Convolution Operators in White Noise Calculus: Revisited

(joint work with H. Ouerdiane)

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Plan

1. Backgrounds

1) ∞-dimensional stochastic analysis (Kreé, Gross, Malliavin, Hida,... since 1970s)

$$\text{(test functions)} \subset L^2\text{(Gaussian space)} \subset \text{(distributions)}$$

Applications to SDEs, quantum field theory, ...

— weighted Fock space approach (Cochran–Kuo–Sengupta)
vs ∞-dimensional holomorphic function approach (Ouerdiane et al.)

2) Wiener–Itô–Segal isomorphism

$$L^2(E^*, \mu) \cong \Gamma(L^2(\mathbb{R})) = \left\{\phi = (f_n)\,;\, f_n \in L^2_{\text{sym}}(\mathbb{R}^n),\ \sum_{n=0}^{\infty} n!|f_n|^2 < \infty\right\}$$

$$B_t \longleftrightarrow (0, 1_{[0,t]}, 0, 0, \dots)$$

3) White noise (Hida) calculus : use of the time derivative of Brownian motion

$$dB_t = \frac{dB_t}{dt}\,dt = W_t\,dt \quad \Longleftrightarrow \quad B_t = \int_0^t W_s\,ds \qquad \{W_t\}\text{: white noise}$$

Analysis of white noise functions (or distributions): $\Phi = \Phi(W_t\,;\, t \in \mathbb{R})$

4) Annihilation and creation processes (unbounded in $\Gamma(L^2(\mathbb{R}))$)

$$A_t : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, n\langle 1_{[0,t]}, \xi\rangle \xi^{\otimes(n-1)}, 0, 0, \dots)$$
$$A_t^* : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, 0, \xi^{\otimes n} \widehat{\otimes} 1_{[0,t]}, 0, \dots)$$

Hudson–Parthasarathy's quantum stochastic calculus (with $\{\Lambda_t\}$)

5) Quantum white noise calculus

$$B_t = A_t + A_t^* \quad \Longrightarrow \quad W_t = a_t + a_t^* \quad \text{(quantum white noise)}$$

A Boson Fock space operator Ξ is considered as a function of quantum white noise:

$$\Xi = \Xi(a_s, a_t^*; s, t \in T)$$

1. N. Obata: "White Noise Calculus and Fock Space," LNM Vol. 1577, Springer, 1994.
2. U. C. Ji and N. Obata: *Quantum white noise calculus*, in "Non-Commutativity, Infinite-Dimensionality and Probability at the Crossroads (N. Obata, T. Matsui and A. Hora, Eds.)," 2002.
3. U. C. Ji and N. Obata: *Annihilation-derivative, creation-derivative and representation of quantum martingales*, CMP 286 (2009).
4. U. C. Ji and N. Obata: *Implementation problem for the canonical commutation relation in terms of quantum white noise derivatives*, JMP 51 (2010).

2. White Noise Distributions

T: a σ-finite measure space,

time parameter space for stochastic analysis ($\mathbb{R}_+$, $\mathbb{R}$, $\mathbf{Z}$, ...)

space-time parameter space for quantum (random) field theory ($\mathbb{R}^n$, M, ...)

The Boson Fock space over $H = L^2(T)$ is defined by

$$\Gamma(H) = \left\{ \phi = (f_n)\,;\ f_n \in H^{\hat{\otimes} n},\ \|\phi\|_0^2 \equiv \sum_{n=0}^{\infty} n! |f_n|_0^2 < \infty \right\},$$

where $H^{\hat{\otimes} n}$ is the symmetric tensor power of H and $H^{\hat{\otimes} 0} = \mathbb{C}$.

The creation and annihilation operators are essential, but unbounded in $\Gamma(H)$

A standard method to avoid unbounded operators is to employ a Gelfand triple:

$$\mathcal{W} \subset \Gamma(H) \subset \mathcal{W}^*,$$

where $\mathcal{W}$ is a nuclear Fréchet space, densely and continuously embedded in $\Gamma(H)$.

1. I. M. Gelfand and N. Ya. Vilenkin: "Generalized Functions, Vol.4," 1964.
2. N. N. Bogolubov *et al.*: "Introduction to Axiomatic Quantum Field Theory," 1975.

