Quantum White Noise Derivatives and Implementation Problem
On the occasion of their 60th birthdays of Professors K. R. Ito and I. Ojima

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The Implementation Problem

\( a(\xi), a^*(\eta) \): annihilation and creation operators on Boson Fock space \( \Gamma(H) \) satisfying

\[
\text{CCR: } [a(\xi), a(\eta)] = [a^*(\xi), a^*(\eta)] = 0, \quad [a(\xi), a^*(\eta)] = \langle \xi, \eta \rangle
\]

Consider transformed annihilation and creation operators:

\[
b(\zeta) = a(S\zeta) + a^*(T\zeta), \quad b^*(\zeta) = a^*(S\zeta) + a(T\zeta).
\]

The implementation problem [Berezin (1966), Ruijsenaars (1977), ...] is to find a (unitary) operator \( U \) on the Boson Fock space \( \Gamma(H) \) satisfying

\[
\begin{align*}
\Gamma(H) & \xrightarrow{U} \Gamma(H) & \Gamma(H) & \xrightarrow{U} \Gamma(H) \\
\downarrow a(\zeta) & & \downarrow b(\zeta) & & \downarrow a^*(\zeta) & & \downarrow b^*(\zeta) \\
\Gamma(H) & \xrightarrow{U} \Gamma(H) & \Gamma(H) & \xrightarrow{U} \Gamma(H)
\end{align*}
\]

Remarks: (1) \([b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0 \iff T^* S = S^* T\)

(2) \([b(\zeta), b^*(\eta)] = \langle \zeta, \eta \rangle \iff S^* S - T^* T = I\)
1. Quantum White Noise Calculus
   - 1.1. Background and Notation
   - 1.2. White Noise Operators
   - 1.3. Quantum White Noise
   - 1.4. Integral Kernel Operators and Fock Expansion

2. Quantum White Noise Derivatives
   - 2.1. Definition
   - 2.2. Examples
   - 2.3. Wick Product
   - 2.4. Wick Derivations

3. Differential Equations for White Noise Operators
   - 3.1. Differential Equations
   - 3.2. Reproducing Irreducibility of CCR
   - 3.3. Linear Equations

4. Implementation Problem for CCR
   - 4.1. The Implementation Problem
   - 4.2. Our Approach
   - 4.3. Solution to the Implementation Problem
1. Quantum White Noise Calculus
1.1. Background and Notation

The Boson Fock space over \( H = L^2(T) \) is defined by

\[
\Gamma(H) = \left\{ \phi = (f_n); f_n \in H^\otimes n, \|\phi\|^2 = \sum_{n=0}^{\infty} n!|f_n|^2_0 < \infty \right\},
\]

where \( T \) is a topological space equipped with a \( \sigma \)-finite Borel measure \( dt \), \( |f_n|_0 \) is the usual \( L^2 \)-norm of \( H^\otimes n = L^2_{\text{sym}}(T^n) \).

The annihilation and creation operator at a point \( t \in T \)

\[
a_t : (0, \ldots, 0, \xi^\otimes n, 0, \ldots) \mapsto (0, \ldots, 0, n\xi(t)\xi^\otimes(n-1), 0, 0, \ldots)
\]
\[
a^*_t : (0, \ldots, 0, \xi^\otimes n, 0, \ldots) \mapsto (0, \ldots, 0, 0, \xi^\otimes n \otimes \delta_t, 0, \ldots)
\]

A “general” Fock space operator takes the form:

\[
\sum_{l,m=0}^{\infty} \int_{T^{l+m}} \kappa_{l,m}(s_1, \ldots, s_l, t_1, \ldots, t_m) a^*_{s_1} \cdots a^*_{s_l} a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m
\]

Quantum field theory: e.g., Haag (1955), Berezin (1966), Krée (1988), etc.
1.2. White Noise Operators

I) Gelfand triple for $H = L^2(T)$:

$$E \subset H = L^2(T) \subset E^*,$$

$$E = \operatorname{proj lim}_{p \to \infty} E_p, \quad E^* = \operatorname{ind lim}_{p \to \infty} E_{-p},$$

where $E_p$ is a dense subspace of $H$ and is a Hilbert space for itself.

