Quantum White Noise Derivatives and Implementation Problem

On the occasion of thier 60th birthdays of Professors K. R. Ito and I. Ojima

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The Implementation Problem

$a(\xi), a^*(\eta)$: annihilation and creation operators on Boson Fock space $\Gamma(H)$ satisfying

$$\text{CCR:} \qquad [a(\xi), a(\eta)] = [a^*(\xi), a^*(\eta)] = 0, \quad [a(\xi), a^*(\eta)] = \langle \xi, \eta \rangle$$

Consider transformed annihilation and creation operators:

$$b(\zeta) = a(S\zeta) + a^*(T\zeta), \quad b^*(\zeta) = a^*(S\zeta) + a(T\zeta).$$

The implementation problem [Berezin (1966), Ruijsenaars (1977), ...]

is to find a (unitary) operator U on the Boson Fock space $\Gamma(H)$ satisfying

$$\begin{array}{ccc} \Gamma(H) & \xrightarrow{\;U\;} & \Gamma(H) \\ {\scriptstyle a(\zeta)}\downarrow & & \downarrow{\scriptstyle b(\zeta)} \\ \Gamma(H) & \xrightarrow[\;U\;]{} & \Gamma(H) \end{array} \qquad\qquad \begin{array}{ccc} \Gamma(H) & \xrightarrow{\;U\;} & \Gamma(H) \\ {\scriptstyle a^*(\zeta)}\downarrow & & \downarrow{\scriptstyle b^*(\zeta)} \\ \Gamma(H) & \xrightarrow[\;U\;]{} & \Gamma(H) \end{array}$$

Remarks: (1) $[b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0 \iff T^*S = S^*T$

(2) $[b(\zeta), b^*(\eta)] = \langle \zeta, \eta \rangle \iff S^*S - T^*T = I$

Plan

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1. Quantum White Noise Calculus

1.1. Background and Notation

The Boson Fock space over $H = L^2(T)$ is defined by

$$\Gamma(H) = \left\{ \phi = (f_n) \, ; \, f_n \in H^{\widehat{\otimes} n}, \; \|\phi\|^2 = \sum_{n=0}^{\infty} n! |f_n|_0^2 < \infty \right\},$$

where T is a topological space equipped with a σ-finite Borel measure dt, $|f_n|_0$ is the usual L^2-norm of $H^{\widehat{\otimes} n} = L^2_{\mathrm{sym}}(T^n)$.

The *annihilation* and *creation operator* at a point $t \in T$

$$a_t : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, n\xi(t)\xi^{\otimes(n-1)}, 0, 0, \dots)$$
$$a_t^* : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, 0, \xi^{\otimes n} \widehat{\otimes} \delta_t, 0, \dots)$$

A "general" Fock space operator takes the form:

$$\sum_{l,m=0}^{\infty} \int_{T^{l+m}} \kappa_{l,m}(s_1, \dots, s_l, t_1, \dots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m$$

Quantum field theory: e.g., Haag (1955), Berezin (1966), Krée (1988), etc.

1.2. White Noise Operators

I) Gelfand triple for $H = L^2(T)$:

$$E \subset H = L^2(T) \subset E^*, \qquad E = \operatorname*{proj\,lim}_{p\to\infty} E_p\,, \quad E^* = \operatorname*{ind\,lim}_{p\to\infty} E_{-p}\,,$$

where E_p is a dense subspace of H and is a Hilbert space for itself.

II) Gelfand triple for $\Gamma(H)$ (e.g., Hida–Kubo–Takenaka space (1980)):

$$(E) \subset \Gamma(H) \subset (E)^*, \qquad (E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p), \quad (E)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p}),$$

Note: (1) $\Gamma(H) \cong L^2(E^*, \mu)$ (Wiener–Itô–Segal isomorphism)
(2) (E) is the space of test functions and $(E)^*$ the space of distributions.

Definition (White noise operator)

A continuous operator from (E) into $(E)^*$ is called a *white noise operator*. Let $\mathcal{L}((E),(E)^*)$ denote the space of white noise operators, equipped with the topology of bounded convergence.

Note: $\mathcal{L}((E),(E))$, $\mathcal{L}((E)^*,(E)^*)$ and $\mathcal{B}(\Gamma(H))$ are subspaces of $\mathcal{L}((E),(E)^*)$.