2.2. White Noise Triples

$$\mathcal{W} \subset \Gamma(H) \subset \mathcal{W}^*,$$

Construction of a white noise triple consists of two steps:

1. constructing a underlying Gelfand triple (nuclear rigging):

$$\cdots \subset E_p \subset \cdots \subset H = L^2(T) \subset \cdots \subset E_{-p} \subset \cdots$$

$$E = \operatorname*{proj\,lim}_{p\to\infty} E_p \subset H \subset \operatorname*{ind\,lim}_{p\to\infty} E_{-p} = E^*$$

2. taking the second quantization

$$\text{“}\Gamma(E)\text{”} \subset \Gamma(H) \subset \text{“}\Gamma(E^*)\text{”}$$

Nuclearity is essential and yields fruitful structures

- Realization of (quantum) white noise ($\approx$ justification of a delta function δ_t)
- S-transform (Laplace transform) and its analytic characterization — a fundamental tool for generalized white noise functions
- Operator theory by means of nuclear kernel approach — operator (or Wick) symbol, Fock expansion, quantum white noise derivative, ...

A: a positive selfadjoint operator in $H_{\mathbb{R}}$ with Hilbert–Schmidt inverse
For $p \geq 0$ define

$$E_{p,\mathbb{R}} = \{\xi \in H_{\mathbb{R}}\,;\, |\xi|_p \equiv |A^p\xi|_0 < \infty\},$$
$$E_{-p,\mathbb{R}} = \text{completion of } H_{\mathbb{R}} \text{ with respect to } |\xi|_{-p} \equiv |A^{-p}\xi|_0.$$

Then we have a Gelfand triple:

$$\cdots \subset E_p \subset \cdots \subset H = L^2(T) \subset \cdots \subset E_{-p} \subset \cdots$$
$$E = \operatorname*{proj\,lim}_{p\to\infty} E_p \subset H = L^2(T) \subset \operatorname*{ind\,lim}_{p\to\infty} E_{-p} = E^*$$

The minimum conditions for (quantum) white noise theory

(i) $\rho^{-1} \equiv \inf \mathrm{Spec}\,(A) > 1$;

(ii) for each $\xi \in E$ there exists a unique continuous function $\widetilde{\xi}$ on T such that $\xi(t) = \widetilde{\xi}(t)$ for almost all $t \in T$;

(iii) for each $t \in T$ a linear functional $\delta_t : \xi \mapsto \widetilde{\xi}(t)$, $\xi \in E$, is continuous, i.e., $\delta_t \in E^*$;

(iv) the map $t \mapsto \delta_t \in E^*$, $t \in T$, is continuous.

Example

A prototype of the underlying Gelfand triple

$$\boldsymbol{\mathcal{S}}(\mathbb{R}) \subset \boldsymbol{L}^{\mathbf{2}}(\mathbb{R}) \subset \boldsymbol{\mathcal{S}}'(\mathbb{R}),$$

which is obtained by means of the operator:

$$\boldsymbol{A} = \mathbf{1} + \boldsymbol{t}^{\mathbf{2}} - \frac{\boldsymbol{d}^{\mathbf{2}}}{\boldsymbol{dt}^{\mathbf{2}}}\,.$$

In fact,

$$\boldsymbol{E} = \boldsymbol{\mathcal{S}}(\mathbb{R}) = \operatorname*{\mathbf{proj\,lim}}_{p\to\infty} \boldsymbol{\mathcal{S}}_{\boldsymbol{p}}(\mathbb{R})$$

$$\boldsymbol{\mathcal{S}}_{\boldsymbol{p}}(\mathbb{R}) = \{\boldsymbol{\xi} \in \boldsymbol{L}^{\mathbf{2}}(\mathbb{R})\,;\, |\boldsymbol{\xi}|_{\boldsymbol{p}} \equiv |\boldsymbol{A}^{\boldsymbol{p}}\boldsymbol{\xi}|_{\mathbf{0}} < \infty\}$$

2.4. Construction of White Noise Triples

We have constructed a underlying Gelfand triple:

$$E = \operatorname*{proj\,lim}_{p\to\infty} E_p \subset H \subset E^* = \operatorname*{ind\,lim}_{p\to\infty} E_{-p}$$

We are going to apply the "second quantization" to get a white noise triple:

$$\mathcal{W} \subset \Gamma(H) \subset \mathcal{W}^*$$

In the recent studies there are two approaches:

(I) Weighted Fock spaces (CKS-spaces)
tracing back to Kubo–Takenaka (1980), Kondratiev–Streit, Cochran–Kuo–Sengupta, Asai–Kuo–Kubo, Chung–Ji–Obata, Ji–Obata, ...