II) Gelfand triple for $\Gamma(H)$ (e.g., Hida–Kubo–Takenaka space (1980)):

$$(E) \subset \Gamma(H) \subset (E)^*, \quad (E) = \operatorname{proj lim}_{p \to \infty} \Gamma(E_p), \quad (E)^* = \operatorname{ind lim}_{p \to \infty} \Gamma(E_{-p}),$$

Note: (1) $\Gamma(H) \cong L^2(E^*, \mu)$ (Wiener–Itô–Segal isomorphism)

(2) $(E)$ is the space of test functions and $(E)^*$ the space of distributions.

Definition (White noise operator)

A continuous operator from $(E)$ into $(E)^*$ is called a white noise operator. Let $\mathcal{L}((E), (E)^*)$ denote the space of white noise operators, equipped with the topology of bounded convergence.

Note: $\mathcal{L}((E), (E))$, $\mathcal{L}((E)^*, (E)^*)$ and $\mathcal{B}(\Gamma(H))$ are subspaces of $\mathcal{L}((E), (E)^*)$. 
1.3. Quantum White Noise

Theorem (Quantum white noise is very regular)

\[ a_t \in \mathcal{L}(\langle E \rangle, \langle E \rangle) \text{ and } a_t^* \in \mathcal{L}(\langle E \rangle^*, \langle E \rangle^*) \text{ for all } t \in \mathbb{R}. \]
Moreover, both maps \( t \mapsto a_t \in \mathcal{L}(\langle E \rangle, \langle E \rangle) \) and \( t \mapsto a_t^* \in \mathcal{L}(\langle E \rangle^*, \langle E \rangle^*) \) are operator-valued rapidly decreasing functions, i.e., belongs to \( E \otimes \mathcal{L}(\langle E \rangle, \langle E \rangle) \) and \( E \otimes \mathcal{L}(\langle E \rangle^*, \langle E \rangle^*) \), respectively. (The pair \( \{a_t, a_t^*; t \in T\} \) is called the quantum white noise on \( T \).)

Smeared operators

\[ a(\zeta) = \int \zeta(t)a_t \, dt, \quad a^*(\zeta) = \int \zeta(t)a_t^* \, dt \]

Traditional approach

1. \( \zeta \) is a test function, e.g., \( \zeta \in \mathcal{S}(\mathbb{R}) \).
2. \( a(\zeta), a^*(\zeta) \) are unbounded operators in \( \Gamma(H) \).

White noise approach

1. \( \zeta \) is a distribution, e.g., \( \zeta \in \mathcal{S}'(\mathbb{R}) \).
2. \( a(\zeta), a^*(\zeta) \) are white noise operators, i.e., belong to \( \mathcal{L}(\langle E \rangle, \langle E \rangle^*) \).
3. In fact, \( a(\zeta) \in \mathcal{L}(\langle E \rangle, \langle E \rangle) \) and \( a^*(\zeta) \in \mathcal{L}(\langle E \rangle^*, \langle E \rangle^*) \).
Definition (Integral kernel operator)

Given $\kappa_{l,m} \in (E \otimes (l+m))^*$, $l, m = 0, 1, 2, \ldots$, the integral kernel operator

$$\Xi_{l,m}(\kappa_{l,m}) = \int_{T^{l+m}} \kappa_{l,m}(s_1, \ldots, s_l, t_1, \ldots, t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m$$

is defined and is a white noise operator, i.e., $\Xi_{l,m}(\kappa_{l,m}) \in \mathcal{L}((E), (E)^*)$.


Every white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ admits the infinite series expansion:

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \quad \kappa_{l,m} \in (E \otimes (l+m))^*,$$

where the right-hand side converges in $\mathcal{L}((E), (E)^*)$. If $\Xi \in \mathcal{L}((E), (E))$, then $\kappa_{l,m} \in E^\otimes l \otimes (E^\otimes m)^*$ and the series converges in $\mathcal{L}((E), (E))$. 
2. Quantum White Noise Derivatives
2.1. Definition

For a Brownian (or white noise) function $\Phi$ stochastic derivatives (gradients) were introduced by Malliavin, Hida, Gross, ...

\[ \nabla \Phi, \quad \frac{\delta \Phi}{\delta \dot{B}(t)}, \quad \partial_t \Phi, \quad a_t \Phi \]

A quantum counterpart

A white noise operator $\Xi$ is considered as a function of quantum white noise:

$\Xi = \Xi(a_s, a_t^*; s, t \in T)$. We should like to define the derivatives with respect to $a_s$ and $a_t^*$:

\[ \frac{\delta \Xi}{\delta a_s} \quad \text{and} \quad \frac{\delta \Xi}{\delta a_t^*} \]

