Recent Developments in Quantum White Noise Calculus: Quantum White Noise Derivatives and Implementation Problem

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1. Quantum White Noise Calculus

The Boson Fock space over $H = L^2(T)$ is defined by

$$\Gamma(H) = \left\{ \phi = (f_n) \,;\, f_n \in H^{\widehat{\otimes} n},\ \|\phi\|^2 = \sum_{n=0}^{\infty} n! |f_n|_0^2 < \infty \right\},$$

where T is a topological space equipped with a σ-finite Borel measure dt, $|f_n|_0$ is the usual L^2-norm of $H^{\widehat{\otimes} n} = L^2_{\mathrm{sym}}(T^n)$.

The *annihilation* and *creation operator* at a point $t \in T$

$$a_t : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, n\xi(t)\xi^{\otimes(n-1)}, 0, 0, \dots)$$
$$a_t^* : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, 0, \xi^{\otimes n} \widehat{\otimes} \delta_t, 0, \dots)$$

A "general" Fock space operator takes the form:

$$\sum_{l,m=0}^{\infty} \int_{T^{l+m}} \kappa_{l,m}(s_1, \dots, s_l, t_1, \dots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m$$

Quantum field theory: e.g., Haag (1955), Berezin (1966), Krée (1988), etc.

1.2. White Noise Approach

I) Gelfand triple for $H = L^2(T)$:

$$E \subset H = L^2(T) \subset E^*, \qquad E = \operatorname*{proj\,lim}_{p\to\infty} E_p\,, \quad E^* = \operatorname*{ind\,lim}_{p\to\infty} E_{-p}\,,$$

where E_p is a dense subspace of H and is a Hilbert space for itself.

II) Gelfand triple for $\Gamma(H)$ (e.g., Hida–Kubo–Takenaka space):

$$(E) \subset \Gamma(H) \subset (E)^*, \qquad (E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p), \quad (E)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p}),$$

Notes: (1) $\Gamma(H) \cong L^2(E^*, \mu)$ (Wiener–Itô–Segal isomorphism)
(2) (E) is the space of test functions and $(E)^*$ the space of distributions.

Definition

A continuous operator from (E) into $(E)^*$ is called a *white noise operator*. Let $\mathcal{L}((E), (E)^*)$ denote the space of white noise operators, equipped with the topology of bounded convergence.

Note: $\mathcal{L}((E), (E))$, $\mathcal{L}((E)^*, (E)^*)$ and $\mathcal{B}(\Gamma(H))$ are subspaces of $\mathcal{L}((E), (E)^*)$.

1.3. Integral Kernel Operators

Theorem

$a_t \in \mathcal{L}((E),(E))$ and $a_t^* \in \mathcal{L}((E)^*,(E)^*)$ for all $t \in \mathbb{R}$. Moreover, both maps $t \mapsto a_t \in \mathcal{L}((E),(E))$ and $t \mapsto a_t^* \in \mathcal{L}((E)^*,(E)^*)$ are operator-valued rapidly decreasing functions, i.e., belongs to $E \otimes \mathcal{L}((E),(E))$ and $E \otimes \mathcal{L}((E)^*,(E)^*)$, respectively. (The pair $\{a_t, a_t^* \,;\, t \in T\}$ is called the quantum white noise on T.)

Definition

Given $\kappa_{l,m} \in (E^{\otimes(l+m)})^*$, $l, m = 0, 1, 2, \dots$, the *integral kernel operator* $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E),(E)^*)$ is defined by

$$\langle\!\langle \Xi_{l,m}(\kappa_{l,m})\varphi_\xi,\, \varphi_\eta \rangle\!\rangle = \langle \kappa_{l,m},\, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle e^{\langle \xi,\, \eta \rangle},$$

where $\varphi_\xi = (1, \xi, \xi^{\otimes 2}/2!, \cdots)$ is the exponential vector. We write

$$\Xi_{l,m}(\kappa_{l,m})$$
$$= \int_{T^{l+m}} \kappa_{l,m}(s_1, \cdots, s_l, t_1, \cdots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m$$