1.3. Quantum White Noise

Theorem (Quantum white noise is very regular)

$a_t \in \mathcal{L}((E),(E))$ and $a_t^* \in \mathcal{L}((E)^*,(E)^*)$ for all $t \in \mathbb{R}$. Moreover, both maps $t \mapsto a_t \in \mathcal{L}((E),(E))$ and $t \mapsto a_t^* \in \mathcal{L}((E)^*,(E)^*)$ are operator-valued rapidly decreasing functions, i.e., belongs to $E \otimes \mathcal{L}((E),(E))$ and $E \otimes \mathcal{L}((E)^*,(E)^*)$, respectively. (The pair $\{a_t, a_t^* \,;\, t \in T\}$ is called the quantum white noise on T.)

Smeared operators

$$a(\zeta) = \int \zeta(t) a_t \, dt, \qquad a^*(\zeta) = \int \zeta(t) a_t^* \, dt$$

Traditional approach

1. ζ is a test function, e.g., $\zeta \in \mathcal{S}(\mathbb{R})$.
2. $a(\zeta)$, $a^*(\zeta)$ are unbounded opereators in $\Gamma(H)$.

White noise approach

1. ζ is a distribution, e.g., $\zeta \in \mathcal{S}'(\mathbb{R})$.
2. $a(\zeta)$, $a^*(\zeta)$ are white noise operators, i.e., belong to $\mathcal{L}((E),(E)^*)$.
3. In fact, $a(\zeta) \in \mathcal{L}((E),(E))$ and $a^*(\zeta) \in \mathcal{L}((E)^*,(E)^*)$.

Definition (Integral kernel operator)

Given $\kappa_{l,m} \in (E^{\otimes(l+m)})^*$, $l, m = 0, 1, 2, \dots$, the *integral kernel operator*

$$\Xi_{l,m}(\kappa_{l,m}) = \int_{T^{l+m}} \kappa_{l,m}(s_1, \cdots, s_l, t_1, \cdots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m$$

is defined and is a white noise operator, i.e., $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E), (E)^*)$.

Theorem (O.(1993), cf. Berezin (1966), Krée (1988))

Every white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ admits the infinite series expansion:

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \qquad \kappa_{l,m} \in (E^{\otimes(l+m)})^*,$$

where the right-hand side converges in $\mathcal{L}((E), (E)^*)$. If $\Xi \in \mathcal{L}((E), (E))$, then $\kappa_{l,m} \in E^{\otimes l} \otimes (E^{\otimes m})^*$ and the series converges in $\mathcal{L}((E), (E))$.

2. Quantum White Noise Derivatives

2.1. Definition

For a Brownian (or white noise) function Φ stochastic derivatives (gradients) were introduced by Malliavin, Hida, Gross, ...

$$\nabla\Phi, \quad \frac{\delta\Phi}{\delta\dot{B}(t)}, \quad \partial_t\Phi, \quad a_t\Phi$$

A quantum counterpart

A white noise operator Ξ is considered as a function of quantum white noise: $\Xi = \Xi(a_s, a_t^*; s, t \in T)$. We should like to define the derivatives with respect to a_s and a_t^*:

$$\frac{\delta\Xi}{\delta a_s} \quad \text{and} \quad \frac{\delta\Xi}{\delta a_t^*}$$

Expected properties:

$$\frac{\delta}{\delta a_s}\int f(t)a_t dt = f(s)I$$

$$\frac{\delta}{\delta a_s}\int f(s,t)a_s a_t ds dt = \int f(s,t)a_t\, dt + \int f(t,s)a_t\, dt$$

$$\frac{\delta}{\delta a_t^*}\int f(s,t)a_s a_t^* ds dt = \int f(s,t)a_s\, ds$$

2.1. Definition

Definition (Ji–Obata (2007))

For $\Xi \in \mathcal{L}((E),(E)^*)$ and $\zeta \in E$ we define $D_\zeta^\pm \Xi \in \mathcal{L}((E),(E)^*)$ by

$$D_\zeta^+ \Xi = [a(\zeta), \Xi], \qquad D_\zeta^- \Xi = -[a^*(\zeta), \Xi].$$

These are called the *creation derivative* and *annihilation derivative* of Ξ, respectively. Both together are called the *quantum white noise derivatives*.

Note: For $\zeta \in E$, both

$$a(\zeta) = \Xi_{0,1}(\zeta) = \int_T \zeta(t) a_t \, dt, \quad a^*(\zeta) = \Xi_{1,0}(\zeta) = \int_T \zeta(t) a_t^* \, dt,$$

belong to $\mathcal{L}((E),(E)) \cap \mathcal{L}((E)^*,(E)^*)$.