Expected properties:

\[ \frac{\delta}{\delta a_s} \int f(t)a_t dt = f(s)I \]
\[ \frac{\delta}{\delta a_s} \int f(s, t)a_s a_t ds dt = \int f(s, t)a_t dt + \int f(t, s)a_t dt \]
\[ \frac{\delta}{\delta a_t^*} \int f(s, t)a_s a_t^* ds dt = \int f(s, t)a_s ds \]
2.1. Definition

Definition (Ji–Obata (2007))

For $\Xi \in \mathcal{L}((E), (E)^*)$ and $\zeta \in E$ we define $D_{\zeta}^\pm \Xi \in \mathcal{L}((E), (E)^*)$ by

$$D_{\zeta}^+ \Xi = [a(\zeta), \Xi], \quad D_{\zeta}^- \Xi = -[a^*(\zeta), \Xi].$$

These are called the creation derivative and annihilation derivative of $\Xi$, respectively. Both together are called the quantum white noise derivatives.

Note: For $\zeta \in E$, both

$$a(\zeta) = \Xi_{0,1}(\zeta) = \int_T \zeta(t)a_t \ dt, \quad a^*(\zeta) = \Xi_{1,0}(\zeta) = \int_T \zeta(t)a_t^* \ dt,$$

belong to $\mathcal{L}((E), (E)) \cap \mathcal{L}((E)^*, (E)^*)$.

1. $(D_{\zeta}^+ \Xi)^* = D_{\zeta}^- (\Xi^*)$ and $(D_{\zeta}^- \Xi)^* = D_{\zeta}^+ (\Xi^*)$.

2. $D_{\zeta}^\pm$ is a continuous linear map from $\mathcal{L}((E), (E)^*)$ into itself.

3. Moreover, $(\zeta, \Xi) \mapsto D_{\zeta}^\pm \Xi$ is a continuous bilinear map from $E \times \mathcal{L}((E), (E)^*)$ into $\mathcal{L}((E), (E)^*)$. 
Recall: The smeared annihilation and creation operators

\[ a(f) = \int_T f(t) a_t \, dt, \quad a^*(f) = \int_T f(t) a_t^* \, dt. \]

It is natural to introduce \( D_t^\pm \) to have

\[ D_t^+ \zeta = \int_T \zeta(t) D_t^+ \, dt, \quad D_t^- \zeta = \int_T \zeta(t) D_t^- \, dt. \]

In fact, this expression is useful for computation.

However, it is not straightforward to define \( D_t^\pm \) for each point \( t \in T \) because

\[ D_t^+ \Xi = [a_t, \Xi] = a_t \Xi - \Xi a_t, \quad D_t^- \Xi = -[a_t^*, \Xi] = -a_t^* \Xi + \Xi a_t^* \]

are not well-defined in general.

Nevertheless, the pointwisely defined quantum white noise derivatives \( D_t^\pm \) are well formulated for admissible white noise operators \( \mathcal{L}(\mathcal{G}, \mathcal{G}^*) \) [Ji–Obata, 2009, to appear].
2.2. Examples

The canonical correspondence (kernel theorem) between $S \in \mathcal{L}(E, E^*)$ and $\tau = \tau_S \in (E \otimes E)^*$ is given by $\langle \tau_S, \eta \otimes \xi \rangle = \langle S\xi, \eta \rangle$ for $\xi, \eta \in E$.

(1) The \textit{generalized Gross Laplacian} associated with $S$ is defined by

$$\Delta_G(S) = \Xi_{0,2}(\tau_S) = \int_{T \times T} \tau_S(s, t)a_s a_t \, ds dt$$

Note that $\Delta_G(S) \in \mathcal{L}((E), (E))$. Then,

$$D_\zeta^+ \Delta_G(S) = 0, \quad D_\zeta^- \Delta_G(S) = a(S\zeta) + a(S^*\zeta)$$