(II) Infinite dimensional holomorphic functions
Lee (1991), Gannoun–Hachaichi–Ouerdiane–Rezgui, Ben Chrouda–Ouerdiane, Da Silva–Erraoui–Ouerdiane, Barhoumi,...

(I) Weighted Fock spaces (CKS-spaces)

Having constructed a underlying Gelfand triple:

$$E \subset \cdots \subset E_p \subset \cdots \subset H = L^2(T) \subset \cdots \subset E_{-p} \subset \cdots \subset \cdots \subset E^*,$$

define the weighted Fock space by

$$\Gamma_\alpha(E_p) = \left\{ \phi = (f_n)\,;\, f_n \in E_p^{\hat{\otimes} n}, \|\phi\|_{p,+}^2 = \sum_{n=0}^{\infty} n!\,\alpha(n)|f_n|_p^2 < \infty \right\}$$

where $\alpha = \{\alpha(n)\}$ is a weight sequence satisfying certain convexity conditions (later), and their limit spaces:

$$\mathcal{W} = \operatorname*{proj\,lim}_{p\to\infty} \Gamma_\alpha(E_p),$$

$$\mathcal{W}^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma_\alpha(E_p)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma_{1/\alpha}(E_{-p}).$$

This is referred to as a *standard CKS-space* [Cochran–Kuo–Sengupta (1998)]

- $\mathcal{W} = (E)$ when $\alpha(n) \equiv 1$ [Kubo–Takenaka (1980)]
- $\mathcal{W} = (E)_\beta$ when $\alpha(n) = (n!)^\beta$ with $0 \le \beta < 1$ [Kondratiev–Streit (1993)]

The canonical bilinear form on $\mathcal{W}^* \times \mathcal{W}$ is defined by

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \qquad \Phi = (F_n), \quad \phi = (f_n).$$

Conditions for the weight sequence $\alpha = \{\alpha(n)\}$

(A1) $\alpha(0) = 1$ and there exists some $\sigma \geq 1$ such that $\inf_{n \geq 0} \alpha(n)\sigma^n > 0$;

(A2) $\lim_{n \to \infty} \left\{ \frac{\alpha(n)}{n!} \right\}^{1/n} = 0$;

(A3) α is equivalent to a positive sequence $\gamma = \{\gamma(n)\}$ such that $\{\gamma(n)/n!\}$ is log-concave;

(A4) α is equivalent to another positive sequence $\delta = \{\delta(n)\}$ such that $\{(n!\delta(n))^{-1}\}$ is log-concave.

Here two sequences $\{\alpha(n)\}, \{\gamma(n)\}$ of positive numbers are said to be *equivalent* if there exist $K_1, K_2, M_1, M_2 > 0$ such that

$$K_1 M_1^n \alpha(n) \leq \gamma(n) \leq K_2 M_2^n \alpha(n), \quad n = 0, 1, 2, \dots.$$

A positive sequence $\beta(n)$ is called *log-concave* if

$$\beta(n)\beta(n+2) \leq \beta(n+1)^2, \quad n = 0, 1, 2, \dots.$$

Remark

The above are almost mimimum requirements for the characterization theorem of S-transform.

Definition

For $\xi \in E$ a *coherent vector* or an *exponential vector* is defined by

$$\phi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \dots, \frac{\xi^{\otimes n}}{n!}, \dots\right).$$

(It is known that $\phi_\xi \in \mathcal{W}$.)

Definition

The S*-transform* of $\Phi = (F_n) \in \mathcal{W}^*$ is defined by

$$S\Phi(\xi) = \langle\langle \Phi, \phi_\xi \rangle\rangle = \sum_{n=0}^{\infty} \langle F_n, \xi^{\otimes n} \rangle, \qquad \xi \in E.$$

Characterization Theorem

$F : E \to \mathbb{C}$ is the S-transform of $\Phi \in \mathcal{W}^*$ if and only if

1. (holomorphy) $z \mapsto F(\xi + z\eta)$ is entire holomorphic for any $\xi, \eta \in E$;
2. (growth condition) there are $C, K, p \ge 0$ such that $|F(\xi)|^2 \le C G_\alpha(K|\xi|_p^2)$, where $G_\alpha(s) = \sum \frac{\alpha(n)}{n!}\, s^n$.