1. $(D_\zeta^+ \Xi)^* = D_\zeta^-(\Xi^*)$ and $(D_\zeta^- \Xi)^* = D_\zeta^+(\Xi^*)$.
2. $D_\zeta^\pm$ is a continuous linear map from $\mathcal{L}((E),(E)^*)$ into itself.
3. Moreover, $(\zeta, \Xi) \mapsto D_\zeta^\pm \Xi$ is a continuous bilinear map from $E \times \mathcal{L}((E),(E)^*)$ into $\mathcal{L}((E),(E)^*)$.

Remark: Pointwisely Defined QWN-Derivatives

1. Recall: The smeared annihilation and creation operators

$$a(f) = \int_T f(t) a_t \, dt, \qquad a^*(f) = \int_T f(t) a_t^* \, dt \, .$$

It is natural to introduce $D_t^{\pm}$ to have

$$D_\zeta^+ = \int_T \zeta(t) D_t^+ \, dt, \qquad D_\zeta^- = \int_T \zeta(t) D_t^- \, dt.$$

In fact, this expression is useful for computation.

2. However, it is not straightforward to define $D_t^{\pm}$ for each point $t \in T$ because

$$D_t^+ \Xi = [a_t, \Xi] = a_t \Xi - \Xi a_t, \qquad D_t^- \Xi = -[a_t^*, \Xi] = -a_t^* \Xi + \Xi a_t^*$$

are not well-defined in general.

3. Nevertheless, the pointwisely defined quantum white noise derivatives $D_t^{\pm}$ are well formulated for admissible white noise operators $\mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ [Ji–Obata, 2009, to appear].

2.2. Examples

The canonical correspondence (kernel theorem) between $S \in \mathcal{L}(E, E^*)$ and $\tau = \tau_S \in (E \otimes E)^*$ is given by $\langle \tau_S, \eta \otimes \xi \rangle = \langle S\xi, \eta \rangle$ for $\xi, \eta \in E$.

(1) The *generalized Gross Laplacian* associated with S is defined by

$$\Delta_G(S) = \Xi_{0,2}(\tau_S) = \int_{T \times T} \tau_S(s,t) a_s a_t \, ds dt$$

Note that $\Delta_G(S) \in \mathcal{L}((E), (E))$. Then,

$$D_\zeta^+ \Delta_G(S) = 0, \qquad D_\zeta^- \Delta_G(S) = a(S\zeta) + a(S^*\zeta)$$

In fact, since

$$D_t^- \Delta_G(S) = \int_T \tau_S(s,t) a_s \, ds + \int_T \tau_S(t,s) a_s \, ds$$

we have

$$\begin{aligned} D_\zeta^- \Delta_G(S) &= \int_{T \times T} \tau_S(s,t) a_s \zeta(t) \, ds dt + \int_{T \times T} \tau_S(t,s) a_s \zeta(t) \, ds dt \\ &= \int_T S\zeta(s) a_s \, ds + \int_T S^*\zeta(s) a_s \, ds = a(S\zeta) + a(S^*\zeta) \end{aligned}$$

2.2. Examples

(2) The adjoint of $\Delta_{\mathrm{G}}(S) \in \mathcal{L}((E)^*, (E)^*)$ is given by

$$\Delta_{\mathrm{G}}^*(S) = \Xi_{2,0}(\tau_S) = \int_{T\times T} \tau_S(s,t) a_s^* a_t^* \, dsdt$$

The quantum white noise derivatives are given by

$$D_\zeta^- \Delta_{\mathrm{G}}^*(S) = 0, \qquad D_\zeta^+ \Delta_{\mathrm{G}}^*(S) = a^*(S\zeta) + a^*(S^*\zeta)$$

(3) The *conservation operator* associated with S is defined by

$$\Lambda(S) = \Xi_{1,1}(\tau_S) = \int_{T\times T} \tau_S(s,t) a_s^* a_t \, dsdt$$

In general, $\Lambda(S) \in \mathcal{L}((E), (E)^*)$.