In fact, since

$$D_t^- \Delta_G(S) = \int_T \tau_S(s, t)a_s \, ds + \int_T \tau_S(t, s)a_s \, ds$$

we have

$$D_\zeta^- \Delta_G(S) = \int_{T \times T} \tau_S(s, t)a_s \zeta(t) \, ds dt + \int_{T \times T} \tau_S(t, s)a_s \zeta(t) \, ds dt$$

$$= \int_T S\zeta(s)a_s \, ds + \int_T S^*\zeta(s)a_s \, ds = a(S\zeta) + a(S^*\zeta)$$
(2) The adjoint of $\Delta_G(S) \in \mathcal{L}((E)^*, (E)^*)$ is given by

$$\Delta^*_G(S) = \Xi_{2,0}(\tau_S) = \int_{T \times T} \tau_S(s, t)a_s^*a_t^* \, dsdt$$

The quantum white noise derivatives are given by

$$D^-_\zeta \Delta^*_G(S) = 0, \quad D^+_\zeta \Delta^*_G(S) = a^*(S\zeta) + a^*(S^*\zeta)$$

(3) The *conservation operator* associated with $S$ is defined by

$$\Lambda(S) = \Xi_{1,1}(\tau_S) = \int_{T \times T} \tau_S(s, t)a_s^*a_t \, dsdt$$

In general, $\Lambda(S) \in \mathcal{L}((E), (E)^*)$.

The quantum white noise derivatives are given by

$$D^-_\zeta \Lambda(S) = a^*(S\zeta), \quad D^+_\zeta \Lambda(S) = a(S^*\zeta).$$
2.3. Wick Product

The **Wick product** of white noise operators $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$, denoted by $\Xi_1 \diamond \Xi_2$, is characterized by

$$a_t \diamond \Xi = \Xi \diamond a_t = \Xi a_t, \quad a_t^* \diamond \Xi = \Xi \diamond a_t^* = a_t^* \Xi.$$

Equipped with the Wick product, $\mathcal{L}((E), (E)^*)$ becomes a commutative algebra.

**Definition (Wick product)**

For $\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)$ the Wick (or normal-ordered) product $\Xi_1 \diamond \Xi_2$ is defined by

$$(\Xi_1 \diamond \Xi_2)(\xi, \eta) = \hat{\Xi}_1(\xi, \eta)\hat{\Xi}_2(\xi, \eta)e^{-\langle \xi, \eta \rangle}, \quad \xi, \eta \in E,$$

where $\hat{\Xi}(\xi, \eta)$ is the symbol of a white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ defined by

$$\hat{\Xi}(\xi, \eta) = \langle \langle \Xi \phi_\xi, \phi_\eta \rangle \rangle, \quad \xi, \eta \in E,$$

where $\phi_\xi = (1, \xi, \cdots, \xi \otimes^n / n!, \cdots)$ is an exponential vector. This is verified by the characterization theorem for operator symbols (see O. LNM 1577 (1994))
2.4. Wick Derivations

\((\mathcal{L}((E), (E)^*), \diamond)\) is a commutative algebra.

**Definition (Wick derivation)**

A continuous linear map \(\mathcal{D} : \mathcal{L}((E), (E)^*) \rightarrow \mathcal{L}((E), (E)^*)\) is called a **Wick derivation** if

\[
\mathcal{D}(\Xi_1 \diamond \Xi_2) = (\mathcal{D}\Xi_1) \diamond \Xi_2 + \Xi_1 \diamond (\mathcal{D}\Xi_2)
\]

for all \(\Xi_1, \Xi_2 \in \mathcal{L}((E), (E)^*)\).

**Theorem**

*The creation and annihilation derivatives* \(D_{\zeta}^\pm\) *are Wick derivations for any* \(\zeta \in E\).

**Note:** It is proved that a general Wick derivation \(\mathcal{D}\) is expressed in the form:

\[
\mathcal{D} = \int_T F(t) \diamond D_t^+ dt + \int_T G(t) \diamond D_t^- dt,
\]

where \(F, G \in E \otimes \mathcal{L}((E), (E)^*)\).
Proof.

