An Introduction to Quantum White Noise Calculus

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Plan

1. Backgrounds

Wiener–Itô–Segal isomorphism

$$L^2(E^*, \mu) \cong \Gamma_{\mathrm{Boson}}(H), \qquad H = L^2(\mathbb{R}) \text{ etc.}$$

$$\phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}: , f_n \rangle \leftrightarrow \phi = (f_n)$$

$$B_t \leftrightarrow (0, 1_{[0,t]}, 0, 0, \dots) \quad \text{Brownian motion}$$

Annihilation and Creation Operators ⇔ Bose Field

$$A(f) : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, n\langle f, \xi \rangle \xi^{\otimes (n-1)}, 0, 0, \dots)$$

$$A^*(f) : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, 0, \xi^{\otimes n} \widehat{\otimes} f, 0, \dots)$$

Quantum Brownian Motion and Quantum White Noise

$$B_t \text{ (as multiplication operator on } L^2(E^*, \mu)) = A(1_{[0,t]}) + A^*(1_{[0,t]})$$

$$W_t = \frac{d}{dt} B_t = a_t + a_t^*$$

though almost all sample paths are nowhere differentiable [Payley–Wiener–Zygmund]

1.2. Our Stand Point: Fusing Hida's and Hudson–Parthasarathy's Calculi

$\{B_t\}$ Quantum Brownian Motion

$\{P_t\}$ Quantum Poisson Process

$\{B_t\}$ Brownian Motion

Ito Calculus (1940s)

SDE $\{dB_t\}$

$dB_t = dA_t + dA_t^*$

Hudson-Parthasarathy Calculus (1984)

QSDE $\{dA_t, dA_t^*, d\Lambda_t\}$

$$W_t = \frac{dB_t}{dt}$$

$$a_t = \frac{dA_t}{dt} \qquad a_t^* = \frac{dA_t^*}{dt}$$

Hida Calculus (1975)

white noise $\{W_t\}$

$W_t = a_t + a_t^*$

Quantum White Noise Calculus

$\{a_t, a_t^*\}$

Classical Probability Theory

Quantum Probability Theory

2. Elements of Quantum White Noise Calculus

2.1. White Noise Distributions

T: a topological space (time interal, space-time manifold, even a discrete space,...)

Gelfand (nuclear) triple for $H = L^2(T)$

$$E \subset H = L^2(T) \subset E^*, \qquad E = \operatorname*{proj\,lim}_{p\to\infty} E_p\,, \quad E^* = \operatorname*{ind\,lim}_{p\to\infty} E_{-p}\,,$$

where E_p is a dense subspace of H and is a Hilbert space for itself.

Example: $E = \mathcal{S}(\mathbb{R}) = \mathrm{proj\,lim}_{p\to\infty} \mathcal{S}_p(\mathbb{R})$

The Boson Fock space over $H = L^2(T)$ is defined by

$$\Gamma(H) = \left\{\phi = (f_n)\,;\, f_n \in H^{\widehat{\otimes} n}\,,\ \|\phi\|^2 = \sum_{n=0}^{\infty} n!|f_n|_0^2 < \infty\right\},$$

where $|f_n|_0$ is the usual L^2-norm of $H^{\widehat{\otimes} n} = L^2_{\mathrm{sym}}(T^n)$.

Gelfand (nuclear) triple for $\Gamma(H)$ [Kubo–Takenaka PJA 56A (1980)]

$$(E) \subset \Gamma(H) \subset (E)^*, \qquad (E) = \operatorname*{proj\,lim}_{p\to\infty} \Gamma(E_p), \quad (E)^* = \operatorname*{ind\,lim}_{p\to\infty} \Gamma(E_{-p})$$

Cochran–Kuo–Sengupta IDAQP 1 (1998)

$$\Gamma_\alpha(E_p) = \left\{ \phi = (f_n)\,;\, f_n \in E_p^{\hat{\otimes} n}, \|\phi\|_{p,+}^2 = \sum_{n=0}^{\infty} n!\, \alpha(n) |f_n|_p^2 < \infty \right\},$$

$$\mathcal{W} = \operatorname*{proj\,lim}_{p\to\infty} \Gamma_\alpha(E_p),$$

Gannoun–Hachaichi–Ouerdiane–Rezgui JFA 171 (2000), also Lee JFA 100 (1991)

$$\mathrm{Exp}(E_p, \theta, m) = \left\{ f : E_p \to \mathbb{C}\,; \begin{array}{l} \text{entire holomorphic,} \\ \|f\|_{\theta,p,m} = \sup_{x\in E_p} |f(x)| e^{-\theta(m|x|_p)} < \infty \end{array} \right\}$$

$$\mathcal{F}_\theta(E^*) = \operatorname*{proj\,lim}_{p\to\infty,\, m\to+0} \mathrm{Exp}(E_{-p}, \theta, m),$$

$$\mathcal{W} = \{\text{Taylor coefficients of } \phi \in \mathcal{F}_\theta(E^*)\}$$

Theorem (Asai–Kubo–Kuo, Hiroshima Math. J. 31 (2001))

The above two classes of Gelfand triples coincide.

2.3. White Noise Operators

Definition

A continuous operator from (E) into $(E)^*$ is called a *white noise operator*. The space of white noise operators is denoted by $\mathcal{L}((E),(E)^*)$ (bounded convergence topology).

The *annihilation* and *creation operator* at a point $t \in T$

$$a_t : (0,\dots,0,\xi^{\otimes n},0,\dots) \mapsto (0,\dots,0,n\xi(t)\xi^{\otimes(n-1)},0,0,\dots)$$
$$a_t^* : (0,\dots,0,\xi^{\otimes n},0,\dots) \mapsto (0,\dots,0,0,\xi^{\otimes n}\widehat{\otimes}\delta_t,0,\dots)$$

The pair $\{a_t, a_t^* \,;\, t \in T\}$ is called the *quantum white noise* on T.

Theorem

$a_t \in \mathcal{L}((E),(E))$ and $a_t^* \in \mathcal{L}((E)^*,(E)^*)$ for all $t \in \mathbb{R}$. Moreover, both maps $t \mapsto a_t \in \mathcal{L}((E),(E))$ and $t \mapsto a_t^* \in \mathcal{L}((E)^*,(E)^*)$ are operator-valued test functions, i.e., belongs to $E \otimes \mathcal{L}((E),(E))$ and $E \otimes \mathcal{L}((E)^*,(E)^*)$, respectively.