Theorem (Obata (1993))

Every white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ *admits the infinite series expansion:*

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \qquad \kappa_{l,m} \in (E^{\otimes(l+m)})^*,$$

where the right-hand side converges in $\mathcal{L}((E),(E)^*)$*. If* $\Xi \in \mathcal{L}((E),(E))$*, then* $\kappa_{l,m} \in E^{\otimes l} \otimes (E^{\otimes m})^*$ *and the series converges in* $\mathcal{L}((E),(E))$*.*

Berezin (1966): for bounded operators in a weak sense

Krée (1988): introduced distribution theory for operators

Our Standpoint

A white noise operator Ξ as a function of quantum white noise:

$$\Xi = \Xi(a_s, a_t^*; s, t \in T)$$

2. Quantum White Noise Derivatives

2.1. Definition

White noise (Hida) derivative: for a white noise functional $\Phi = \Phi(\dot{B}(t)\,;\, t \in T)$,

$$\frac{\delta \Phi}{\delta \dot{B}(t)} \quad \Longrightarrow \quad \partial_t \Phi \ \text{ or } \ a_t \Phi$$

A quantum counterpart: for a white noise operator as a function of quantum white noise:

$$\Xi = \Xi(a_s, a_t^*; s, t \in T)$$

We should like to define the derivatives with respect to a_s and a_t^*:

$$\frac{\delta \Xi}{\delta a_s} \quad \text{and} \quad \frac{\delta \Xi}{\delta a_t^*}$$

Expected properties:

$$\frac{\delta}{\delta a_s} \int f(t) a_t dt = f(s) I$$

$$\frac{\delta}{\delta a_s} \int f(s,t) a_s a_t ds dt = \int f(s,t) a_t \, dt + \int f(t,s) a_t \, dt$$

$$\frac{\delta}{\delta a_t^*} \int f(s,t) a_s a_t^* ds dt = \int f(s,t) a_s \, ds$$

2.1. Definition

Definition (Ji–Obata (2007))

For $\Xi \in \mathcal{L}((E),(E)^*)$ and $\zeta \in E$ we define $D_\zeta^{\pm}\Xi \in \mathcal{L}((E),(E)^*)$ by

$$D_\zeta^{+}\Xi = [a(\zeta),\Xi], \qquad D_\zeta^{-}\Xi = -[a^*(\zeta),\Xi].$$

These are called the *creation derivative* and *annihilation derivative* of Ξ, respectively. Both together are called the *quantum white noise derivatives*.

Note: For $\zeta \in E$, both

$$a(\zeta) = \Xi_{0,1}(\zeta) = \int_T \zeta(t) a_t \, dt, \quad a^*(\zeta) = \Xi_{1,0}(\zeta) = \int_T \zeta(t) a_t^* \, dt,$$

belong to $\mathcal{L}((E),(E)) \cap \mathcal{L}((E)^*,(E)^*)$.

Some properties:

1. $(D_\zeta^{+}\Xi)^* = D_\zeta^{-}(\Xi^*)$ and $(D_\zeta^{-}\Xi)^* = D_\zeta^{+}(\Xi^*)$.
2. $D_\zeta^{\pm}$ is a continuous linear map from $\mathcal{L}((E),(E)^*)$ into itself.
3. Moreover, $(\zeta,\Xi) \mapsto D_\zeta^{\pm}\Xi$ is a continuous bilinear map from $E \times \mathcal{L}((E),(E)^*)$ into $\mathcal{L}((E),(E)^*)$.

Remark: Pointwisely Defined QWN-Derivatives

Recall: The annihilation and creation operators

$$a(f) = \int_T f(t) a_t \, dt, \qquad a^*(f) = \int_T f(t) a_t^* \, dt \, .$$

It is natural to write

$$D_\zeta^+ = \int_T \zeta(t) D_t^+ \, dt, \qquad D_\zeta^- = \int_T \zeta(t) D_t^- \, dt.$$

In fact, this expression is useful for computation.