In general, for $\Xi \in \mathcal{L}((E), (E)^*)$ we have

$$
(D_\zeta^+ \Xi)(\xi, \eta) = \langle \langle a(\zeta)\Xi - \Xi a(\zeta) \rangle \phi_\xi, \phi_\eta \rangle
$$

$$
= \langle \langle \Xi \phi_\xi, a^*(\zeta) \phi_\eta \rangle - \langle \langle \Xi a(\zeta) \phi_\xi, \phi_\eta \rangle \rangle
$$

$$
= \left. \frac{d}{dt} \right|_{t=0} \langle \langle \Xi \phi_\xi, \phi_{\eta + t\zeta} \rangle - \langle \xi, \zeta \rangle \langle \langle \Xi \phi_\xi, \phi_\eta \rangle \rangle
$$

$$
= \left. \frac{d}{dt} \right|_{t=0} \Xi(\xi, \eta + t\zeta) - \langle \xi, \zeta \rangle \Xi(\xi, \eta).
$$

(1)

Then for $\Xi = \Xi_1 \diamond \Xi_2$ we have

$$
(D_\zeta^+ \Xi)(\xi, \eta) = \left. \frac{d}{dt} \right|_{t=0} \Xi_1(\xi, t\zeta + \eta) \Xi_2(\xi, t\zeta + \eta) e^{-\langle \xi, t\zeta + \eta \rangle}
$$

$$
- \langle \xi, \zeta \rangle \Xi_1(\xi, \eta) \Xi_2(\xi, \eta) e^{-\langle \xi, \eta \rangle}
$$

$$
= \left( \left. \frac{d}{dt} \right|_{t=0} \Xi_1(\xi, t\zeta + \eta) \right) \Xi_2(\xi, \eta) e^{-\langle \xi, \eta \rangle}
$$

$$
+ \Xi_1(\xi, \eta) \left( \left. \frac{d}{dt} \right|_{t=0} \Xi_2(\xi, t\zeta + \eta) \right) e^{-\langle \xi, \eta \rangle}
$$

$$
- 2 \langle \xi, \zeta \rangle \Xi_1(\xi, \eta) \Xi_2(\xi, \eta) e^{-\langle \xi, \eta \rangle}.
$$

Viewing (1) once again, we obtain

$$
(D_\zeta^+ \Xi)(\xi, \eta) = ((D_\zeta^+ \Xi_1) \diamond \Xi_2)(\xi, \eta) + (\Xi_1 \diamond (D_\zeta^+ \Xi_2))(\xi, \eta).
$$
3. Differential Equations for White Noise Operators
3.1. Differential Equations

A general form

\[ \mathcal{D} : \mathcal{L}((E), (E)^*) \rightarrow \mathcal{L}((E), (E)^*) : \text{a Wick derivation} \]
\[ f : \mathcal{L}((E), (E)^*) \rightarrow \mathcal{L}((E), (E)^*) : \text{a map} \]

\[ \mathcal{D}\Xi = f(\Xi) \]

Simple cases:

1. \[ \mathcal{D}\Xi = 0 \] ("constant" with respect to \( \mathcal{D} \))
2. \[ \mathcal{D}\Xi = G \diamond \Xi \text{ with } G \in \mathcal{L}((E), (E)^*) \] (linear equation)

General cases: interesting for characterizing white noise operators (future problem)?
3.2. Reproducing Irreducibility of CCR

Let us consider the (system of) differential equations:

\[
D_\zeta^+ \Xi = 0, \quad \zeta \in E. \tag{2}
\]

We expect easily that \( \Xi = \Xi(a_s, a_t^*; s, t \in T) \) does not depend on the creation operators. In fact, by Fock expansion we see that the solutions to (2) are given by

\[
\Xi = \sum_{m=0}^{\infty} \Xi_{0,m}(\kappa_{0,m}).
\]

In a similar manner, the solutions to

\[
D_\zeta^- \Xi = 0, \quad \zeta \in E, \tag{3}
\]

are given by

\[
\Xi = \sum_{l=0}^{\infty} \Xi_{l,0}(\kappa_{l,0}).
\]

Consequently, a white noise operator satisfying both (2) and (3) are the scalar operators. Thus, \textit{the irreducibility of the canonical commutation relation is reproduced.}
Given a Wick derivation $\mathcal{D}$ and $G \in \mathcal{L}((E), (E)^*)$, consider

$$\mathcal{D} \Xi = G \diamond \Xi$$

(4)

The **Wick exponential** is defined by

$$\text{wexp } Y = \sum_{n=0}^{\infty} \frac{1}{n!} Y \diamond^n, \quad Y \in \mathcal{L}((E), (E)^*),$$

whenever the series converges in $\mathcal{L}((E), (E)^*)$.