The quantum white noise derivatives are given by

$$D_\zeta^- \Lambda(S) = a^*(S\zeta), \qquad D_\zeta^+ \Lambda(S) = a(S^*\zeta).$$

2.3. Wick Product

The *Wick product* of white noise operators $\Xi_1, \Xi_2 \in \mathcal{L}((E),(E)^*)$, denoted by $\Xi_1 \diamond \Xi_2$, is characterized by

$$a_t \diamond \Xi = \Xi \diamond a_t = \Xi a_t\,, \qquad a_t^* \diamond \Xi = \Xi \diamond a_t^* = a_t^* \Xi.$$

Equipped with the Wick product, $\mathcal{L}((E),(E)^*)$ becomes a commutative algebra.

Definition (Wick product)

For $\Xi_1, \Xi_2 \in \mathcal{L}((E),(E)^*)$ the Wick (or normal-ordered) product $\Xi_1 \diamond \Xi_2$ is defined by

$$(\Xi_1 \diamond \Xi_2)\widehat{\ }(\xi,\eta) = \widehat{\Xi}_1(\xi,\eta)\widehat{\Xi}_2(\xi,\eta)e^{-\langle \xi,\eta\rangle}, \qquad \xi,\eta \in E,$$

where $\widehat{\Xi}(\xi,\eta)$ is the symbol of a white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ defined by

$$\widehat{\Xi}(\xi,\eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi,\eta \in E,$$

where $\phi_\xi = (1, \xi, \cdots, \xi^{\otimes n}/n!, \cdots)$ is an *exponential vector*. This is verified by the characterization theorem for operator symbols (see O. LNM 1577 (1994)]

2.4. Wick Derivations

$(\mathcal{L}((E),(E)^*),\diamond)$ is a commutative algebra.

Definition (Wick derivation)

A continuous linear map $\mathcal{D}:\mathcal{L}((E),(E)^*)\to\mathcal{L}((E),(E)^*)$ is called a *Wick derivation* if

$$\mathcal{D}(\Xi_1\diamond\Xi_2)=(\mathcal{D}\Xi_1)\diamond\Xi_2+\Xi_1\diamond(\mathcal{D}\Xi_2)$$

for all $\Xi_1,\Xi_2\in\mathcal{L}((E),(E)^*)$.

Theorem

The creation and annihilation derivatives $D_\zeta^\pm$ are Wick derivations for any $\zeta\in E$.

Note: It is proved that a general Wick derivation $\mathcal{D}$ is expressed in the form:

$$\mathcal{D}=\int_T F(t)\diamond D_t^+ dt+\int_T G(t)\diamond D_t^- dt,$$

where $F,G\in E\otimes\mathcal{L}((E),(E)^*)$.

Proof.

In general, for $\Xi \in \mathcal{L}((E),(E)^*)$ we have

$$
\begin{aligned}
(D_\zeta^+ \Xi)\widehat{}\,(\xi,\eta) &= \langle\langle (a(\zeta)\Xi - \Xi a(\zeta))\phi_\xi, \phi_\eta \rangle\rangle \\
&= \langle\langle \Xi\phi_\xi, a^*(\zeta)\phi_\eta \rangle\rangle - \langle\langle \Xi a(\zeta)\phi_\xi, \phi_\eta \rangle\rangle \\
&= \frac{d}{dt}\bigg|_{t=0} \langle\langle \Xi\phi_\xi, \phi_{\eta+t\zeta} \rangle\rangle - \langle \xi, \zeta\rangle \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle \\
&= \frac{d}{dt}\bigg|_{t=0} \widehat{\Xi}(\xi, \eta + t\zeta) - \langle \xi, \zeta\rangle \widehat{\Xi}(\xi,\eta).
\end{aligned} \tag{1}
$$

Then for $\Xi = \Xi_1 \diamond \Xi_2$ we have

$$
\begin{aligned}
(D_\zeta^+ \Xi)\widehat{}\,(\xi,\eta) = &\frac{d}{dt}\bigg|_{t=0} \widehat{\Xi}_1(\xi, t\zeta+\eta)\widehat{\Xi}_2(\xi, t\zeta+\eta) e^{-\langle \xi,\, t\zeta+\eta\rangle} \\
&\quad - \langle \xi, \zeta\rangle \widehat{\Xi}_1(\xi,\eta)\widehat{\Xi}_2(\xi,\eta) e^{-\langle \xi,\, \eta\rangle} \\
= &\left(\frac{d}{dt}\bigg|_{t=0} \widehat{\Xi}_1(\xi, t\zeta+\eta) \right) \widehat{\Xi}_2(\xi,\eta) e^{-\langle \xi,\, \eta\rangle} \\
&\quad + \widehat{\Xi}_1(\xi,\eta) \left(\frac{d}{dt}\bigg|_{t=0} \widehat{\Xi}_2(\xi, t\zeta+\eta) \right) e^{-\langle \xi,\, \eta\rangle} \\
&\quad - 2\langle \xi, \zeta\rangle \widehat{\Xi}_1(\xi,\eta)\widehat{\Xi}_2(\xi,\eta) e^{-\langle \xi,\, \eta\rangle}.
\end{aligned}
$$

