+_{\zeta} \Delta_G = 0$, we have

\[ D^+_{\zeta} \Xi = (\text{wexp } \Delta_G) \diamond D^+_{\zeta} F = 0. \]

Hence $D^+_{\zeta} F = 0$ for all $\zeta \in E$, so $F$ is a scalar operator (Example (1)). Consequently, the solution to (9) is of the form:

\[ \Xi = C \text{ wexp } \Delta_G, \quad C \in \mathbb{C}. \]
4. Implementation Problem for CCR
4.1. The Implementation Problem

Let \( S, T \in \mathcal{L}(E, E) \) and consider transformed annihilation and creation operators:

\[
\begin{align*}
    b(\zeta) &= a(S\zeta) + a^*(T\zeta), \\
    b^*(\zeta) &= a^*(S\zeta) + a(T\zeta),
\end{align*}
\]

where \( \zeta \in E \). We know that \( b(\zeta), b^*(\zeta) \in \mathcal{L}((E), (E)) \cap \mathcal{L}((E)^*, (E)^*) \).

The implementation problem

is to find a white noise operator \( U \in \mathcal{L}((E), (E)^*) \) satisfying

\[
\begin{align*}
    (E) \xrightarrow{U} (E)^* \quad &\quad (E) \xrightarrow{U} (E)^* \\
    a(\zeta) \downarrow \quad &\quad b(\zeta) \downarrow \quad &\quad a^*(\zeta) \downarrow \quad &\quad b^*(\zeta) \downarrow \\
    (E) \xrightarrow{U} (E)^* \quad &\quad (E) \xrightarrow{U} (E)^* \\
\end{align*}
\]

Remarks: (1) \( T^* S = S^* T \) is equivalent to

\[
[b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0, \quad \zeta, \eta \in E.
\]

(2) \( S^* S - T^* T = I \) is equivalent to

\[
[b(\zeta), b^* (\eta)] = \langle \zeta, \eta \rangle, \quad \zeta, \eta \in E.
\]
4.2. Our Approach

\[ Ua(\zeta) = b(\zeta)U \]
\[ = (a(S\zeta) + a^*(T\zeta))U \]
\[ = D^+_S U + Ua(S\zeta) + a^*(T\zeta)U, \]

\[ D^+_S U = Ua(\zeta) - Ua(S\zeta) - a^*(T\zeta)U \]
\[ = Ua(\zeta - S\zeta) - a^*(T\zeta)U \]
\[ = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U. \]

Thus,

\[ Ua(\zeta) = b(\zeta)U \iff D^+_S U = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U. \]

Similarly,

\[ Ua^*(\zeta) = b^*(\zeta)U \iff (D^-_\zeta - D^+_T \zeta)U = [a^*(S\zeta - \zeta) + a(T\zeta)] \diamond U. \]
4.3. Solution to the Implementation Problem (1)

Theorem

Assume that \( S \) is invertible and that \( T^*S = S^*T \). Then a white noise operator \( U \in \mathcal{L}((E), (E)^*) \) satisfies the intertwining property:

\[
U a(\zeta) = b(\zeta) U, \quad \zeta \in E,
\]

if and only if \( U \) is of the form

\[
U = \text{wexp} \left\{ -\frac{1}{2} \Delta_G^*(TS^{-1}) + \Lambda((S^{-1})^* - I) \right\} \diamond F, \tag{10}
\]

where \( F \in \mathcal{L}((E), (E)^*) \) fulfills \( D^+_\zeta F = 0 \) for all \( \zeta \in E \).

Remark: Note that

\[
\text{wexp} \left\{ -\frac{1}{2} \Delta_G^*(TS^{-1}) \right\} = e^{-\frac{1}{2} \Delta_G^*(TS^{-1})}, \quad \text{wexp} \left\{ \Lambda((S^{-1})^* - I) \right\} = \Gamma((S^{-1})^*),
\]

where \( \Gamma((S^{-1})^*) \) is the second quantization of \((S^{-1})^*\). Hence, (10) becomes

\[
U = e^{-\frac{1}{2} \Delta_G^*(TS^{-1})} \diamond \Gamma((S^{-1})^*) \diamond F = e^{-\frac{1}{2} \Delta_G^*(TS^{-1})} \Gamma((S^{-1})^*) F.
\]
Proof.

(1) We only need to solve the differential equation

\[ D_{S\zeta}^+ U = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U. \]  

(11)

(2) We readily know that

\[ D_{S\zeta}^+ \Lambda((S^{-1})^* - I) = a(\zeta - S\zeta), \quad D_{S\zeta}^+ \Delta_G^*(TS^{-1}) = 2a^*(T\zeta). \]

(3) Then by the general result a general form of the solutions to (11) is given by

\[ U = \text{wexp} \left\{ -\frac{1}{2} \Delta_G^*(TS^{-1}) + \Lambda((S^{-1})^* - I) \right\} \diamond F, \]

where \( F \in \mathcal{L}((E), (E)^*) \) is an arbitrary white noise operator satisfying \( D_{S\zeta}^+ F = 0 \) for all \( \zeta \in E \).