**Theorem**

*Every solution to (4) is of the form:*

$$\Xi = (\text{wexp } Y) \diamond F,$$

(5)

*where (i) $Y \in \mathcal{L}((E), (E)^*)$ is a solution to $\mathcal{D} Y = G$;*

*(ii) $\text{wexp } Y$ should be defined in $\mathcal{L}((E), (E)^*)$;*

*(iii) $F \in \mathcal{L}((E), (E)^*)$ is arbitrary satisfying $\mathcal{D} F = 0$.***
It is straightforward to see that

$$\Xi = (\text{wexp } Y) \diamond F$$

is a solution to

$$\mathcal{D}\Xi = G \diamond \Xi$$

To prove the converse, let $\Xi$ be an arbitrary solution to (6). Set

$$F = (\text{wexp } (-Y)) \diamond \Xi.$$ 

Obviously, $F \in \mathcal{L}((E), (E)^*)$ and $\Xi = (\text{wexp } Y) \diamond F$. We only need to show that $\mathcal{D}F = 0$. In fact,

$$\mathcal{D}F = -\mathcal{D}Y \diamond (\text{wexp } (-Y)) \diamond \Xi + (\text{wexp } (-Y)) \diamond \mathcal{D}\Xi$$

$$= -G \diamond (\text{wexp } (-Y)) \diamond \Xi + (\text{wexp } (-Y)) \diamond G \diamond \Xi = 0.$$ 

This completes the proof. \qed
Example (1)

\[ D_\zeta \Xi = 2a(\zeta) \diamond \Xi, \quad \zeta \in E. \quad (7) \]

1. We need to find \( Y \in \mathcal{L}((E), (E)^*) \) satisfying \( D_\zeta Y = 2a(\zeta). \)
2. In fact, \( Y = \Delta_G = \int a_t^2 \, dt \)

is a solution.
3. Moreover, it is easily verified that \( \text{weexp} \, \Delta_G \) is defined in \( \mathcal{L}((E), (E)). \)
4. Then, a general solution to (7) is of the form:

\[ \Xi = (\text{weexp} \, \Delta_G) \diamond F, \quad (8) \]

where \( D_\zeta F = 0 \) for all \( \zeta \in E. \)
Example (2)

\[
\begin{cases}
D^-_\zeta \Xi = 2a(\zeta) \diamond \Xi, & \zeta \in E, \\
D^+_\zeta \Xi = 0.
\end{cases}
\] (9)

1. By Example (1) the solution is of the form:

\[\Xi = (\text{wexp } \Delta_G) \diamond F, \quad D^-_\zeta F = 0 \text{ for all } \zeta \in E.\]

2. We need only to find additional conditions for \(F\) satisfying \(D^+_\zeta \Xi = 0\).

3. Noting that \(D^+_\zeta \Delta_G = 0\), we have

\[D^+_\zeta \Xi = (\text{wexp } \Delta_G) \diamond D^+_\zeta F = 0.\]

Hence \(D^+_\zeta F = 0\) for all \(\zeta \in E\). Consequently, \(F\) is a scalar operator (irreducibility of CCR).

4. Finally, the solution to (9) is of the form:

\[\Xi = C \text{ wexp } \Delta_G, \quad C \in \mathbb{C}.\]
4. Implementation Problem for CCR
4.1. The Implementation Problem

Let $S, T \in \mathcal{L}(E, E)$ and consider transformed annihilation and creation operators:

$$
\begin{align*}
    b(\zeta) &= a(S\zeta) + a^*(T\zeta), \\
    b^*(\zeta) &= a^*(S\zeta) + a(T\zeta),
\end{align*}
$$

where $\zeta \in E$. We know that $b(\zeta), b^*(\zeta) \in \mathcal{L}((E), (E)) \cap \mathcal{L}((E)^*, (E)^*)$.

The implementation problem is to find a white noise operator $U \in \mathcal{L}((E), (E)^*)$ satisfying

$$
\begin{align*}
    (E) &\xrightarrow{U} (E)^* \\
    a(\zeta) &\downarrow \quad b(\zeta) \\
    (E) &\xrightarrow{U} (E)^*
\end{align*}
\quad
\begin{align*}
    (E) &\xrightarrow{U} (E)^* \\
    a^*(\zeta) &\downarrow \quad b^*(\zeta) \\
    (E) &\xrightarrow{U} (E)^*
\end{align*}
$$

Remarks: (1) $T^* S = S^* T$ is equivalent to

$$
[b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0, \quad \zeta, \eta \in E.
$$