Remark: Without a Gelfand triple, a_t and a_t^* are considered as (unbounded) operator-valued distribution,

$$a(\xi) = \int_T \xi(t)a_t\,dt, \quad a^*(\xi) = \int_T \xi(t)a_t^*\,dt \qquad \text{(smeared operators)}$$

For $\Xi \in \mathcal{L}((E),(E)^*)$ the *symbol* is defined by

$$\widehat{\Xi}(\xi,\eta) = \langle\langle \Xi\varphi_\xi,\, \varphi_\eta \rangle\rangle\,, \qquad \xi,\eta \in E,$$

where $\varphi_\xi = \left(1, \xi, \dfrac{\xi^{\otimes 2}}{2!}, \cdots\right)$ is an *exponential vector*.

$\mathcal{E} = \{\varphi_\xi\,;\, \xi \in E\} \subset (E)$ is linearly independent dense set
$\Longrightarrow$ A linear operator Ξ is uniquely specified by the action on $\mathcal{E}$

Characterization theorem for symbols [O. JMSJ 45 (1993)]

Let Θ be a $\mathbb{C}$-valued function on $E \times E$. Then there exists a white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ such that $\Theta = \widehat{\Xi}$ if and only if

1. (analyticity) for any ξ, ξ_1, η, $\eta_1 \in E$, the function $\Theta(z\xi+\xi_1, w\eta+\eta_1)$ is an entire holomorphic function of $(z,w) \in \mathbb{C}\times\mathbb{C}$;
2. (growth condition) there exist constant numbers $C \ge 0$, $K \ge 0$ and $p \ge 0$ such that
$$|\Theta(\xi,\eta)| \le C\exp\left\{K(|\xi|_p^2 + |\eta|_p^2)\right\}, \qquad \xi,\eta \in E.$$

We have similar conditions for $\Xi \in \mathcal{L}((E),(E))$.

Definition (Integral kernel operator)

Given $\kappa_{l,m} \in (E^{\otimes(l+m)})^*$, $l, m = 0, 1, 2, \dots$,

$$\Xi_{l,m}(\kappa_{l,m}) = \int_{T^{l+m}} \kappa_{l,m}(s_1, \cdots, s_l, t_1, \cdots, t_m)$$
$$a^*_{s_1} \cdots a^*_{s_l} a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m$$

is a well-defined white noise operator and is called an *integral kernel operator*.

In fact, $\Xi_{l,m}(\kappa_{l,m})$ is defined by its symbol:

$$\langle\!\langle \Xi_{l,m}(\kappa_{l,m})\varphi_\xi,\, \varphi_\eta \rangle\!\rangle = \langle \kappa_{l,m},\, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle e^{\langle \xi,\, \eta \rangle},$$

where

$$\varphi_\xi = \left(1, \xi, \frac{\xi^{\otimes 2}}{2!}, \cdots\right) \quad \text{(exponential vector)}.$$

Theorem (O. JMSJ 45 (1993); tracing back to Haag, Berezin, Krée,...)

Every white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ *admits the Fock expansion:*

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \qquad \kappa_{l,m} \in (E^{\otimes(l+m)})^*,$$

where the right-hand side converges in $\mathcal{L}((E),(E)^*)$. *If* $\Xi \in \mathcal{L}((E),(E))$, *then* $\kappa_{l,m} \in E^{\otimes l} \otimes (E^{\otimes m})^*$ *and the series converges in* $\mathcal{L}((E),(E))$.

Applications:

1. rotation-invariant operators (N, Δ_G, Δ_G^*) [O. MZ 210 (1992)]
2. derivations and Wick derivations [Chung–Chung (1996), Chung–Chung–Ji (1998), Huang–Luo (1998), etc.]
3. irreducibility of energy representation of gauge groups [Shimada IDAQP 8 (2005)]

Our Standpoint

A white noise operator Ξ as a function of quantum white noise:

$$\Xi = \Xi(a_s, a_t^*; s,t \in T)$$

$\Longrightarrow$ We treat $\{a_s,\, a_t^*\,;\, s,t \in T\}$ as a *coordinate system* for white noise operators.

2.5. Wick Product

Let us introduce a product of operators, different from the usual composition.

Definition (Wick (normal-ordered) product)

For $\Xi_1, \Xi_2 \in \mathcal{L}((E),(E)^*)$ the Wick (or normal-ordered) product $\Xi_1 \diamond \Xi_2$ is defined by

$$(\Xi_1 \diamond \Xi_2)\widehat{\ }(\xi,\eta) = \widehat{\Xi}_1(\xi,\eta)\widehat{\Xi}_2(\xi,\eta)e^{-\langle \xi,\eta\rangle}, \qquad \xi,\eta \in E,$$

where $\widehat{\Xi}(\xi,\eta)$ is the *symbol* of a white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ defined by

$$\widehat{\Xi}(\xi,\eta) = \langle\langle \Xi\phi_\xi, \phi_\eta \rangle\rangle, \qquad \xi,\eta \in E,$$

where $\phi_\xi = (1, \xi, \cdots, \xi^{\otimes n}/n!, \cdots)$ is an *exponential vector*.

Some properties:

1. For any $\Xi \in \mathcal{L}((E),(E)^*)$ we have
$$a_t \diamond \Xi = \Xi \diamond a_t = \Xi a_t\,, \qquad a_t^* \diamond \Xi = \Xi \diamond a_t^* = a_t^* \Xi.$$
2. Equipped with the Wick product, $\mathcal{L}((E),(E)^*)$ becomes a commutative algebra.

2.5. Convolution Product = Wick Product

Definition (Ben Chrouda–El Oued–Ouerdiane Soochow JM 28 (2002))

With each $\Phi \in \mathcal{W}^*$ we associate the *convolution operator* $C_\Phi \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ defined by

$$[H(C_\Phi \phi)](x) = \langle\langle \Phi, T_{-x}\phi \rangle\rangle, \qquad x \in E^*.$$

Theorem (O.–Ouerdiane IDAQP 14 (2011))

$$C_\Phi = (M_\Phi^\diamond)^*, \qquad M_\Phi^\diamond = (C_\Phi)^*, \qquad \Phi \in \mathcal{W}^*,$$

where $M_\Phi^\diamond \in \mathcal{L}(\mathcal{W}^*, \mathcal{W}^*)$ *is the Wick multiplication operator defined by*

$$M_\Phi^\diamond \Psi = \Phi \diamond \Psi, \qquad \Psi \in \mathcal{W}^*.$$

In some literatures, the "convolution product" of $\Phi, \Psi \in \mathcal{W}^*$ is defined by

$$\langle\langle \Phi \star \Psi, \phi \rangle\rangle = \langle\langle \Psi, C_\Phi \phi \rangle\rangle.$$