However, it is not straightforward to define $D_t^\pm$ for each point $t \in T$ because

$$D_t^+ \Xi = [a_t, \Xi] = a_t \Xi - \Xi a_t, \qquad D_t^- \Xi = -[a_t^*, \Xi] = -a_t^* \Xi + \Xi a_t^*$$

are not well-defined in general.

Nevertheless, the pointwisely defined quantum white noise derivatives $D_t^\pm$ are well formulated for admissible white noise operators (Ji–Obata, 2009, to appear).

The canonical correspondence (kernel theorem) between $S \in \mathcal{L}(E, E^*)$ and $\tau = \tau_S \in (E \otimes E)^*$ is given by $\langle \tau_S, \eta \otimes \xi \rangle = \langle S\xi, \eta \rangle$ for $\xi, \eta \in E$.

(1) The *generalized Gross Laplacian* associated with S is defined by

$$\Delta_{\mathrm{G}}(S) = \Xi_{0,2}(\tau_S) = \int_{T\times T} \tau_S(s,t) a_s a_t \, ds dt$$

Note that $\Delta_{\mathrm{G}}(S) \in \mathcal{L}((E), (E))$. Then,

$$D_\zeta^+ \Delta_{\mathrm{G}}(S) = 0, \qquad D_\zeta^- \Delta_{\mathrm{G}}(S) = a(S\zeta) + a(S^*\zeta)$$

In fact, since

$$D_t^- \Delta_{\mathrm{G}}(S) = \int_T \tau_S(s,t) a_s \, ds + \int_T \tau_S(t,s) a_s \, ds$$

we have

$$\begin{aligned} D_\zeta^- \Delta_{\mathrm{G}}(S) &= \int_{T\times T} \tau_S(s,t) a_s \zeta(t) \, ds dt + \int_{T\times T} \tau_S(t,s) a_s \zeta(t) \, ds dt \\ &= \int_T S\zeta(s) a_s \, ds + \int_T S^*\zeta(s) a_s \, ds = a(S\zeta) + a(S^*\zeta) \end{aligned}$$

(2) The adjoint of $\Delta_{\mathrm{G}}(S) \in \mathcal{L}((E)^*, (E)^*)$ is given by

$$\Delta_{\mathrm{G}}^*(S) = \Xi_{2,0}(\tau_S) = \int_{T\times T} \tau_S(s,t) a_s^* a_t^* \, dsdt$$

The quantum white noise derivatives are given by

$$D_\zeta^- \Delta_{\mathrm{G}}^*(S) = 0, \qquad D_\zeta^+ \Delta_{\mathrm{G}}^*(S) = a^*(S\zeta) + a^*(S^*\zeta)$$

(3) The *conservation operator* associated with S is defined by

$$\Lambda(S) = \Xi_{1,1}(\tau_S) = \int_{T\times T} \tau_S(s,t) a_s^* a_t \, dsdt$$

In general, $\Lambda(S) \in \mathcal{L}((E), (E)^*)$.
The quantum white noise derivatives are given by

$$D_\zeta^- \Lambda(S) = a^*(S\zeta), \qquad D_\zeta^+ \Lambda(S) = a(S^*\zeta).$$

2.3. Wick Products

Definition

For $\Xi_1, \Xi_2 \in \mathcal{L}((E),(E)^*)$ the Wick (or normal-ordered) product $\Xi_1 \diamond \Xi_2$ is defined by

$$(\Xi_1 \diamond \Xi_2)\widehat{\ }(\xi,\eta) = \widehat{\Xi}_1(\xi,\eta)\widehat{\Xi}_2(\xi,\eta)e^{-\langle \xi,\eta \rangle}, \qquad \xi,\eta \in E,$$

where $\widehat{\Xi}(\xi,\eta)$ is the symbol of a white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ defined by

$$\widehat{\Xi}(\xi,\eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi,\eta \in E,$$

where $\phi_\xi = (1, \xi, \cdots, \xi^{\otimes n}/n!, \cdots)$ is an *exponential vector*.