(2) $S^* S - T^* T = I$ is equivalent to

$$
[b(\zeta), b^*(\eta)] = \langle \zeta, \eta \rangle, \quad \zeta, \eta \in E.
$$
4.2. Our Approach

\[ U a(\zeta) = b(\zeta) U \]
\[ = (a(S\zeta) + a^*(T\zeta)) U \]
\[ = D_{S\zeta}^+ U + U a(S\zeta) + a^*(T\zeta) U, \]
\[ D_{S\zeta}^+ U = U a(\zeta) - U a(S\zeta) - a^*(T\zeta) U \]
\[ = U a(\zeta - S\zeta) - a^*(T\zeta) U \]
\[ = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U. \]

Thus,

\[ U a(\zeta) = b(\zeta) U \iff D_{S\zeta}^+ U = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U. \]

Similarly,

\[ U a^*(\zeta) = b^*(\zeta) U \iff (D_{\zeta}^- - D_{T\zeta}^+) U = [a^*(S\zeta - \zeta) + a(T\zeta)] \diamond U. \]
4.3. Solution to the Implementation Problem (1)

**Theorem**

Assume that $S$ is invertible and that $T^* S = S^* T$. Then a white noise operator $U \in \mathcal{L}((E), (E)^\ast)$ satisfies the intertwining property:

$$Ua(\zeta) = b(\zeta)U, \quad \zeta \in E,$$

if and only if $U$ is of the form

$$U = \text{wexp} \left\{-\frac{1}{2} \Delta^*_G (TS^{-1}) + \Lambda((S^{-1})^\ast - I)\right\} \diamond F,$$

(10)

where $F \in \mathcal{L}((E), (E)^\ast)$ fulfills $D^+_\zeta F = 0$ for all $\zeta \in E$.

**Remark:**

$$\text{wexp} \left\{-\frac{1}{2} \Delta^*_G (TS^{-1})\right\} = e^{-\frac{1}{2} \Delta^*_G (TS^{-1})}$$

$$\text{wexp} \left\{\Lambda((S^{-1})^\ast - I)\right\} = \Gamma((S^{-1})^\ast)$$

$$U = e^{-\frac{1}{2} \Delta^*_G (TS^{-1})} \Gamma((S^{-1})^\ast)F \quad \text{(usual composition)}$$
Proof.

1. We only need to solve the differential equation

\[ D_{S\zeta}^+ U = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U. \]  \hspace{1cm} (11)

2. We readily know that

\[ D_{S\zeta}^+ \Lambda((S^{-1})^* - I) = a(\zeta - S\zeta), \quad D_{S\zeta}^+ \Delta^{*}_G(TS^{-1}) = 2a^*(T\zeta). \]

3. Then by the general result a general form of the solutions to (11) is given by

\[ U = \text{wexp} \left\{ -\frac{1}{2} \Delta^{*}_G(TS^{-1}) + \Lambda((S^{-1})^* - I) \right\} \diamond F, \]

where \( F \in \mathcal{L}((E), (E)^*) \) is an arbitrary white noise operator satisfying \( D_{S\zeta}^+ F = 0 \) for all \( \zeta \in E \).

4. Since \( S \) is invertible, the last condition for \( F \) is equivalent to that \( D_{\zeta}^+ F = 0 \) for all \( \zeta \in E \).
Theorem

Assume the following conditions:

(i) \( S \) is invertible;
(ii) \( T^* S = S^* T \);
(iii) \( S^* S - T^* T = I \);
(iv) \( ST^* = TS^* \).

Then a white noise operator \( U \in \mathcal{L}((E), (E)^*) \) satisfies the intertwining property:

\[
Ua^*(\zeta) = b^*(\zeta)U, \quad \zeta \in E,
\]

if and only if \( U \) is of the form:

\[
U = \text{wexp} \left\{ -\frac{1}{2} \Delta_G^* (TS^{-1}) + \Lambda ((S^{-1})^* - I) + \frac{1}{2} \Delta_G (S^{-1} T) \right\} \odot G,
\]

where \( G \in \mathcal{L}((E), (E)^*) \) is an arbitrary white noise operator satisfying

\[
(D_{\zeta}^- - D_{T\zeta}^+) G = 0 \quad \text{for all} \ \zeta \in E.
\]
Our task is to solve the differential equation:

\[(D^- - D^+_T)U = [a^*(S\zeta - \zeta) + a(T\zeta)] \diamond U.\]