Using $C_\Phi = (M_\Phi^\diamond)^*$ we see that

$$\langle\langle \Psi, C_\Phi \phi \rangle\rangle = \langle\langle M_\Phi^\diamond \Psi, \phi \rangle\rangle = \langle\langle \Phi \diamond \Psi, \phi \rangle\rangle$$

Therefore, the convolution product = the Wick product: $\Phi \star \Psi = \Phi \diamond \Psi$

3. Quantum Stochastic Gradients

Motivation

Quantum stochastic integrals of Itô type [Hudson–Parthasarathy (1984)]

$$\Xi_t = \int_0^t E dA_s + \int_0^t F dA_s^* + \int_0^t G d\Lambda_s \quad \text{for adapted integrands}$$

1) taking the actions on exponential vectors (operator symbols)
2) and using parallel arguments as in the case of classical Itô integrals

Generalizations to non-adapted integrands [Belavkin (1991), Lindsay (1993)]

$$\delta^+(\Xi)\phi = \delta(\Xi\phi), \quad \delta^-(\Xi)\phi = \int_{\mathbb{R}} \Xi(t)(\nabla\phi(t))dt, \quad \delta^0(\Xi\phi) = \delta(\Xi\nabla\phi),$$

1) $\nabla : \mathbf{D} \to L^2(\mathbb{R}, \Gamma(H))$ is the classical stochastic gradient: $\phi(t) = a_t\phi$,
2) $\delta = \nabla^* : L^2(\mathbb{R}, \Gamma(H)) \to \mathbf{D}^*$ is the divergence operator,
3) and ϕ is a nice vector.

- The divergence operator $\delta = \nabla^*$ defines (non-adapted) stochastic integrals which generalize the Itô integrals (Hitsuda-Skorokhod, Zakai-Nualart-Pardoux).
- How about introducing the quantum stochastic gradients?

Classical Stochastic Gradient and Hitsuda–Skorohod Integral

$\nabla : (E) \to \mathcal{S}(\mathbb{R}) \otimes (E) \cong \mathcal{S}(\mathbb{R}, (E))$ defined by

$$\nabla\phi(t) = a_t\phi, \qquad \phi \in (E), \quad t \in \mathbb{R}.$$

Classical stochastic gradients with several domains of ∇

$$\begin{array}{ccccccccc}
(E) & \longrightarrow & \mathcal{G} & \longrightarrow & \mathrm{D} & \longrightarrow & \Gamma(H) & \longrightarrow & \mathcal{G}^* \\
{\scriptstyle\nabla}\downarrow & & {\scriptstyle\nabla}\downarrow & & {\scriptstyle\nabla}\downarrow & & {\scriptstyle\nabla}\downarrow & & {\scriptstyle\nabla}\downarrow \\
\mathcal{S}(\mathbb{R}, (E)) & \longrightarrow & L^2(\mathbb{R}, \mathcal{G}) & \longrightarrow & L^2(\mathbb{R}, \Gamma(H)) & \longrightarrow & L^2(\mathbb{R}, \mathrm{D}^*) & \longrightarrow & L^2(\mathbb{R}, \mathcal{G}^*)
\end{array}$$

Adjoint actions of classical stochastic gradients

$$\begin{array}{ccccccccc}
L^2(\mathbb{R}, \mathcal{G}) & \longrightarrow & L^2(\mathbb{R}, \mathrm{D}) & \longrightarrow & L^2(\mathbb{R}, \Gamma(H)) & \longrightarrow & L^2(\mathbb{R}, \mathcal{G}^*) & \longrightarrow & \mathcal{S}'(\mathbb{R}, (E)^*) \\
\downarrow{\scriptstyle\delta=\nabla^*} & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow \\
\mathcal{G} & \longrightarrow & \Gamma(H) & \longrightarrow & \mathrm{D}^* & \longrightarrow & \mathcal{G}^* & \longrightarrow & (E)^*
\end{array}$$

$\delta(\Psi)$: the Hitsuda–Skorohod integral, generalizing the stochastic integral of Itô type.

Creation Gradient ∇^+ and Creation Integral δ^+

Classical stochastic gradient ∇ : (random variables) $\to$ (stochastic processes)
Quantum stochastic gradient

$$\nabla^\epsilon : \text{(quantum random variables)} \to \text{(quantum stochastic processes)}$$

- (quantum random variables) $\approx \mathcal{L}((E),(E)^*)$
- Choose suitable domains for ∇^ϵ, based on the inclusion relations:

$$(E) \subset \mathcal{G} \subset \mathrm{D} \subset \Gamma(H) \subset \mathrm{D}^* \subset \mathcal{G}^* \subset (E)^*$$

Consider ∇^+ acting on $\mathcal{L}((E),\mathrm{D})$

$$\begin{aligned}
\mathcal{L}((E),\mathrm{D}) &\xrightarrow{\cong} \mathrm{D} \otimes (E)^* \xrightarrow{\nabla \otimes I} L^2(\mathbb{R},\Gamma(H)) \otimes (E)^* \\
&\cong \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbb{R},\Gamma(H)) \otimes \Gamma(E_{-p}) \\
&\cong \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbb{R},\Gamma(H)\otimes\Gamma(E_{-p})) \stackrel{\text{def}}{=} L^2(\mathbb{R},\Gamma(H)\otimes(E)^*) \\
&\cong \operatorname*{ind\,lim}_{p\to\infty} L^2(\mathbb{R},\mathcal{L}_2(\Gamma(E_p),\Gamma(H))) \stackrel{\text{def}}{=} L^2(\mathbb{R},\mathcal{L}((E),\Gamma(H)))
\end{aligned}$$