Some properties:

1. For any $\Xi \in \mathcal{L}((E),(E)^*)$ we have

$$a_t \diamond \Xi = \Xi \diamond a_t = \Xi a_t \,, \qquad a_t^* \diamond \Xi = \Xi \diamond a_t^* = a_t^* \Xi.$$

2. Equipped with the Wick product, $\mathcal{L}((E),(E)^*)$ becomes a commutative algebra.

Definition

A continuous linear map $\mathcal{D} : \mathcal{L}((E), (E)^*) \to \mathcal{L}((E), (E)^*)$ is called a *Wick derivation* if

$$\mathcal{D}(\Xi_1 \diamond \Xi_2) = (\mathcal{D}\Xi_1) \diamond \Xi_2 + \Xi_1 \diamond (\mathcal{D}\Xi_2)$$

for all $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$.

Theorem

The creation and annihilation derivatives $D_\zeta^\pm$ are Wick derivations for any $\zeta \in E$.

Moreover, it is proved that a general Wick derivation $\mathcal{D}$ is expressed in the form:

$$\mathcal{D} = \int_T F(t) \diamond D_t^+ dt + \int_T G(t) \diamond D_t^- dt,$$

where $F, G \in E \otimes \mathcal{L}((E), (E)^*)$.

Proof.

In general, for $\Xi \in \mathcal{L}((E),(E)^*)$ we have

$$\begin{aligned}
(D_\zeta^+ \Xi)\widehat{\ }(\xi,\eta) &= \langle\!\langle (a(\zeta)\Xi - \Xi a(\zeta))\phi_\xi, \phi_\eta \rangle\!\rangle \\
&= \langle\!\langle \Xi\phi_\xi, a^*(\zeta)\phi_\eta \rangle\!\rangle - \langle\!\langle \Xi a(\zeta)\phi_\xi, \phi_\eta \rangle\!\rangle \\
&= \left.\frac{d}{dt}\right|_{t=0} \langle\!\langle \Xi\phi_\xi, \phi_{\eta+t\zeta} \rangle\!\rangle - \langle \xi, \zeta\rangle \langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle \\
&= \left.\frac{d}{dt}\right|_{t=0} \widehat{\Xi}(\xi, \eta + t\zeta) - \langle \xi,\zeta\rangle \widehat{\Xi}(\xi,\eta). \qquad (1)
\end{aligned}$$

Then for $\Xi = \Xi_1 \diamond \Xi_2$ we have

$$\begin{aligned}
(D_\zeta^+ \Xi)\widehat{\ }(\xi,\eta) = & \left.\frac{d}{dt}\right|_{t=0} \widehat{\Xi}_1(\xi, t\zeta+\eta)\widehat{\Xi}_2(\xi, t\zeta+\eta) e^{-\langle \xi, t\zeta+\eta\rangle} \\
& \qquad - \langle \xi,\zeta\rangle \widehat{\Xi}_1(\xi,\eta)\widehat{\Xi}_2(\xi,\eta) e^{-\langle \xi,\eta\rangle} \\
= & \left(\left.\frac{d}{dt}\right|_{t=0} \widehat{\Xi}_1(\xi, t\zeta+\eta) \right) \widehat{\Xi}_2(\xi,\eta) e^{-\langle \xi,\eta\rangle} \\
& \qquad + \widehat{\Xi}_1(\xi,\eta) \left(\left.\frac{d}{dt}\right|_{t=0} \widehat{\Xi}_2(\xi, t\zeta+\eta) \right) e^{-\langle \xi,\eta\rangle} \\
& \qquad - 2\langle \xi,\zeta\rangle \widehat{\Xi}_1(\xi,\eta)\widehat{\Xi}_2(\xi,\eta) e^{-\langle \xi,\eta\rangle}.
\end{aligned}$$