Viewing (1) once again, we obtain

$$
(D_\zeta^+ \Xi)\widehat{}\,(\xi,\eta) = ((D_\zeta^+ \Xi_1) \diamond \Xi_2)\widehat{}\,(\xi,\eta) + (\Xi_1 \diamond (D_\zeta^+ \Xi_2))\widehat{}\,(\xi,\eta).
$$

3. Differential Equations for White Noise Operators

3.1. Differential Equations

A general form

$\mathcal{D} : \mathcal{L}((E), (E)^*) \to \mathcal{L}((E), (E)^*)$: a Wick derivation
$f : \mathcal{L}((E), (E)^*) \to \mathcal{L}((E), (E)^*)$: a map

$$\mathcal{D}\Xi = f(\Xi)$$

Simple cases:

1. $\mathcal{D}\Xi = 0$ ("constant" with respect to $\mathcal{D}$)
2. $\mathcal{D}\Xi = G \diamond \Xi$ with $G \in \mathcal{L}((E), (E)^*)$ (linear equation)

General cases: interesting for characterizing white noise operators (future problem)?

3.2. Reproducing Irreducubility of CCR

Let us consider the (system of) differential equations:

$$D_\zeta^+ \Xi = 0, \qquad \zeta \in E. \tag{2}$$

We expect easily that $\Xi = \Xi(a_s, a_t^*; s, t \in T)$ does not depend on the creation operators. In fact, by Fock expansion we see that the solutions to (2) are given by

$$\Xi = \sum_{m=0}^{\infty} \Xi_{0,m}(\kappa_{0,m}).$$

In a similar manner, the solutions to

$$D_\zeta^- \Xi = 0, \qquad \zeta \in E, \tag{3}$$

are given by

$$\Xi = \sum_{l=0}^{\infty} \Xi_{l,0}(\kappa_{l,0}).$$

Consequently, a white noise operator satisfying both (2) and (3) are the scalar operators. Thus, *the irreducibility of the canonical commutation relation is reproduced*.

3.2. Linear Equations

Given a Wick derivation $\mathcal{D}$ and $G \in \mathcal{L}((E),(E)^*)$, consider

$$\mathcal{D}\Xi = G \diamond \Xi \tag{4}$$

The *Wick exponential* is defined by

$$\mathrm{wexp}\ Y = \sum_{n=0}^{\infty} \frac{1}{n!} Y^{\diamond n}, \qquad Y \in \mathcal{L}((E),(E)^*),$$

whenever the series converges in $\mathcal{L}((E),(E)^*)$.

Theorem

Every solution to (4) *is of the form:*

$$\Xi = (\mathrm{wexp}\ Y) \diamond F, \tag{5}$$

where (i) $Y \in \mathcal{L}((E),(E)^*)$ *is a solution to* $\mathcal{D}Y = G$*;*
(ii) $\mathrm{wexp}\ Y$ *should be defined in* $\mathcal{L}((E),(E)^*)$*;*
(iii) $F \in \mathcal{L}((E),(E)^*)$ *is arbitrary satisfying* $\mathcal{D}F = 0$.

Proof.

It is straightforward to see that

$$\boldsymbol{\Xi} = (\mathbf{wexp}\ \boldsymbol{Y}) \diamond \boldsymbol{F}$$

is a solution to

$$\boldsymbol{\mathcal{D}}\boldsymbol{\Xi} = \boldsymbol{G} \diamond \boldsymbol{\Xi} \tag{6}$$

To prove the converse, let $\boldsymbol{\Xi}$ be an arbitrary solution to (6). Set

$$\boldsymbol{F} = (\mathbf{wexp}\,(-\boldsymbol{Y})) \diamond \boldsymbol{\Xi}.$$