— For $\mathcal{W} = (E)$ ($\alpha(n) \equiv 1$), $G_\alpha(s) = e^s$ [Potthoff–Streit (1991)]

(II) ∞-Dim Holomorphic Functions

In this subsection we set $N = E$ by notational convention.

θ: a Young function,

i.e., it is a continuous, convex, and increasing function defined on $[0, \infty)$ such that

$$\theta(0) = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{\theta(x)}{x} = \infty.$$

1. For each $p \in \mathbb{Z}$ and $m > 0$ define a Banach space:

$$\mathrm{Exp}(N_p, \theta, m) = \left\{ f : N_p \to \mathbb{C}\,; \begin{array}{l} \text{entire holomorphic,} \\ \|f\|_{\theta,p,m} = \sup_{x\in N_p} |f(x)| e^{-\theta(m|x|_p)} < \infty \end{array} \right.$$

2. We define

$$\mathcal{F}_\theta(N^*) = \operatorname*{proj\,lim}_{p\to\infty,\, m\to+0} \mathrm{Exp}(N_{-p}, \theta, m).$$

3. Define the Taylor map $T : f \mapsto (f_n)$ by Taylor expansion of $f \in \mathcal{F}_\theta(N^*)$:

$$f(x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, f_n \rangle, \qquad x \in N^*.$$

Theorem (Ouerdiane et al. (2000))

Let $F_\theta(N)$ be the space of Taylor coefficients of $f \in \mathcal{F}_\theta(N^*)$. Then

$$F_\theta(N) = \bigcap_{p\ge 0, m>0} F_{\theta,m}(N_p),$$

$$F_{\theta,m}(N_p) = \left\{ \phi = (f_n)\,;\, f_n \in N_p^{\widehat{\otimes} n},\, \sum_{n=0}^{\infty} \theta_n^{-2} m^{-n} |f_n|_p^2 < \infty \right\},$$

$$\theta_n = \inf_{r>0} \frac{e^{\theta(r)}}{r^n}. \quad n = 0, 1, 2, \dots .$$

Moreover, equipped with the projective limit topology, $F_\theta(N)$ is a nuclear Fréchet space and $T : \mathcal{F}_\theta(N^*) \to F_\theta(N)$ is a topological isomorphism.

The dual space of $F_\theta(N)$ is given by

$$G_\theta(N^*) = \bigcup_{p\ge 0, m>0} G_{\theta,m}(N_{-p}),$$

$$G_{\theta,m}(N_{-p}) = \left\{ \Phi = (F_n)\,;\, F_n \in N_{-p}^{\widehat{\otimes} n}, \sum_{n=0}^{\infty} (n!\theta_n)^2 m^n |F_n|_{-p}^2 < \infty \right\}.$$

The canonical $\mathbb{C}$-bilinear form on $G_\theta(N^*) \times F_\theta(N)$ is given by

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n\ge 0} n! \langle F_n, f_n \rangle, \qquad \Phi = (F_n), \quad \phi = (f_n).$$

In order to get a white noise triple,

Lemma

If $\displaystyle\lim_{x\to+\infty}\frac{\theta(x)}{x^2}<\infty$, then we have

$$F_\theta(N)\subset\Gamma(H).$$

Thus, we have obtained a white noise triple:

$$F_\theta(N)\subset\Gamma(H)\subset F_\theta(N)^*=G_\theta(N^*)$$

Remark

The condition in the above lemma is important to get a white noise triple, but many properties on the duality between $F_\theta(N)$ and $G_\theta(N^*)$ are derived without assuming the above condition.

Comparison

(I) Given $\alpha = \{\alpha(n)\}$ we have

$$\mathcal{W} \subset \Gamma(H) \subset \mathcal{W}^*$$

Wick product — Wick calculus or symbol calculus (applications to QSDEs)

(II) Given θ we have

$$F_\theta(N) \subset \Gamma(H) \subset F_\theta(N)^* = G_\theta(N^*)$$

Convolution calculus — convolution product, convolution operator, translation, Laplacians, derivation, ...