Viewing (1) once again, we obtain

$$(D_\zeta^+ \Xi)\widehat{\ }(\xi,\eta) = ((D_\zeta^+ \Xi_1) \diamond \Xi_2)\widehat{\ }(\xi,\eta) + (\Xi_1 \diamond (D_\zeta^+ \Xi_2))\widehat{\ }(\xi,\eta).$$

3. Differential Equations for White Noise Operators

Given a Wick derivation $\mathcal{D}$ and a white noise operator $G \in \mathcal{L}((E),(E)^*)$, consider

$$\mathcal{D}\Xi = G \diamond \Xi \tag{2}$$

The *Wick exponential* is defined by

$$\mathrm{wexp}\, Y = \sum_{n=0}^{\infty} \frac{1}{n!} Y^{\diamond n}, \qquad Y \in \mathcal{L}((E),(E)^*),$$

whenever the series converges in $\mathcal{L}((E),(E)^*)$.

Theorem

Assume that there exists an operator $\boldsymbol{Y} \in \boldsymbol{\mathcal{L}}((\boldsymbol{E}), (\boldsymbol{E})^*)$ *such that* $\boldsymbol{\mathcal{D}Y} = \boldsymbol{G}$ *and* $\mathbf{wexp}\ \boldsymbol{Y}$ *is defined in* $\boldsymbol{\mathcal{L}}((\boldsymbol{E}), (\boldsymbol{E})^*)$*. Then every solution to*

$$\boldsymbol{\mathcal{D}\Xi} = \boldsymbol{G} \diamond \boldsymbol{\Xi} \tag{3}$$

is of the form:

$$\boldsymbol{\Xi} = (\mathbf{wexp}\ \boldsymbol{Y}) \diamond \boldsymbol{F}, \tag{4}$$

where $\boldsymbol{F} \in \boldsymbol{\mathcal{L}}((\boldsymbol{E}), (\boldsymbol{E})^*)$ *satisfying* $\boldsymbol{\mathcal{D}F} = \mathbf{0}$*.*

Proof.

It is straightforward to see that (4) is a solution to (3). To prove the converse, let $\boldsymbol{\Xi}$ be an arbitrary solution to (3). Set

$$\boldsymbol{F} = (\mathbf{wexp}\,(-\boldsymbol{Y})) \diamond \boldsymbol{\Xi}.$$

Obviously, $\boldsymbol{F} \in \boldsymbol{\mathcal{L}}((\boldsymbol{E}), (\boldsymbol{E})^*)$ and $\boldsymbol{\Xi} = (\mathbf{wexp}\ \boldsymbol{Y}) \diamond \boldsymbol{F}$. We only need to show that $\boldsymbol{\mathcal{D}F} = \mathbf{0}$. In fact,

$$\begin{aligned}\boldsymbol{\mathcal{D}F} &= -\boldsymbol{\mathcal{D}Y} \diamond (\mathbf{wexp}\,(-\boldsymbol{Y})) \diamond \boldsymbol{\Xi} + (\mathbf{wexp}\,(-\boldsymbol{Y})) \diamond \boldsymbol{\mathcal{D}\Xi} \\ &= -\boldsymbol{G} \diamond (\mathbf{wexp}\,(-\boldsymbol{Y})) \diamond \boldsymbol{\Xi} + (\mathbf{wexp}\,(-\boldsymbol{Y})) \diamond \boldsymbol{G} \diamond \boldsymbol{\Xi} = \mathbf{0}.\end{aligned}$$

This completes the proof. □

3.2. Example (1)

Let us consider the (system of) differential equations:

$$D_{\zeta}^{+}\Xi = 0, \qquad \zeta \in E. \tag{5}$$

We expect easily that $\Xi = \Xi(a_s, a_t^*; s, t \in T)$ does not depend on the creation operators. In fact, by Fock expansion we see that the solutions to (5) are given by

$$\Xi = \sum_{m=0}^{\infty} \Xi_{0,m}(\kappa_{0,m}).$$

In a similar manner, the solutions to

$$D_{\zeta}^{-}\Xi = 0, \qquad \zeta \in E, \tag{6}$$

are given by

$$\Xi = \sum_{l=0}^{\infty} \Xi_{l,0}(\kappa_{l,0}).$$

Consequently, a white noise operator satisfying both (5) and (6) are the scalar operators. Thus, *the irreducibility of the canonical commutation relation is reproduced*.