Obviously, $\boldsymbol{F} \in \boldsymbol{\mathcal{L}}((\boldsymbol{E}), (\boldsymbol{E})^*)$ and $\boldsymbol{\Xi} = (\mathbf{wexp}\ \boldsymbol{Y}) \diamond \boldsymbol{F}$. We only need to show that $\boldsymbol{\mathcal{D}}\boldsymbol{F} = \mathbf{0}$. In fact,

$$\begin{aligned} \boldsymbol{\mathcal{D}}\boldsymbol{F} &= -\boldsymbol{\mathcal{D}}\boldsymbol{Y} \diamond (\mathbf{wexp}\,(-\boldsymbol{Y})) \diamond \boldsymbol{\Xi} + (\mathbf{wexp}\,(-\boldsymbol{Y})) \diamond \boldsymbol{\mathcal{D}}\boldsymbol{\Xi} \\ &= -\boldsymbol{G} \diamond (\mathbf{wexp}\,(-\boldsymbol{Y})) \diamond \boldsymbol{\Xi} + (\mathbf{wexp}\,(-\boldsymbol{Y})) \diamond \boldsymbol{G} \diamond \boldsymbol{\Xi} = \mathbf{0}. \end{aligned}$$

This completes the proof. □

Example (1)

$$D_\zeta^- \Xi = 2a(\zeta) \diamond \Xi, \qquad \zeta \in E. \tag{7}$$

1. We need to find $Y \in \mathcal{L}((E),(E)^*)$ satisfying $D_\zeta^- Y = 2a(\zeta)$.
2. In fact,
$$Y = \Delta_G = \int a_t^2 \, dt$$
is a solution.
3. Moreover, it is easily verified that $\mathrm{wexp}\ \Delta_G$ is defined in $\mathcal{L}((E),(E))$.
4. Then, a general solution to (7) is of the form:
$$\Xi = (\mathrm{wexp}\ \Delta_G) \diamond F, \tag{8}$$
where $D_\zeta^- F = 0$ for all $\zeta \in E$.

Example (2)

$$\begin{cases} D_\zeta^- \Xi = 2a(\zeta) \diamond \Xi, & \zeta \in E, \\ D_\zeta^+ \Xi = 0. \end{cases} \tag{9}$$

1. By Example (1) the solution is of the form:

$$\Xi = (\mathrm{wexp}\ \Delta_G) \diamond F, \qquad D_\zeta^- F = 0 \text{ for all } \zeta \in E.$$

2. We need only to find additional conditions for F satisfying $D_\zeta^+ \Xi = 0$.
3. Noting that $D_\zeta^+ \Delta_G = 0$, we have

$$D_\zeta^+ \Xi = (\mathrm{wexp}\ \Delta_G) \diamond D_\zeta^+ F = 0.$$

   Hence $D_\zeta^+ F = 0$ for all $\zeta \in E$. Consequently, F is a scalar operator (irreducubility of CCR).
4. Finally, the solution to (9) is of the form:

$$\Xi = C\ \mathrm{wexp}\ \Delta_G, \qquad C \in \mathbb{C}.$$

4. Implementation Problem for CCR

Let $S, T \in \mathcal{L}(E, E)$ and consider transformed annihilation and creation operators:

$$b(\zeta) = a(S\zeta) + a^*(T\zeta), \quad b^*(\zeta) = a^*(S\zeta) + a(T\zeta),$$

where $\zeta \in E$. We know that $b(\zeta), b^*(\zeta) \in \mathcal{L}((E), (E)) \cap \mathcal{L}((E)^*, (E)^*)$.

The implementation problem

is to find a white noise operator $U \in \mathcal{L}((E), (E)^*)$ satisfying

$$\begin{array}{ccc} (E) & \xrightarrow{\;U\;} & (E)^* \\ {\scriptstyle a(\zeta)}\big\downarrow & & \big\downarrow{\scriptstyle b(\zeta)} \\ (E) & \xrightarrow[\;U\;]{} & (E)^* \end{array} \qquad\qquad \begin{array}{ccc} (E) & \xrightarrow{\;U\;} & (E)^* \\ {\scriptstyle a^*(\zeta)}\big\downarrow & & \big\downarrow{\scriptstyle b^*(\zeta)} \\ (E) & \xrightarrow[\;U\;]{} & (E)^* \end{array}$$

Remarks: (1) $T^*S = S^*T$ is equivalent to

$$[b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0, \qquad \zeta, \eta \in E.$$