Thus, the creation gradient is defined:

$$\nabla^+ : \mathcal{L}((E),\mathrm{D}) \to L^2(\mathbb{R},\Gamma(H)\otimes(E)^*) \cong L^2(\mathbb{R},\mathcal{L}((E),\Gamma(H)))$$

$$\begin{array}{ccccc}
\mathcal{L}((E)^*, \mathrm{D}) & \longrightarrow & \mathcal{L}_2(\Gamma(E_p), \mathrm{D}) & \longrightarrow & \mathcal{L}((E), \mathrm{D}) \\
\nabla^+ \big\downarrow & & \nabla^+ \big\downarrow & & \nabla^+ \big\downarrow \\
L^2(\mathbb{R}, \mathcal{L}((E)^*, \Gamma(H))) & \longrightarrow & L^2(\mathbb{R}, \mathcal{L}_2(\Gamma(E_p), \Gamma(H))) & \longrightarrow & L^2(\mathbb{R}, \mathcal{L}((E), \Gamma(H)))
\end{array}$$

$$\begin{array}{ccccc}
\mathcal{L}((E)^*, \Gamma(H)) & \longrightarrow & \mathcal{L}_2(\Gamma(E_p), \Gamma(H)) & \longrightarrow & \mathcal{L}((E), \Gamma(H)) \\
\nabla^+ \big\downarrow & & \nabla^+ \big\downarrow & & \nabla^+ \big\downarrow \\
L^2(\mathbb{R}, \mathcal{L}((E)^*, \mathrm{D}^*)) & \longrightarrow & L^2(\mathbb{R}, \mathcal{L}_2(\Gamma(E_p), \mathrm{D}^*)) & \longrightarrow & L^2(\mathbb{R}, \mathcal{L}((E), \mathrm{D}^*))
\end{array}$$

Explicit norm estimates are possible, for example,

Theorem (Ji–Obata IIS 15 (2009))

For any $\Xi \in \mathcal{L}_2(\Gamma(H), \mathrm{D})$ *we have*

$$\int_{\mathbb{R}} \|\nabla^+ \Xi(t)\|^2_{\mathcal{L}_2(\Gamma(H),\Gamma(H))} dt \le \|\Xi\|^2_{\mathcal{L}_2(\Gamma(H),\mathrm{D})}.$$

Therefore, $\nabla^+ \Xi(t)$ *is a Hilbert–Schmidt operator on* $\Gamma(H)$ *for a.e.* $t \in \mathbb{R}$.

Definition (Creation integral $\delta^+ = (\nabla^+)^*$)

$$\begin{tikzcd}
L^2(\mathbb{R}, \mathcal{L}((E)^*, \Gamma(H))) \arrow[r] \arrow[d, "\delta^+"'] & L^2(\mathbb{R}, \mathcal{L}_2(\Gamma(E_p), \Gamma(H))) \arrow[r] \arrow[d, "\delta^+"'] & L^2(\mathbb{R}, \mathcal{L}((E), \Gamma(H))) \arrow[d, "\delta^+"'] \\
\mathcal{L}((E)^*, \mathrm{D}^*) \arrow[r] & \mathcal{L}_2(\Gamma(E_p), \mathrm{D}^*) \arrow[r] & \mathcal{L}((E), \mathrm{D}^*)
\end{tikzcd}$$

$$\begin{tikzcd}
L^2(\mathbb{R}, \mathcal{L}((E)^*, \mathrm{D})) \arrow[r] \arrow[d, "\delta^+"'] & L^2(\mathbb{R}, \mathcal{L}_2(\Gamma(E_p), \mathrm{D})) \arrow[r] \arrow[d, "\delta^+"'] & L^2(\mathbb{R}, \mathcal{L}((E), \mathrm{D})) \arrow[d, "\delta^+"'] \\
\mathcal{L}((E)^*, \Gamma(H)) \arrow[r] & \mathcal{L}_2(\Gamma(E_p), \Gamma(H)) \arrow[r] & \mathcal{L}((E), \Gamma(H)).
\end{tikzcd}$$

Relation to the classical HS-integral

For $\Xi \in \mathbf{Dom}\,(\delta^+)$ in the above diagrams we have

$$\langle\langle \delta^+(\Xi)\phi, \psi \rangle\rangle = \int_{\mathbb{R}} \langle\langle \Xi(t)\phi, \nabla\psi(t) \rangle\rangle \, dt$$

for a pair ϕ, ψ in the domains. Therefore, denoting $(\Xi\phi)(t) = \Xi(t)\phi$ we have

$$\delta^+(\Xi)\phi = \delta\,(\Xi\phi)\,, \qquad \phi \in (E).$$

Therefore, $\delta^+(\Xi)$ coincides with the non-adapted quantum stochastic integrals defined by Belavkin (1991) and Lindsay (1993) when Ξ is in the common domain.

Regularity of the creation integrals follows from the corresponding norm estimates of the creation gradient.
Estimates for the creation gradient is often more straightforward than the integrals.

Theorem (Ji–Obata IIS 15 (2009))

(1) *We have norm estimates of* $\delta^+(\Xi)$, *for example,*

$$\|\delta^+(\Xi)\|^2_{\mathcal{L}_2(\Gamma(E_p),\mathrm{D}^*)} \le \int_{\mathbb{R}} \|\Xi(t)\|^2_{\mathcal{L}_2(\Gamma(E_p),\Gamma(H))}\, dt$$

$$\|\delta^+(\Xi)\|^2_{\mathcal{L}_2(\Gamma(E_p),\Gamma(H))} \le \int_{\mathbb{R}} \|\Xi(t)\|^2_{\mathcal{L}_2(\Gamma(E_p),\mathrm{D})}\, dt$$

(2) [Hilbert–Schmidt criterion] *For any* $\Xi \in L^2(\mathbb{R}, \mathcal{L}_2(\Gamma(H),\mathrm{D}))$ *the creation integral* $\delta^+(\Xi)$ *is a Hilbert–Schmidt operator on* $\Gamma(H)$.

(3) [Boundedness criterion] *For any* $\Xi \in L^2(\mathbb{R}, \mathcal{L}(\Gamma(H),\mathrm{D}))$ *the creation integral* $\delta^+(\Xi)$ *is a bounded operator on* $\Gamma(H)$.

creation gradient

$$\nabla^+ : \mathcal{L}((E), \mathrm{D}) \xrightarrow{\cong} \mathrm{D} \otimes (E)^* \xrightarrow{\nabla \otimes I} L^2(\mathbb{R}, \Gamma(H)) \otimes (E)^* \cong \cdots$$

annihilation gradient

$$\nabla^- : \mathcal{L}(\mathrm{D}^*, (E)^*) \xrightarrow{\cong} (E)^* \otimes \mathrm{D} \xrightarrow{I \otimes \nabla} (E)^* \otimes L^2(\mathbb{R}, \Gamma(H)) \cong \cdots$$

conservation gradient

$$\nabla^0 : \mathcal{L}((E)^*, \mathrm{D}) \xrightarrow{\cong} \mathrm{D} \otimes (E) \xrightarrow{\nabla \oslash \nabla} L^2(\mathbb{R}, \Gamma(H) \otimes (E)) \xrightarrow{\cong} L^2(\mathbb{R}, \mathcal{L}((E)^*, \Gamma(H$$

where $[(\nabla \oslash \nabla)\phi \otimes \psi](t) = \nabla\phi(t) \otimes \nabla\psi(t)$ ("diagonalized" tensor product).