3.3. Example (2)

Let us consider the differential equation:

$$D_\zeta^- \Xi = 2a(\zeta) \diamond \Xi, \qquad \zeta \in E. \tag{7}$$

We need to find $Y \in \mathcal{L}((E),(E)^*)$ satisfying $D_\zeta^- Y = 2a(\zeta)$.

In fact, $Y = \Delta_G$ is a solution.

Moreover, it is easily verified that $\mathbf{wexp}\ \Delta_G$ is defined in $\mathcal{L}((E),(E))$.

Then, a general solution to (7) is of the form:

$$\Xi = (\mathbf{wexp}\ \Delta_G) \diamond F, \tag{8}$$

where $D_\zeta^- F = 0$ for all $\zeta \in E$.

3.3. Example (3)

Now we consider the differential equation:

$$\begin{cases} D_\zeta^+ \Xi = 0, \\ D_\zeta^- \Xi = 2a(\zeta) \diamond \Xi, \qquad \zeta \in E. \end{cases} \tag{9}$$

By Example (2) the solution is of the form:

$$\Xi = (\mathrm{wexp}\ \Delta_{\mathrm{G}}) \diamond F, \qquad D_\zeta^- F = 0 \text{ for all } \zeta \in E.$$

We need only to find additional conditions for F satisfying $D_\zeta^+ \Xi = 0$.
Noting that $D_\zeta^+ \Delta_{\mathrm{G}} = 0$, we have

$$D_\zeta^+ \Xi = (\mathrm{wexp}\ \Delta_{\mathrm{G}}) \diamond D_\zeta^+ F = 0.$$

Hence $D_\zeta^+ F = 0$ for all $\zeta \in E$, so F is a scalar operator (Example (1)).
Consequently, the solution to (9) is of the form:

$$\Xi = C\ \mathrm{wexp}\ \Delta_{\mathrm{G}}, \qquad C \in \mathbb{C}.$$

4. Implementation Problem for CCR

4.1. The Implementation Problem

Let $S, T \in \mathcal{L}(E, E)$ and consider transformed annihilation and creation operators:

$$b(\zeta) = a(S\zeta) + a^*(T\zeta), \quad b^*(\zeta) = a^*(S\zeta) + a(T\zeta),$$

where $\zeta \in E$. We know that $b(\zeta), b^*(\zeta) \in \mathcal{L}((E), (E)) \cap \mathcal{L}((E)^*, (E)^*)$.

The implementation problem

is to find a white noise operator $U \in \mathcal{L}((E), (E)^*)$ satisfying

$$\begin{tikzcd} (E) \arrow[r, "U"] \arrow[d, "a(\zeta)"'] & (E)^* \arrow[d, "b(\zeta)"] \\ (E) \arrow[r, "U"'] & (E)^* \end{tikzcd} \qquad \begin{tikzcd} (E) \arrow[r, "U"] \arrow[d, "a^*(\zeta)"'] & (E)^* \arrow[d, "b^*(\zeta)"] \\ (E) \arrow[r, "U"'] & (E)^* \end{tikzcd}$$

Remarks: (1) $T^*S = S^*T$ is equivalent to

$$[b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0, \qquad \zeta, \eta \in E.$$

(2) $S^*S - T^*T = I$ is equivalent to

$$[b(\zeta), b^*(\eta)] = \langle \zeta, \eta \rangle, \qquad \zeta, \eta \in E.$$