(2) $S^*S - T^*T = I$ is equivalent to

$$[b(\zeta), b^*(\eta)] = \langle \zeta, \eta \rangle, \qquad \zeta, \eta \in E.$$

$$\begin{aligned} Ua(\zeta) &= b(\zeta)U \\ &= (a(S\zeta) + a^*(T\zeta))\,U \\ &= D^+_{S\zeta}U + Ua(S\zeta) + a^*(T\zeta)U, \\ D^+_{S\zeta}U &= Ua(\zeta) - Ua(S\zeta) - a^*(T\zeta)U \\ &= Ua(\zeta - S\zeta) - a^*(T\zeta)U \\ &= [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U. \end{aligned}$$

Thus,

$$Ua(\zeta) = b(\zeta)U \quad \Longleftrightarrow \quad D^+_{S\zeta}U = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U.$$

Similarly,

$$Ua^*(\zeta) = b^*(\zeta)U \quad \Longleftrightarrow \quad (D^-_{\zeta} - D^+_{T\zeta})U = [a^*(S\zeta - \zeta) + a(T\zeta)] \diamond U.$$

Theorem

Assume that S *is invertible and that* $T^*S = S^*T$. *Then a white noise operator* $U \in \mathcal{L}((E),(E)^*)$ *satisfies the intertwining property:*

$$Ua(\zeta) = b(\zeta)U, \qquad \zeta \in E,$$

if and only if U *is of the form*

$$U = \mathrm{wexp}\left\{-\frac{1}{2}\Delta_G^*(TS^{-1}) + \Lambda((S^{-1})^* - I)\right\} \diamond F, \tag{10}$$

where $F \in \mathcal{L}((E),(E)^*)$ *fulfills* $D_\zeta^+ F = 0$ *for all* $\zeta \in E$.

Remark:

$$\mathrm{wexp}\left\{-\frac{1}{2}\Delta_G^*(TS^{-1})\right\} = e^{-\frac{1}{2}\Delta_G^*(TS^{-1})}$$

$$\mathrm{wexp}\left\{\Lambda((S^{-1})^* - I)\right\} = \Gamma((S^{-1})^*)$$

$$U = e^{-\frac{1}{2}\Delta_G^*(TS^{-1})}\Gamma((S^{-1})^*)F \quad \text{(usual composition)}$$

Proof.

1. We only need to solve the differential equation

$$D_{S\zeta}^{+} U = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U. \tag{11}$$

2. We readily know that

$$D_{S\zeta}^{+} \Lambda((S^{-1})^* - I) = a(\zeta - S\zeta), \qquad D_{S\zeta}^{+} \Delta_{\mathrm{G}}^*(TS^{-1}) = 2a^*(T\zeta).$$

3. Then by the general result a general form of the solutions to (11) is given by

$$U = \mathbf{wexp}\left\{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1}) + \Lambda((S^{-1})^* - I)\right\} \diamond F,$$

where $F \in \mathcal{L}((E), (E)^*)$ is an arbitrary white noise operator satisfying $D_{S\zeta}^{+} F = 0$ for all $\zeta \in E$.

4. Since S is invertible, the last condition for F is equivalent to that $D_{\zeta}^{+} F = 0$ for all $\zeta \in E$.

□

Solution to the Implementation Problem (2)

Theorem

Assume the following conditions:

(i) S *is invertible;*

(ii) $T^*S = S^*T$;

(iii) $S^*S - T^*T = I$;

(iv) $ST^* = TS^*$.

Then a white noise operator $U \in \mathcal{L}((E),(E)^*)$ *satisfies the intertwining property:*

$$Ua^*(\zeta) = b^*(\zeta)U, \qquad \zeta \in E,$$

if and only if U *is of the form:*

$$U = \mathrm{wexp}\left\{-\frac{1}{2}\Delta_G^*(TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2}\Delta_G(S^{-1}T)\right\} \diamond G,$$

where $G \in \mathcal{L}((E),(E)^*)$ *is an arbitrary white noise operator satisfying*

$$(D_\zeta^- - D_{T\zeta}^+)G = 0 \quad \text{for all } \zeta \in E.$$

Proof.