Annihilation integrals $\delta^- = (\nabla^-)^*$

$$\begin{array}{ccccc}
L^2(\mathbb{R}, \mathcal{L}(\Gamma(H), (E))) & \longrightarrow & L^2(\mathbb{R}, \mathcal{L}_2(\Gamma(H), \Gamma(E_p))) & \longrightarrow & L^2(\mathbb{R}, \mathcal{L}(\Gamma(H), (E)^*)) \\
\delta^- \big\downarrow & & \delta^- \big\downarrow & & \delta^- \big\downarrow \\
\mathcal{L}(\mathrm{D}, (E)) & \longrightarrow & \mathcal{L}_2(\mathrm{D}, \Gamma(E_p)) & \longrightarrow & \mathcal{L}(\mathrm{D}, (E)^*)
\end{array}$$

Conservation integrals $\delta^0 = (\nabla^0)^*$

$$\begin{array}{ccc}
L^2(\mathbb{R}, \mathcal{L}((E), \mathrm{D})) & \longrightarrow & L^2(\mathbb{R}, \mathcal{L}((E), \Gamma(H))) \\
\delta^0 \big\downarrow & & \delta^0 \big\downarrow \\
\mathcal{L}((E), \Gamma(H)) & \longrightarrow & \mathcal{L}((E), \mathrm{D}^*).
\end{array}$$

4. Quantum White Noise Derivatives

Aim

For a white noise operator

$$\Xi = \Xi(a_s, a_t^*; s, t \in T)$$

we should like to define the derivatives with respect to a_s and a_t^*:

$$\frac{\delta \Xi}{\delta a_s} \quad \text{and} \quad \frac{\delta \Xi}{\delta a_t^*}$$

Expected properties:

$$\frac{\delta}{\delta a_s} \int f(t) a_t dt = f(s) I$$

$$\frac{\delta}{\delta a_s} \int f(s,t) a_s a_t ds dt = \int f(s,t) a_t \, dt + \int f(t,s) a_t \, dt$$

$$\frac{\delta}{\delta a_t^*} \int f(s,t) a_s a_t^* ds dt = \int f(s,t) a_s \, ds$$

Definition (Ji–O. Sem. et Congres 16 (2008))

For $\Xi \in \mathcal{L}((E),(E)^*)$ and $\zeta \in E$ we define $D_\zeta^\pm \Xi \in \mathcal{L}((E),(E)^*)$ by

$$D_\zeta^+ \Xi = [a(\zeta), \Xi], \qquad D_\zeta^- \Xi = -[a^*(\zeta), \Xi].$$

These are called the *creation derivative* and *annihilation derivative* of Ξ, respectively. Both together are called the *quantum white noise derivatives*.

Note: For $\zeta \in E$, both

$$a(\zeta) = \Xi_{0,1}(\zeta) = \int_T \zeta(t) a_t \, dt, \quad a^*(\zeta) = \Xi_{1,0}(\zeta) = \int_T \zeta(t) a_t^* \, dt,$$

belong to $\mathcal{L}((E),(E)) \cap \mathcal{L}((E)^*,(E)^*)$.

Some properties:

1. $(D_\zeta^+ \Xi)^* = D_\zeta^-(\Xi^*)$ and $(D_\zeta^- \Xi)^* = D_\zeta^+(\Xi^*)$.
2. $D_\zeta^\pm$ is a continuous linear map from $\mathcal{L}((E),(E)^*)$ into itself.
3. Moreover, $(\zeta, \Xi) \mapsto D_\zeta^\pm \Xi$ is a continuous bilinear map from $E \times \mathcal{L}((E),(E)^*)$ into $\mathcal{L}((E),(E)^*)$.

4.3. Examples

The canonical correspondence (kernel theorem) between $S \in \mathcal{L}(E, E^*)$ and $\tau = \tau_S \in (E \otimes E)^*$ is given by $\langle \tau_S, \eta \otimes \xi \rangle = \langle S\xi, \eta \rangle$ for $\xi, \eta \in E$.

(1) The *generalized Gross Laplacian* associated with S is defined by

$$\Delta_G(S) = \Xi_{0,2}(\tau_S) = \int_{T \times T} \tau_S(s,t) a_s a_t \, ds dt$$

Note that $\Delta_G(S) \in \mathcal{L}((E),(E))$. Then,

$$D_\zeta^+ \Delta_G(S) = 0, \qquad D_\zeta^- \Delta_G(S) = a(S\zeta) + a(S^*\zeta)$$

In fact, since

$$D_t^- \Delta_G(S) = \int_T \tau_S(s,t) a_s \, ds + \int_T \tau_S(t,s) a_s \, ds$$

we have

$$\begin{aligned} D_\zeta^- \Delta_G(S) &= \int_{T \times T} \tau_S(s,t) a_s \zeta(t) \, ds dt + \int_{T \times T} \tau_S(t,s) a_s \zeta(t) \, ds dt \\ &= \int_T S\zeta(s) a_s \, ds + \int_T S^*\zeta(s) a_s \, ds = a(S\zeta) + a(S^*\zeta) \end{aligned}$$

(2) The adjoint of $\Delta_{\mathrm{G}}(S) \in \mathcal{L}((E)^*, (E)^*)$ is given by

$$\Delta_{\mathrm{G}}^*(S) = \Xi_{2,0}(\tau_S) = \int_{T\times T} \tau_S(s,t) a_s^* a_t^* \, dsdt$$

The quantum white noise derivatives are given by

$$D_\zeta^- \Delta_{\mathrm{G}}^*(S) = 0, \qquad D_\zeta^+ \Delta_{\mathrm{G}}^*(S) = a^*(S\zeta) + a^*(S^*\zeta)$$

(3) The *conservation operator* associated with S is defined by

$$\Lambda(S) = \Xi_{1,1}(\tau_S) = \int_{T\times T} \tau_S(s,t) a_s^* a_t \, dsdt$$

In general, $\Lambda(S) \in \mathcal{L}((E), (E)^*)$.
The quantum white noise derivatives are given by

$$D_\zeta^- \Lambda(S) = a^*(S\zeta), \qquad D_\zeta^+ \Lambda(S) = a(S^*\zeta).$$

$(\mathcal{L}((E),(E)^*),\diamond)$ is a commutative algebra.