4.2. Our Approach

$$
\begin{aligned}
Ua(\zeta) &= b(\zeta)U \\
&= (a(S\zeta) + a^*(T\zeta))\,U \\
&= D^+_{S\zeta}U + Ua(S\zeta) + a^*(T\zeta)U,
\end{aligned}
$$

$$
\begin{aligned}
D^+_{S\zeta}U &= Ua(\zeta) - Ua(S\zeta) - a^*(T\zeta)U \\
&= Ua(\zeta - S\zeta) - a^*(T\zeta)U \\
&= [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U.
\end{aligned}
$$

Thus,

$$
Ua(\zeta) = b(\zeta)U \quad \Longleftrightarrow \quad D^+_{S\zeta}U = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U.
$$

Similarly,

$$
Ua^*(\zeta) = b^*(\zeta)U \quad \Longleftrightarrow \quad (D^-_{\zeta} - D^+_{T\zeta})U = [a^*(S\zeta - \zeta) + a(T\zeta)] \diamond U.
$$

4.3. Solution to the Implementation Problem (1)

Theorem

Assume that S is invertible and that $T^*S = S^*T$. Then a white noise operator $U \in \mathcal{L}((E),(E)^*)$ satisfies the intertwining property:

$$Ua(\zeta) = b(\zeta)U, \qquad \zeta \in E,$$

if and only if U is of the form

$$U = \mathrm{wexp}\left\{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1}) + \Lambda((S^{-1})^* - I)\right\} \diamond F, \tag{10}$$

where $F \in \mathcal{L}((E),(E)^*)$ fulfills $D_\zeta^+ F = 0$ for all $\zeta \in E$.

Remark: Note that

$$\mathrm{wexp}\left\{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1})\right\} = e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1})}, \quad \mathrm{wexp}\left\{\Lambda((S^{-1})^* - I)\right\} = \Gamma((S^{-1})^*),$$

where $\Gamma((S^{-1})^*)$ is the second quantization of $(S^{-1})^*$. Hence, (10) becomes

$$\begin{aligned} U &= e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1})} \diamond \Gamma((S^{-1})^*) \diamond F \\ &= e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1})}\Gamma((S^{-1})^*)F. \end{aligned}$$

Proof.

(1) We only need to solve the differential equation

$$D^+_{S\zeta} U = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U. \tag{11}$$

(2) We readily know that

$$D^+_{S\zeta} \Lambda((S^{-1})^* - I) = a(\zeta - S\zeta), \qquad D^+_{S\zeta} \Delta^*_{\mathrm{G}}(TS^{-1}) = 2a^*(T\zeta).$$

(3) Then by the general result a general form of the solutions to (11) is given by

$$U = \mathrm{wexp}\left\{-\frac{1}{2}\Delta^*_{\mathrm{G}}(TS^{-1}) + \Lambda((S^{-1})^* - I)\right\} \diamond F,$$

where $F \in \mathcal{L}((E),(E)^*)$ is an arbitrary white noise operator satisfying $D^+_{S\zeta} F = 0$ for all $\zeta \in E$.

(4) Since S is invertible, the last condition for F is equivalent to that $D^+_\zeta F = 0$ for all $\zeta \in E$. $\square$

Solution to the Implementation Problem (2)

Theorem

Assume the following conditions:

(i) S *is invertible;*

(ii) $T^*S = S^*T$;

(iii) $S^*S - T^*T = I$;

(iv) $ST^* = TS^*$.

Then a white noise operator $U \in \mathcal{L}((E),(E)^*)$ *satisfies the intertwining property:*

$$Ua^*(\zeta) = b^*(\zeta)U, \qquad \zeta \in E,$$

if and only if U *is of the form:*

$$U = \mathrm{wexp}\left\{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T)\right\} \diamond G,$$

where $G \in \mathcal{L}((E),(E)^*)$ *is an arbitrary white noise operator satisfying*

$$(D_\zeta^- - D_{T\zeta}^+)G = 0 \quad \textit{for all } \zeta \in E.$$

Proof.