Theorem (Asai–Kubo–Kuo (2001))

If $G_\alpha(s) = \exp\{2\theta(\sqrt{r})\}$, we have

$$\mathcal{W} = F_\theta(N)$$

3. White Noise Operators

$$\mathcal{W} \subset \Gamma(H) \subset \mathcal{W}^*$$

Definition

A continuous operator from $\mathcal{W}$ into $\mathcal{W}^*$ is called a *white noise operator*. The space of white noise operators is denoted by $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ (bounded convergence topology).

The *annihilation* and *creation operator* at a point $t \in T$

$$a_t : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, n\xi(t)\xi^{\otimes(n-1)}, 0, 0, \dots)$$
$$a_t^* : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, 0, \xi^{\otimes n} \widehat{\otimes} \delta_t, 0, \dots)$$

The pair $\{a_t, a_t^* \,;\, t \in T\}$ is called the *quantum white noise* on T.

Theorem

$a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ and $a_t^* \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$ for all $t \in \mathbb{R}$. Moreover, both maps $t \mapsto a_t \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ and $t \mapsto a_t^* \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$ are operator-valued rapidly decreasing functions, i.e., belongs to $E \otimes \mathcal{L}(\mathcal{W}, \mathcal{W})$ and $E \otimes \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$, respectively.

For $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ the *symbol* is defined by

$$\widehat{\Xi}(\xi, \eta) = \langle\langle \Xi\varphi_\xi, \varphi_\eta \rangle\rangle , \qquad \xi, \eta \in E,$$

where $\varphi_\xi = \left(1, \xi, \dfrac{\xi^{\otimes 2}}{2!}, \cdots\right)$ is an *exponential vector*.

$\mathcal{E} = \{\varphi_\xi \,;\, \xi \in E\} \subset \mathcal{W}$ is linearly independent dense set

$\Longrightarrow$ A linear operator Ξ is uniquely specified by the action on $\mathcal{E}$

Characterization theorem for symbols [Obata (1993), Ji–Obata,...]

Let Θ be a $\mathbb{C}$-valued function on $E \times E$. Then there exists a white noise operator $\Xi \in \mathcal{L}((E), (E)^*)$ such that $\Theta = \widehat{\Xi}$ if and only if

1. (holomorphy) for any ξ, ξ_1, η, $\eta_1 \in E$, the function $\Theta(z\xi + \xi_1, w\eta + \eta_1)$ is an entire holomorphic function of $(z, w) \in \mathbb{C} \times \mathbb{C}$;
2. (growth condition) there exist constant numbers $C \geq 0$, $K \geq 0$ and $p \geq 0$ such that
$$|\Theta(\xi, \eta)| \leq C \exp\left\{K(|\xi|_p^2 + |\eta|_p^2)\right\}, \qquad \xi, \eta \in E.$$

(We have similar conditions for $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$.)

3.3. Integral Kernel Operators and Fock Expansion

Definition (Integral kernel operator)

Given $\kappa_{l,m} \in (E^{\otimes(l+m)})^*$, $l, m = 0, 1, 2, \dots$,

$$\Xi_{l,m}(\kappa_{l,m}) = \int_{T^{l+m}} \kappa_{l,m}(s_1, \cdots, s_l, t_1, \cdots, t_m) \\ a^*_{s_1} \cdots a^*_{s_l} a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m$$

is a well-defined white noise operator and is called an *integral kernel operator*.

Theorem (Haag, Berezin, Krée, O.(1993),...)

Every white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ *admits the Fock expansion:*

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \qquad \kappa_{l,m} \in (E^{\otimes(l+m)})^*,$$

where the right-hand side converges in $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$. *If* $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W})$, *then* $\kappa_{l,m} \in E^{\otimes l} \otimes (E^{\otimes m})^*$ *and the series converges in* $\mathcal{L}(\mathcal{W}, \mathcal{W})$.