First we need to find a solution to the differential equation:

\[(D^- - D^+_T)Y = a^*(S\zeta - \zeta) + a(T\zeta).\]  \hspace{1cm} (12)

As is easily verified,

\[Y = \Delta^*_G(K) + \Lambda(L) + \Delta_G(M), \quad K = K^*, \quad M = M^*,\]

satisfies (12) if and only if

\[2M - L^*T = T, \quad L - 2KT = S - I.\]

Thanks to the conditions (i)–(iv) we may choose

\[K = -\frac{1}{2} TS^{-1}, \quad L = (S^{-1})^* - I, \quad M = \frac{1}{2} S^{-1}T.\]

Then the assertion follows immediately from our general theorem.
Theorem

Assume the following conditions:

(i) $S$ is invertible;

(ii) $T^* S = S^* T \iff [b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0$;

(iii) $S^* S - T^* T = I \iff [b(\zeta), b^*(\eta)] = \langle \zeta, \eta \rangle$;

(iv) $ST^* = TS^*$.

A white noise operator $U \in \mathcal{L}((E), (E)^*)$ satisfies the following intertwining properties:

$$U a(\zeta) = b(\zeta) U, \quad U a^*(\zeta) = b^*(\zeta) U, \quad \zeta \in E,$$

if and only if $U$ is of the form:

$$U = C \ \text{wexp} \left\{ -\frac{1}{2} \Delta^*_G(TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2} \Delta_G(S^{-1}T) \right\}$$

$$= C e^{-\frac{1}{2} \Delta^*_G(TS^{-1})} \Gamma((S^{-1})^*) e^{\frac{1}{2} \Delta_G(S^{-1}T)},$$

where $C \in \mathbb{C}$. 
Proof.

By the above two theorems, \( U \) is of the form

\[
U = \text{wexp} \left\{ -\frac{1}{2} \Delta^*_G (TS^{-1}) + \Lambda((S^{-1})^* - I) \right\} \diamond F
\]

\[
= \text{wexp} \left\{ -\frac{1}{2} \Delta^*_G (TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2} \Delta_G (S^{-1}T) \right\} \diamond G,
\]

where \( F, G \in \mathcal{L}((E), (E)^*) \) satisfy

\[
D^+_{\zeta} F = 0, \quad (D^-_{\zeta} - D^+_{T\zeta}) G = 0, \quad \text{for all } \zeta \in E.
\]

We see from the above identity that

\[
G = F \diamond \text{wexp} \left\{ -\frac{1}{2} \Delta_G (S^{-1}T) \right\}.
\]

Since the right hand side contains no creation operators, we have

\[
D^+_{\zeta} G = 0, \quad \zeta \in E.
\] (13)

Then,

\[
0 = (D^-_{\zeta} - D^+_{T\zeta}) G = D^-_{\zeta} G, \quad \zeta \in E,
\] (14)

so \( G \) is a scalar operator.
We have derived a general form of $U$ by means of a new type of a differential equation for white noise operators:

$$U = C \text{ wexp } \left\{ -\frac{1}{2} \Delta^*_G(TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2} \Delta_G(S^{-1}T) \right\}$$

$$= C e^{-\frac{1}{2} \Delta^*_G(TS^{-1})} \Gamma((S^{-1})^*) e^{\frac{1}{2} \Delta_G(S^{-1}T)}$$

This is the normal-ordered exponential of a quadratic function of quantum white noise (Bogoliubov Hamiltonian).

We can derive conditions for unitarity (e.g., by using complex white noise).

$U$ is the composition of the generalized Fourier–Mehler and Fourier–Gauss transforms. Unitarity conditions with respect to another inner product? (Some results for $G_{U,V}$, see [Ji–Obata (2006)]).

**Definition (Chung–Ji (1997))**

$$G_{U,V} = \Gamma(V) e^{\Delta_G(U)}, \quad U \in \mathcal{L}(E, E^*), \quad V \in \mathcal{L}(E, E^*)$$

is called a generalized Fourier–Gauss transform and its adjoint operator $G_{U,V}^*$ a generalized Fourier–Mehler transform.
Final Remarks (2)

We discussed

- Quantum white noise derivatives and their applications to the implementation problem for CCR.


Another applications of quantum white noise derivatives

1. Hitsuda-Skorohod quantum stochastic integrals — adjoint action of derivatives


2. Representations of quantum martingales — a direct formula for the integrands