(1) Our task is to solve the differential equation:

$$(D_\zeta^- - D_{T\zeta}^+)U = [a^*(S\zeta - \zeta) + a(T\zeta)] \diamond U.$$

(2) First we need to find a solution to the differential equation:

$$(D_\zeta^- - D_{T\zeta}^+)Y = a^*(S\zeta - \zeta) + a(T\zeta). \tag{12}$$

(3) As is easily verified,

$$Y = \Delta_{\mathrm{G}}^*(K) + \Lambda(L) + \Delta_{\mathrm{G}}(M), \qquad K = K^*, \quad M = M^*,$$

satisfies (12) if and only if

$$2M - L^*T = T, \qquad L - 2KT = S - I.$$

Thanks to the conditions (i)–(iv) we may choose

$$K = -\frac{1}{2}\,TS^{-1}, \quad L = (S^{-1})^* - I, \quad M = \frac{1}{2}\,S^{-1}T.$$

(4) Then the assertion follows immediately from our general theorem. □

Solution to the Implementation Problem (3)

Theorem

Assume the following conditions:

(i) S *is invertible;*

(ii) $T^*S = S^*T \iff [b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0$*;*

(iii) $S^*S - T^*T = I \iff [b(\zeta), b^*(\eta)] = \langle \zeta, \eta \rangle$*;*

(iv) $ST^* = TS^*$*.*

A white noise operator $U \in \mathcal{L}((E), (E)^*)$ *satisfies the following intertwining properties:*

$$Ua(\zeta) = b(\zeta)U, \qquad Ua^*(\zeta) = b^*(\zeta)U, \qquad \zeta \in E,$$

if and only if U *is of the form:*

$$\begin{aligned} U &= C \text{ wexp} \left\{ -\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T) \right\} \\ &= C\, e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1})} \Gamma((S^{-1})^*)\, e^{\frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T)}, \end{aligned}$$

where $C \in \mathbb{C}$*.*

Proof.

By the above two theorems, U is of the form

$$\begin{aligned} U &= \mathrm{wexp}\left\{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1}) + \Lambda((S^{-1})^* - I)\right\} \diamond F \\ &= \mathrm{wexp}\left\{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T)\right\} \diamond G, \end{aligned}$$

where $F, G \in \mathcal{L}((E), (E)^*)$ satisfy

$$D_\zeta^+ F = 0, \quad (D_\zeta^- - D_{T\zeta}^+)G = 0, \quad \text{for all } \zeta \in E.$$

We see from the above relation that

$$G = F \diamond \mathrm{wexp}\left\{-\frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T)\right\}.$$

Since the right hand side contains no creation operators, we have

$$D_\zeta^+ G = 0, \qquad \zeta \in E. \tag{13}$$

Then,

$$0 = (D_\zeta^- - D_{T\zeta}^+)G = D_\zeta^- G, \qquad \zeta \in E, \tag{14}$$

so G is a scalar operator. □

Definition (Chung–Ji (1997))

For $U \in \mathcal{L}(E, E^*)$ we have $e^{\Delta_{\mathrm{G}}(U)} \in \mathcal{L}((E),(E))$ and for $V \in \mathcal{L}(E, E^*)$ we have $\Gamma(V) \in \mathcal{L}((E),(E)^*)$. Then their composition

$$\mathcal{G}_{U,V} = \Gamma(V)\, e^{\Delta_{\mathrm{G}}(U)}$$

becomes a white noise operator. This is called a *generalized Fourier–Gauss transform* and its adjoint operator $\mathcal{G}_{U,V}^*$ a *generalized Fourier–Mehler transform*.

(1) The solution to the implementation problem:

$$U = C\, e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1})}\Gamma((S^{-1})^*)\, e^{\frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T)}$$

is the composition of the generalized Fourier–Mehler and Fourier–Gauss transforms.

$\Longrightarrow$ a new (white noise) approach to Bogoliubov transform

(2) Integral transform (lecture by H. S. Chung)

$$\int_{E^*} \phi(ax+by)\mu(dx) = \mathcal{G}_{(a^2+b^2-1)/2,b}\phi(y)$$

A systematic study of the transformations will be presented by Un Cig Ji.

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