4. Convolution Operators

N. Obata and H. Ouerdiane: *A note on convolution operators in white noise calculus*, preprint, 2011.

Let $\phi = (f_n) \in \mathcal{W}$ and consider the *holomorphic realization*

$$[H\phi](x) = \sum_{n=0}^{\infty} \langle x^{\otimes n}, f_n \rangle, \qquad x \in E^*.$$

($H\phi \in \mathcal{F}_\theta(N^*)$.) With each $y \in E^*$ we associate a translation operator t_y defined by

$$(t_y[H\phi])(x) = [H\phi](x-y), \qquad x \in E^*.$$

Lemma

There exists $\psi = T_y\phi \in \mathcal{W}$ such that

$$t_y[H\phi] = H\psi.$$

Moreover, $T_y \in \mathcal{L}(\mathcal{W}, \mathcal{W})$.

Proof. By definition we have

$$
\begin{aligned}
(t_y[H\phi])(x) &= \sum_{n=0}^{\infty} \langle (x-y)^{\otimes n}, f_n \rangle \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \langle x^{\otimes (n-k)} \otimes y^{\otimes k}, f_n \rangle \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \binom{n+k}{k} (-1)^k \langle x^{\otimes n} \otimes y^{\otimes k}, f_{n+k} \rangle \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \binom{n+k}{k} (-1)^k \langle x^{\otimes n}, f_{n+k} \otimes_k y^{\otimes k} \rangle,
\end{aligned}
$$

where $\otimes_k$ is the contraction of tensor product. It is easily verified that

$$
g_n = \sum_{k=0}^{\infty} \binom{n+k}{k} (-1)^k f_{n+k} \otimes_k y^{\otimes k} \tag{1}
$$

converges in $E^{\otimes n}$, $\psi = (g_n)$ belongs to $\mathcal{W}$ and

$$
t_y[H\phi] = H\psi.
$$

Then by definition

$$
\psi = T_y \phi, \qquad \phi \in \mathcal{W}.
$$

That $T_y \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ is verified by straightforward estimates of norms.

Lemma

Let $\Phi = (F_n) \in \mathcal{W}^*$ *and* $\phi = (f_n) \in \mathcal{W}$. *Then there exists* $\psi = C_\Phi \phi \in \mathcal{W}$ *such that*

$$[H\psi](x) = \langle\langle \Phi, T_{-x}\phi \rangle\rangle, \qquad x \in E^*.$$

Moreover, $C_\Phi \in \mathcal{L}(\mathcal{W}, \mathcal{W})$.

Definition

C_Φ is called the *convolution operator* associated with $\Phi \in \mathcal{W}^*$.

Proof. For $x \in E^*$ we observe that

$$
\begin{aligned}
\langle\langle \Phi, T_{-x}\phi \rangle\rangle &= \sum_{n=0}^{\infty} n! \left\langle F_n, \sum_{k=0}^{\infty} \binom{n+k}{k} (-1)^k f_{n+k} \otimes_k (-x)^{\otimes k} \right\rangle \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} \langle F_n, f_{n+k} \otimes_k x^{\otimes k} \rangle \\
&= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(n+k)!}{k!} \langle x^{\otimes k}, F_n \otimes_n f_{n+k} \rangle .
\end{aligned}
$$

This should coincide with $H\psi(x)$ so $\psi = (h_k)$. It is easily shown that

$$
h_k = \sum_{n=0}^{\infty} \frac{(n+k)!}{k!} F_n \otimes_n f_{n+k}
$$

converges in $E^{\hat{\otimes} k}$, hence $\psi = (h_k)$ belongs to $\mathcal{W}$.

Theorem (Fock expansion of convolution operator)

For $\Phi = (F_n) \in \mathcal{W}^*$ *we have*

$$C_\Phi = \sum_{m=0}^{\infty} \Xi_{0,m}(F_m). \tag{2}$$

Proof. By definition we have

$$C_\Phi \phi = (h_k), \qquad h_k = \sum_{n=0}^{\infty} \frac{(n+k)!}{k!} F_n \otimes_n f_{n+k}\,.$$

Taking an exponential vector $\phi = \phi_\xi$, $\xi \in E$, we see easily that $C_\Phi \phi_\xi = \langle\langle \Phi, \phi_\xi \rangle\rangle \, \phi_\xi$. Hence

$$\widehat{C_\Phi}(\xi, \eta) = \langle\langle C_\Phi \phi_\xi, \phi_\eta \rangle\rangle = \langle\langle \Phi, \phi_\xi \rangle\rangle \langle\langle \phi_\xi, \phi_\eta \rangle\rangle = \sum_{m=0}^{\infty} \langle F_m, \xi^{\otimes m} \rangle \, e^{\langle \xi, \eta \rangle},$$

from which the Fock expansion (2) follows.