Definition (Wick derivation)

A continuous linear map $\mathcal{D}:\mathcal{L}((E),(E)^*)\to\mathcal{L}((E),(E)^*)$ is called a *Wick derivation* if

$$\mathcal{D}(\Xi_1\diamond\Xi_2)=(\mathcal{D}\Xi_1)\diamond\Xi_2+\Xi_1\diamond(\mathcal{D}\Xi_2),\qquad \Xi_1,\Xi_2\in\mathcal{L}((E),(E)^*).$$

Theorem (Ji–O. JMP 51 (2010))

The creation and annihilation derivatives $D_\zeta^\pm$ are Wick derivations for any $\zeta\in E$.

Theorem (Ji–O. JMP 51 (2010))

A general Wick derivation $\mathcal{D}$ is expressed in the form:

$$\mathcal{D}=\int_T F(t)\diamond D_t^+\,dt+\int_T G(t)\diamond D_t^-\,dt,$$

where $F,G\in E\otimes\mathcal{L}((E),(E)^*)$.

Wick derivations for white noise functions [Chung–Chung JKMS 33 (1996)].

5. Wick Type Differential Equations for White Noise Operators

Theorem (Ji–O. JMP 51 (2010))

Given a Wick derivation $\mathcal{D}$ *and a white noise operator* $G \in \mathcal{L}((E),(E)^*)$*, consider a* **Wick type differential equation for white noise operators***:*

$$\mathcal{D}\Xi = G \diamond \Xi. \tag{*}$$

Assume that there exists an operator $Y \in \mathcal{L}((E),(E)^*)$ *such that* $\mathcal{D}Y = G$ *and* $\mathrm{wexp}\ Y$ *is defined in* $\mathcal{L}((E),(E)^*)$*. Then every solution to (*) is of the form:*

$$\Xi = (\mathrm{wexp}\ Y) \diamond F,$$

where $F \in \mathcal{L}((E),(E)^*)$ *satisfying* $\mathcal{D}F = 0$*.*

Wick exponential

$$\mathrm{wexp}\ Y = \sum_{n=0}^{\infty} \frac{1}{n!} Y^{\diamond n}, \qquad Y \in \mathcal{L}((E),(E)^*),$$

whenever the series converges in $\mathcal{L}((E),(E)^*)$.

Let us consider the (system of) differential equations:

$$D_\zeta^+ \Xi = 0, \qquad \zeta \in E. \tag{1}$$

By Fock expansion we see that the solutions to (1) are given by

$$\Xi = \sum_{m=0}^{\infty} \Xi_{0,m}(\kappa_{0,m}) \qquad \text{contains no creations}$$

In a similar manner, the solutions to

$$D_\zeta^- \Xi = 0, \qquad \zeta \in E, \tag{2}$$

are given by

$$\Xi = \sum_{l=0}^{\infty} \Xi_{l,0}(\kappa_{l,0}) \qquad \text{contains no annihilations}$$

Consequently, a white noise operator satisfying both (1) and (2) are the scalar operators (*irreducibility of the canonical commutation relation*).

Let us consider the differential equation:

$$D_\zeta^- \Xi = 2a(\zeta) \diamond \Xi, \qquad \zeta \in E. \tag{3}$$

Apply our general result (Theorem).

1. We need to find $Y \in \mathcal{L}((E), (E)^*)$ satisfying $D_\zeta^- Y = 2a(\zeta)$.
2. In fact, $Y = \Delta_G$ is a solution.
3. Moreover, it is easily verified that $\mathbf{wexp}\ \Delta_G$ is defined in $\mathcal{L}((E), (E))$.
4. Then, a general solution to (3) is of the form:

$$\Xi = (\mathbf{wexp}\ \Delta_G) \diamond F,$$

where $D_\zeta^- F = 0$ for all $\zeta \in E$.

5.2. Example (3)

Now we consider the differential equation:

$$\begin{cases} D_\zeta^- \Xi = 2a(\zeta) \diamond \Xi, & \zeta \in E, \\ D_\zeta^+ \Xi = 0. \end{cases} \tag{4}$$

By Example (2) the solution is of the form:

$$\Xi = (\mathrm{wexp}\ \Delta_{\mathrm{G}}) \diamond F, \qquad D_\zeta^- F = 0 \text{ for all } \zeta \in E.$$

We need only to find additional conditions for F satisfying $D_\zeta^+ \Xi = 0$.
Noting that $D_\zeta^+ \Delta_{\mathrm{G}} = 0$, we have

$$D_\zeta^+ \Xi = (\mathrm{wexp}\ \Delta_{\mathrm{G}}) \diamond D_\zeta^+ F = 0.$$

Hence $D_\zeta^+ F = 0$ for all $\zeta \in E$, so F is a scalar operator (Example (1)).
Consequently, the solution to (4) is of the form:

$$\Xi = C\ \mathrm{wexp}\ \Delta_{\mathrm{G}}, \qquad C \in \mathbb{C}.$$

6. Applications

6.1. Integral Representation of Quantum Martingales

Theorem (Ji–O. CMP 286 (2009))

Let $\zeta \in E$ *and* $\Xi \in L^2(\mathbb{R}, \mathcal{L}((E),(E)^*))$. *Then we have*

$$D_\zeta^+(\delta^+(\Xi)) = \delta^+(D_\zeta^+\Xi) + \int_{\mathbb{R}} \zeta(t)\Xi(t)dt, \qquad D_\zeta^-(\delta^+(\Xi)) = \delta^+(D_\zeta^-\Xi).$$

$$D_\zeta^+(\delta^-(\Xi)) = \delta^-(D_\zeta^+\Xi), \qquad D_\zeta^-(\delta^-(\Xi)) = \delta^-(D_\zeta^-\Xi) + \int_{\mathbb{R}} \zeta(t)\Xi(t)dt,$$

$$D_\zeta^+(\delta^0(\Xi)) = \delta^0(D_\zeta^+\Xi) + \delta^-(\zeta\Xi), \qquad D_\zeta^-(\delta^0(\Xi)) = \delta^0(D_\zeta^-\Xi) + \delta^+(\zeta\Xi).$$

⋆ We wish to put $\zeta = \delta_t$ to obtain more direct formulas.