1. Our task is to solve the differential equation:

$$(D_\zeta^- - D_{T\zeta}^+)U = [a^*(S\zeta - \zeta) + a(T\zeta)] \diamond U.$$

2. First we need to find a solution to the differential equation:

$$(D_\zeta^- - D_{T\zeta}^+)Y = a^*(S\zeta - \zeta) + a(T\zeta). \qquad (12)$$

3. As is easily verified,

$$Y = \Delta_G^*(K) + \Lambda(L) + \Delta_G(M), \qquad K = K^*, \quad M = M^*,$$

satisfies (12) if and only if

$$2M - L^*T = T, \qquad L - 2KT = S - I.$$

Thanks to the conditions (i)–(iv) we may choose

$$K = -\frac{1}{2}TS^{-1}, \quad L = (S^{-1})^* - I, \quad M = \frac{1}{2}S^{-1}T.$$

4. Then the assertion follows immediately from our general theorem.

□

Solution to the Implementation Problem (3)

Theorem

Assume the following conditions:

(i) S *is invertible;*

(ii) $T^*S = S^*T \iff [b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0$;

(iii) $S^*S - T^*T = I \iff [b(\zeta), b^*(\eta)] = \langle \zeta, \eta \rangle$;

(iv) $ST^* = TS^*$.

A white noise operator $U \in \mathcal{L}((E), (E)^*)$ *satisfies the following intertwining properties:*

$$Ua(\zeta) = b(\zeta)U, \qquad Ua^*(\zeta) = b^*(\zeta)U, \qquad \zeta \in E,$$

if and only if U *is of the form:*

$$\begin{aligned} U &= C \,\mathrm{wexp}\left\{ -\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T) \right\} \\ &= C\, e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1})}\Gamma((S^{-1})^*)\, e^{\frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T)}, \end{aligned}$$

where $C \in \mathbb{C}$.

Proof.

1. By the above two theorems, U is of the form

$$\begin{aligned} U &= \mathrm{wexp}\left\{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1}) + \Lambda((S^{-1})^* - I)\right\} \diamond F \\ &= \mathrm{wexp}\left\{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T)\right\} \diamond G, \end{aligned}$$

where $F, G \in \mathcal{L}((E), (E)^*)$ satisfy

$$D_\zeta^+ F = 0, \quad (D_\zeta^- - D_{T\zeta}^+)G = 0, \quad \text{for all } \zeta \in E.$$

2. We see from the above identity that

$$G = F \diamond \mathrm{wexp}\left\{-\frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T)\right\}.$$

3. Since the right hand side contains no creation operators, we have

$$D_\zeta^+ G = 0, \qquad \zeta \in E. \tag{13}$$

4. Then,

$$0 = (D_\zeta^- - D_{T\zeta}^+)G = D_\zeta^- G, \qquad \zeta \in E, \tag{14}$$

so G is a scalar operator.

Final Remarks (1)

1. We have derived a general form of U by means of a new type of a differential equation for white noise operators:

$$U = C \operatorname{wexp}\left\{-\frac{1}{2}\Delta_G^*(TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2}\Delta_G(S^{-1}T)\right\}$$
$$= C\, e^{-\frac{1}{2}\Delta_G^*(TS^{-1})}\Gamma((S^{-1})^*)\, e^{\frac{1}{2}\Delta_G(S^{-1}T)}$$

   This is the normal-ordered exponential of a quadratic function of quantum white noise (Bogoliubov Hamiltonian).

2. We can derive conditions for unitarity (e.g., by using complex white noise)
3. U is the composition of the generalized Fourier–Mehler and Fourier–Gauss transforms. Unitarity conditions with respect to another inner product? (Some results for $\mathcal{G}_{U,V}$, see [Ji–Obata (2006)]).

Definition (Chung–Ji (1997))

$$\mathcal{G}_{U,V} = \Gamma(V)\, e^{\Delta_G(U)}, \quad U \in \mathcal{L}(E, E^*), \quad V \in \mathcal{L}(E, E^*)$$

is called a *generalized Fourier–Gauss transform* and its adjoint operator $\mathcal{G}_{U,V}^*$ a *generalized Fourier–Mehler transform*.

Final Remarks (2)

We discussed

- Quantum white noise derivatives and their applications to the implementation problem for CCR.

  U. C. Ji and N. Obata: A new approach to implementation problem in terms of quantum white noise derivatives, preprint, 2009.

Another applications of quantum white noise derivatives

1. Hitsuda-Skorohod quantum stochastic integrals — adjoint action of derivatives

   U. C. Ji and N. Obata: Quantum stochastic integral representations of Fock space operators, Stochastics 81 (2009), 367–384.

2. Representations of quantum martingales — a direct formula for the integrands

   U. C. Ji and N. Obata: Annihilation-derivative, creation-derivative and representation of quantum martingales, Commun. Math. Phys. **286** *(2009), 751–775.*