Example (the Gross Laplacian is a convolution operator)

Let $\tau \in (E \times E)^*$ be the trace, i.e., the integral kernel corresponding to the identity operator. For $\tilde{\tau} = (0, 0, \tau, 0, \dots)$ we have

$$\Delta_G = \Xi_{0,2}(\tau) = C_{\tilde{\tau}}$$

4.3. Wick Multiplication Operator

For two $\Phi_1, \Phi_2 \in \mathcal{W}^*$ the Wick product $\Phi_1 \diamond \Phi_2 \in \mathcal{W}^*$ is characterized by

$$S(\Phi_1 \diamond \Phi_2)(\xi) = S\Phi_1(\xi) S\Phi_2(\xi), \qquad \xi \in E.$$

With each $\Phi \in \mathcal{W}^*$ we associate the *Wick multiplication operator* $M_\Phi^\diamond$ by

$$M_\Phi^\diamond \Psi = \Phi \diamond \Psi, \qquad \Psi \in \mathcal{W}^*.$$

Theorem (Fock expansion of Wick multiplication operator)

For $\Phi = (F_n) \in \mathcal{W}^*$ we have

$$M_\Phi^\diamond = \sum_{l=0}^\infty \Xi_{l,0}(F_l).$$

Proof. By definition of the Wick product we have

$$\widehat{M_\Phi^\diamond}(\xi, \eta) = \langle\langle M_\Phi^\diamond \phi_\xi, \phi_\eta \rangle\rangle = \langle\langle \Phi \diamond \phi_\xi, \phi_\eta \rangle\rangle = \langle\langle \Phi, \phi_\eta \rangle\rangle \langle\langle \phi_\xi, \phi_\eta \rangle\rangle.$$

Hence

$$\widehat{M_\Phi^\diamond}(\xi, \eta) = \langle\langle \Phi, \phi_\eta \rangle\rangle e^{\langle \xi, \eta \rangle} = \sum_{l=0}^\infty \langle F_l, \eta^{\otimes l} \rangle e^{\langle \xi, \eta \rangle},$$

from which the Fock expansion follows.

Theorem

For $\Phi \in \mathcal{W}^*$ *we have*

$$C_\Phi = (M_\Phi^\diamond)^*, \qquad M_\Phi^\diamond = (C_\Phi)^*.$$

Proof. By comparing the Fock expansions of the Wick multiplication and convolution operators.

Theorem (Characterization of convolution operators)

For a white noise operator $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ *the following conditions are equivalent:*

(i) Ξ *is a convolution operator, i.e.,* $\Xi = C_\Phi$ *for some* $\Phi \in \mathcal{W}^*$*;*

(ii) Ξ *commutes with all annihilation operators* $a(y) = \Xi_{0,1}(y)$, $y \in E^*$*;*

(iii) Ξ *commutes with all translations* T_y, $y \in E^*$*;*

(iv) Ξ *is the adjoint operator of a Wick multiplication.*

Wick Product of White Noise Operators

For two white noise operators $\Xi_1, \Xi_2 \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ the Wick product, denoted by $\Xi \diamond \Xi_2$, is defined to be the unique white noise operator satisfying

$$\langle\!\langle (\Xi_1 \diamond \Xi_2)\phi_\xi, \phi_\eta \rangle\!\rangle = \langle\!\langle \Xi_1\phi_\xi, \phi_\eta \rangle\!\rangle \langle\!\langle \Xi_2\phi_\xi, \phi_\eta \rangle\!\rangle e^{-\langle \xi, \eta \rangle}, \qquad \xi, \eta \in E,$$

or equivalently in terms of symbols:

$$(\Xi_1 \diamond \Xi_2)\widehat{\ }(\xi, \eta) = \widehat{\Xi}_1(\xi, \eta)\widehat{\Xi}_2(\xi, \eta)e^{-\langle \xi, \eta \rangle}, \qquad \xi, \eta \in E.$$

It is also known that $\mathcal{L}(\mathcal{W}, \mathcal{W})$ is closed under the Wick product.