Definition

A white noise operator $\Xi \in \mathcal{L}((E),(E)^*)$ is called *pointwisely differentiable* if there exists a measurable map $t \mapsto D_t^\pm\Xi \in \mathcal{L}((E),(E)^*)$ such that

$$\langle\langle (D_\zeta^\pm\Xi)\phi_\xi,\, \phi_\eta\rangle\rangle = \int_{\mathbb{R}_+} \langle\langle (D_t^\pm\Xi)\phi_\xi,\, \phi_\eta\rangle\rangle \zeta(t)dt, \qquad \zeta \in H,\ \xi,\eta \in E.$$

Then $D_t^+\Xi$ ($D_t^-\Xi$) is called the pointwise creation- (annihilation-) derivative of Ξ.

Definition (Belavkin (1991), Lindsay (1993))

Admissible white noise functions:

$$\mathcal{G} = \operatorname*{proj\,lim}_{p\to\infty} \mathcal{G}_p, \qquad \mathcal{G}_p = \left\{ \phi = (f_n) \in \Gamma(H)\,;\, \sum_{n=0}^{\infty} n! e^{2pn} |f_n|_0^2 < \infty \right\},$$

where $\mathcal{G}$ is a countably Hilbert space but not nuclear. We have
$(E) \subset \mathcal{G} \subset \Gamma(H) \subset \mathcal{G}^* \subset (E)^*$ and $\mathcal{L}(\mathcal{G}, \mathcal{G}^*) \subset \mathcal{L}((E), (E)^*)$.

Theorem

Every $\Xi \in \mathcal{L}(\mathcal{G}, \mathcal{G}^*)$ is pointwisely differentiable and $D_t^{\pm}\Xi \in \mathcal{L}((E), (E)^*)$ is determined for a.e. $t \in \mathbb{R}$.

Theorem

For $\Xi \in L^2(\mathbb{R}, \mathcal{L}(\mathcal{G}, \mathcal{G}^*))$, the quantum stochastic integrals $\delta^\epsilon(\Xi)$ are pointwisely differentiable. Moreover, for a.e. $t \in \mathbb{R}_+$ we have

$$\begin{aligned}
D_t^+(\delta^+(\Xi)) &= \delta^+(D_t^+\Xi) + \Xi(t)\,, & D_t^-(\delta^+(\Xi)) &= \delta^+(D_t^-\Xi), \\
D_t^+(\delta^-(\Xi)) &= \delta^-(D_t^+\Xi), & D_t^-(\delta^-(\Xi)) &= \delta^-(D_t^-\Xi) + \Xi(t)\,, \\
D_t^+(\delta^0(\Xi)) &= \delta^0(D_t^+\Xi) + \Xi(t)a_t\,, & D_t^-(\delta^0(\Xi)) &= \delta^0(D_t^-\Xi) + a_t^*\Xi(t)\,.
\end{aligned}$$

Theorem (Ji JFA 201 (2003), also Parthasarathy–Sinha JFA 67 (1986))

A regular quantum martingale $\{M_t\}_{t\in\mathbb{R}_+} \subset \mathcal{L}(\mathcal{G}_p(\mathbb{R}_+), \mathcal{G}_q(\mathbb{R}_+))$ *admits an integral representation:*

$$M_t = \lambda I + \int_0^t (E dA + F dA^* + G d\Lambda),$$

where $\{E_t\}, \{F_t\}, \{G_t\}$ *in* $\mathcal{L}(\mathcal{G}_p(\mathbb{R}_+), \mathcal{G}_q(\mathbb{R}_+))$ *are adapted processes and* $\lambda \in \mathbb{C}$.

Theorem (Ji–O. CMP 286 (2009))

The integrands of M_t *is obtained by*

$$E_s = D_s^- \left[M_s - \int_0^s a_u^* (D_u^+ M_u) du \right],$$

$$F_s = D_s^+ \left[M_s - \int_0^s (D_u^- M_u) a_u du \right],$$

$$G_s = D_s^+ \left[\int_0^s \left\{ D_u^- \left(M_u - \int_0^u E_v a_v dv - \int_0^u a_v^* F_v dv \right) \right\} du \right].$$

Question

Find the normal-ordered form of $e^{\Delta_G(S)}e^{\Delta_G^*(T)}$ ($S = S^*$, $T = T^*$).

$$\text{i.e.,} \qquad e^{\Delta_G(S)}e^{\Delta_G^*(T)} = \text{(creation part)(annihilation part)},$$

Set $\Xi = e^{\Delta_G(S)}e^{\Delta_G^*(T)}$ and derive a Wick type differential equation.

$$\begin{aligned}
D_\zeta^+ \Xi &= e^{\Delta_G(S)} \cdot 2a^*(T\zeta) \cdot e^{\Delta_G^*(T)} \\
&= \left\{ -2[a^*(T\zeta), e^{\Delta_G(S)}] + 2a^*(T\zeta)e^{\Delta_G(S)} \right\} e^{\Delta_G^*(T)} \\
&= \left\{ 2D_{T\zeta}^- e^{\Delta_G(S)} + 2a^*(T\zeta)e^{\Delta_G(S)} \right\} e^{\Delta_G^*(T)} \\
&= 2 \cdot 2a(ST\zeta)e^{\Delta_G(S)} \cdot e^{\Delta_G^*(T)} + 2a^*(T\zeta)\Xi \\
&= 4a(ST\zeta)\Xi + 2a^*(T\zeta)\Xi \\
&= 4D_{ST\zeta}^+ \Xi + 4\Xi a(ST\zeta) + 2a^*(T\zeta)\Xi
\end{aligned}$$

Hence

$$D_{(I-4ST)\zeta}^+ \Xi = (4a(ST\zeta) + 2a^*(T\zeta)) \diamond \Xi$$

Assume that $I - 4ST$ is invertible. Then we obtain

$$D_\zeta^+ \Xi = \left\{ a(((I - 4ST)^{-1} - I)\zeta) + 2a^*(T(I - 4ST)^{-1}\zeta) \right\} \diamond \Xi \qquad (1)$$

Similarly,

$$D_\zeta^- \Xi = \left\{ a^*(((I - 4TS)^{-1} - I)\zeta) + 2a(S(I - 4TS)^{-1}\zeta) \right\} \diamond \Xi \qquad (2)$$

General solutions to (1) and (2) are obtained by our general result before:

$$\begin{aligned}\Xi &= \mathbf{wexp}\ \Delta_G^*(T(I - 4ST)^{-1}) \diamond \mathbf{wexp}\ \Lambda((I - 4ST)^{-1} - I) \diamond (\text{annihilations}) \\ &= (\text{creations}) \diamond \mathbf{wexp}\ \Lambda((I - 4TS)^{-1} - I) \diamond \mathbf{wexp}\ \Delta_G(S(I - 4TS)^{-1})\end{aligned}$$

Assuming that $ST = TS$, we obtain

$$\begin{aligned}\Xi = C\, \mathbf{wexp}\ &\Delta_G^*(T(I - 4ST)^{-1}) \\ &\diamond \mathbf{wexp}\ \Lambda((I - 4ST)^{-1} - I) \diamond \mathbf{wexp}\ \Delta_G(S(I - 4ST)^{-1}).\end{aligned}$$

Consequently,

$$e^{\Delta_G(S)} e^{\Delta_G^*(T)} = C e^{\Delta_G^*(T(I-4ST)^{-1})}\, \Gamma((I - 4ST)^{-1})\, e^{\Delta_G(S(I-4ST)^{-1})},$$

where the constant C is obtained from vacuum expectation $(C = \det(1 - 4ST)^{-1/2})$.