(4) Since \( S \) is invertible, the last condition for \( F \) is equivalent to that \( D_{\zeta}^+ F = 0 \) for all \( \zeta \in E \).
Theorem

Assume the following conditions:

(i) $S$ is invertible;
(ii) $T^* S = S^* T$;
(iii) $S^* S - T^* T = I$;
(iv) $ST^* = TS^*$.

Then a white noise operator $U \in \mathcal{L}((E), (E)^*)$ satisfies the intertwining property:

$$U a^*(\zeta) = b^*(\zeta) U, \quad \zeta \in E,$$

if and only if $U$ is of the form:

$$U = \text{wexp} \left\{ -\frac{1}{2} \Delta^*_G (TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2} \Delta_G (S^{-1} T) \right\} \diamond G,$$

where $G \in \mathcal{L}((E), (E)^*)$ is an arbitrary white noise operator satisfying

$$(D^-_\zeta - D^+_T \zeta) G = 0 \quad \text{for all} \quad \zeta \in E.$$
Proof.

(1) Our task is to solve the differential equation:

\[(D_{\zeta} - D_{T\zeta})U = [a^*(S\zeta - \zeta) + a(T\zeta)] \ast U.\]

(2) First we need to find a solution to the differential equation:

\[(D_{\zeta} - D_{T\zeta})Y = a^*(S\zeta - \zeta) + a(T\zeta).\] (12)

(3) As is easily verified,

\[Y = \Delta_g^*(K) + \Lambda(L) + \Delta_g(M), \quad K = K^*, \quad M = M^*,\]

satisfies (12) if and only if

\[2M - L^*T = T, \quad L - 2KT = S - I.\]

Thanks to the conditions (i)–(iv) we may choose

\[K = -\frac{1}{2} TS^{-1}, \quad L = (S^{-1})^* - I, \quad M = \frac{1}{2} S^{-1}T.\]

(4) Then the assertion follows immediately from our general theorem.
Theorem

Assume the following conditions:

(i) $S$ is invertible;

(ii) $T^* S = S^* T \iff [b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0$;

(iii) $S^* S - T^* T = I \iff [b(\zeta), b^*(\eta)] = \langle \zeta, \eta \rangle$;

(iv) $ST^* = TS^*$.

A white noise operator $U \in \mathcal{L}((E), (E)^*)$ satisfies the following intertwining properties:

$$Ua(\zeta) = b(\zeta)U, \quad Ua^*(\zeta) = b^*(\zeta)U, \quad \zeta \in E,$$

if and only if $U$ is of the form:

$$U = C \exp \left\{-\frac{1}{2} \Delta^*_G(TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2} \Delta_G(S^{-1}T)\right\}$$

$$= C e^{-\frac{1}{2} \Delta^*_G(TS^{-1})} \Gamma((S^{-1})^*) e^{\frac{1}{2} \Delta_G(S^{-1}T)},$$

where $C \in \mathbb{C}$. 
Proof.

By the above two theorems, $U$ is of the form

$$U = \text{wexp} \left\{ -\frac{1}{2} \Delta^*_G (TS^{-1}) + \Lambda((S^{-1})^* - I) \right\} \Diamond F$$

$$= \text{wexp} \left\{ -\frac{1}{2} \Delta^*_G (TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2} \Delta_G (S^{-1}T) \right\} \Diamond G,$$

where $F, G \in \mathcal{L}((E), (E)^*)$ satisfy

$$D^+_\zeta F = 0, \quad (D^-_\zeta - D^+_T \zeta) G = 0, \quad \text{for all } \zeta \in E.$$

We see from the above relation that

$$G = F \Diamond \text{wexp} \left\{ -\frac{1}{2} \Delta_G (S^{-1}T) \right\}.$$ 

Since the right hand side contains no creation operators, we have

$$D^+_\zeta G = 0, \quad \zeta \in E. \quad (13)$$

Then,

$$0 = (D^-_\zeta - D^+_T \zeta) G = D^-_\zeta G, \quad \zeta \in E, \quad (14)$$

so $G$ is a scalar operator.
Final Remarks

Definition (Chung–Ji (1997))

For $U \in \mathcal{L}(E, E^*)$ we have $e^{\Delta_G(U)} \in \mathcal{L}((E), (E))$ and for $V \in \mathcal{L}(E, E^*)$ we have $\Gamma(V) \in \mathcal{L}((E), (E)^*)$. Then their composition

$$\mathcal{G}_{U,V} = \Gamma(V) e^{\Delta_G(U)}$$

becomes a white noise operator. This is called a generalized Fourier–Gauss transform and its adjoint operator $\mathcal{G}_{U,V}^*$ a generalized Fourier–Mehler transform.

(1) The solution to the implementation problem:

$$U = C e^{-\frac{1}{2} \Delta_G^*(TS^{-1})} \Gamma((S^{-1})^*) e^{\frac{1}{2} \Delta_G(S^{-1}T)}$$

is the composition of the generalized Fourier–Mehler and Fourier–Gauss transforms.  

$\Rightarrow$ a new (white noise) approach to Bogoliubov transform

(2) Integral transform (lecture by H. S. Chung)

$$\int_{E^*} \phi(ax + by) \mu(dx) = \mathcal{G}_{(a^2+b^2-1)/2,b\phi(y)}$$

A systematic study of the transformations will be presented by Un Cig Ji.