Theorem

The map $\Phi \mapsto C_\Phi$ gives rise to a continuous, injective homomorphism from $(\mathcal{W}^*, \diamond)$ into $(\mathcal{L}(\mathcal{W}, \mathcal{W}), \diamond)$.

Proof. We need only to note that

$$C_{\Phi_1} \diamond C_{\Phi_2} = C_{\Phi_1} C_{\Phi_2} = C_{\Phi_1 \diamond \Phi_2}, \qquad \Phi_1, \Phi_2 \in \mathcal{W}^*.$$

The first relation is due to the fact that C_Φ contains only annihilation operators (Theorem **??**) and the second term by Theorem **??**. The rest is a routine work.

Convolution Product in (II) = Wick Product in (I)

Within the framework (II) $F_\theta(N) \subset \Gamma(H) \subset F_\theta(N)^* = G_\theta(N^*)$

the "convolution product" of two white noise distributions Φ, Ψ is defined by

$$\langle\langle \Phi \star \Psi, \phi \rangle\rangle = \langle\langle \Psi, C_\Phi \phi \rangle\rangle.$$

Using $C_\Phi = (M_\Phi^\diamond)^*$ we see that

$$\langle\langle \Psi, C_\Phi \phi \rangle\rangle = \langle\langle M_\Phi^\diamond \Psi, \phi \rangle\rangle = \langle\langle \Phi \diamond \Psi, \phi \rangle\rangle$$

Therefore,

$$\Phi \star \Psi = \Phi \diamond \Psi$$

Namely,

The "convolution product" in (II) coincides with the Wick product in (I).

Kuo's Convolution Operator (1992)

Start with standard definition: *convolution* of measures on the additive group E^*:

$$\nu \star \sigma(A) = \int_{E^*} \nu(A - x)\sigma(dx), \qquad A \subset E^*$$

This defines *convolution* of "density functions," i.e., letting μ the Gaussian measure,

$$\nu = \Phi(x)\mu, \quad \sigma = \Psi(x)\mu, \quad \Longrightarrow \quad \nu \star \sigma = (\Phi \star \Psi)(x)\mu$$

By direct calculation we obtain

$$\int_{E^*} 1_A(x)(\Phi \star \Psi)(x)\mu(dx) = \int_{E^*}\int_{E^*} 1_A(x+y)\Phi(x)\Psi(y)\mu(dx)\mu(dy)$$

Replacing 1_A with an exponential vector ϕ_ξ we have

$$\begin{aligned} S(\Phi \star \Psi)(\xi) &= S\Phi(\xi)S\Psi(\xi)e^{\langle \xi, \xi \rangle/2} \\ &= S\Phi(\xi)S\Psi(\xi)Sg_{-2}(\xi) \\ &= S(\Phi \diamond \Psi \diamond g_{-2})(\xi) \end{aligned}$$

The last expression is valid for any pair of $\Phi, \Psi \in \mathcal{W}^*$.

Definition (Kuo (1992))

For $\Phi, \Psi \in \mathcal{W}^*$ their convolution is defined by

$$\Phi \star \Psi = \Phi \diamond \Psi \diamond g_{-2}$$

(Different from the convolution in (II) and from the Wick product (I).)

Kuo's "convolution" operator is defined by

$$M_\Phi^\star \Psi = \Phi \star \Psi$$

Then, $M_\Phi^\star \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$ and

$$\begin{aligned} M^\star &= M_\Phi^\diamond M_{g_2}^\diamond \\ &= (C_\Phi)^* e^{\Delta_G^*/2} \end{aligned}$$

In particular,

$$(M^\star)^* = e^{\Delta_G/2} C_\Phi$$

Summary

1. We reviewed two methods of constructing a white noise triple.
   (I) weighted Fock spaces (CKS-spaces)
   (II) ∞-dim holomorphic functions
2. We obtained characterization of the convolution operators.
3. We showed the Wick product in (I) coincides with the convolution product in (II), namely,

$$\text{Wick calculus} = \text{convolution calculus}$$

The main results are found in

N. Obata and H. Ouerdiane: A note on convolution operators in white noise calculus, preprint, 2011.