6.3. The Implementation Problem for CCR

Let $S, T \in \mathcal{L}(E, E)$ and consider transformed annihilation and creation operators:

$$b(\zeta) = a(S\zeta) + a^*(T\zeta), \quad b^*(\zeta) = a^*(S\zeta) + a(T\zeta),$$

where $\zeta \in E$. We know that $b(\zeta), b^*(\zeta) \in \mathcal{L}((E), (E)) \cap \mathcal{L}((E)^*, (E)^*)$.

The implementation problem

is to find a white noise operator $U \in \mathcal{L}((E), (E)^*)$ satisfying

$$\begin{array}{ccc} (E) & \xrightarrow{U} & (E)^* \\ {\scriptstyle a(\zeta)}\downarrow & & \downarrow{\scriptstyle b(\zeta)} \\ (E) & \xrightarrow[U]{} & (E)^* \end{array} \qquad\qquad \begin{array}{ccc} (E) & \xrightarrow{U} & (E)^* \\ {\scriptstyle a^*(\zeta)}\downarrow & & \downarrow{\scriptstyle b^*(\zeta)} \\ (E) & \xrightarrow[U]{} & (E)^* \end{array}$$

Key observation

$$Ua(\zeta) = b(\zeta)U \quad \Longleftrightarrow \quad D^+_{S\zeta}U = [a(\zeta - S\zeta) - a^*(T\zeta)] \diamond U,$$

$$Ua^*(\zeta) = b^*(\zeta)U \quad \Longleftrightarrow \quad (D^-_{\zeta} - D^+_{T\zeta})U = [a^*(S\zeta - \zeta) + a(T\zeta)] \diamond U.$$

6.3. Solution to the Implementation Problem (1)

Theorem (Ji–O. JMP 51 (2010))

Assume that S *is invertible and that* $T^*S = S^*T$ ($\Longleftrightarrow$ $[b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0$). *Then a white noise operator* $U \in \mathcal{L}((E), (E)^*)$ *satisfies the intertwining property:*

$$Ua(\zeta) = b(\zeta)U, \qquad \zeta \in E,$$

if and only if U *is of the form*

$$U = \mathrm{wexp}\left\{-\frac{1}{2}\Delta_G^*(TS^{-1}) + \Lambda((S^{-1})^* - I)\right\} \diamond F, \tag{5}$$

where $F \in \mathcal{L}((E), (E)^*)$ *fulfills* $D_\zeta^+ F = 0$ *for all* $\zeta \in E$.

Remark: Note that

$$\mathrm{wexp}\left\{-\frac{1}{2}\Delta_G^*(TS^{-1})\right\} = e^{-\frac{1}{2}\Delta_G^*(TS^{-1})}, \quad \mathrm{wexp}\left\{\Lambda((S^{-1})^* - I)\right\} = \Gamma((S^{-1})^*),$$

where $\Gamma((S^{-1})^*)$ is the second quantization of $(S^{-1})^*$. Hence, (5) becomes

$$U = e^{-\frac{1}{2}\Delta_G^*(TS^{-1})} \diamond \Gamma((S^{-1})^*) \diamond F = e^{-\frac{1}{2}\Delta_G^*(TS^{-1})}\Gamma((S^{-1})^*)F.$$

6.3. Solution to the Implementation Problem (2)

Theorem (Ji–O. JMP 51 (2010))

Assume the following conditions:

(i) S *is invertible;*

(ii) $T^*S = S^*T \iff [b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0$;

(iii) $S^*S - T^*T = I \iff [b(\zeta), b^*(\eta)] = \langle \zeta, \eta \rangle$;

(iv) $ST^* = TS^*$.

Then a white noise operator $U \in \mathcal{L}((E), (E)^*)$ *satisfies the intertwining property:*

$$Ua^*(\zeta) = b^*(\zeta)U, \qquad \zeta \in E,$$

if and only if U *is of the form:*

$$U = \mathrm{wexp}\left\{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T)\right\} \diamond G,$$

where $G \in \mathcal{L}((E), (E)^*)$ *is an arbitrary white noise operator satisfying*

$$(D_\zeta^- - D_{T\zeta}^+)G = 0 \quad \textit{for all } \zeta \in E.$$

6.4. Solution to the Implementation Problem (3)

Theorem (Ji–O. JMP 51 (2010))

Assume the following conditions:

(i) S *is invertible;*

(ii) $T^*S = S^*T \iff [b(\zeta), b(\eta)] = [b^*(\zeta), b^*(\eta)] = 0$;

(iii) $S^*S - T^*T = I \iff [b(\zeta), b^*(\eta)] = \langle \zeta, \eta \rangle$;

(iv) $ST^* = TS^*$.

A white noise operator $U \in \mathcal{L}((E), (E)^*)$ *satisfies the following intertwining properties:*

$$Ua(\zeta) = b(\zeta)U, \qquad Ua^*(\zeta) = b^*(\zeta)U, \qquad \zeta \in E,$$

if and only if U *is of the form:*

$$\begin{aligned} U &= C \ \mathrm{wexp}\left\{ -\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1}) + \Lambda((S^{-1})^* - I) + \frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T) \right\} \\ &= C\, e^{-\frac{1}{2}\Delta_{\mathrm{G}}^*(TS^{-1})}\Gamma((S^{-1})^*)\, e^{\frac{1}{2}\Delta_{\mathrm{G}}(S^{-1}T)}, \end{aligned}$$

where $C \in \mathbb{C}$. *This is a (generalization of) Bogolubov transformation.*

References

- We introduced the concept of *quantum white noise derivatives*
- The work together with applications is now in progress.

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