Distance k-Graphs of Direct Product Graphs and their Asymptotic Spectral Distributions

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Plan

1. Asymptotic Spectral Analysis of Growing Graphs

2. Distance k-Graphs of Direct Product Graphs

3. Asymptotic Spectral Distribution of $G^{[N,k]}$

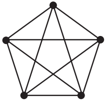
4. q-Deformation

1. Asymptotic Spectral Analysis of Growing Graphs

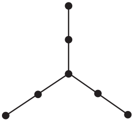
1.1. Graphs and Spectra

Definition (graph)

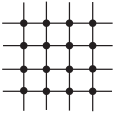
A *graph* is a pair $G = (V, E)$, where V is the set of *vertices* and E the set of *edges*. We write $x \sim y$ (adjacent) if they are connected by an edge.



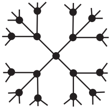
complete graph K_5



star graph



2-dim lattice



homogeneous tree T_4

- In this talk we consider only *finite graphs*.

1.1. Graphs and Spectra

$G = (V, E)$: a finite graph, i.e., $|V| < \infty$

Definition (adjacency matrix and spectrum)

The *adjacency matrix* of a graph $G = (V, E)$ is defined by

$$A = [A_{xy}]_{x,y \in V} \qquad A_{xy} = \begin{cases} 1, & x \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

The *spectrum* of G is defined by

$$\mathrm{Spec}\,(G) = \begin{pmatrix} \cdots & \lambda_i & \cdots \\ \cdots & m_i & \cdots \end{pmatrix} \qquad \begin{array}{l} \lambda_i : \text{distinct eigenvalues of } A \\ m_i : \text{multiplicities} \end{array}$$

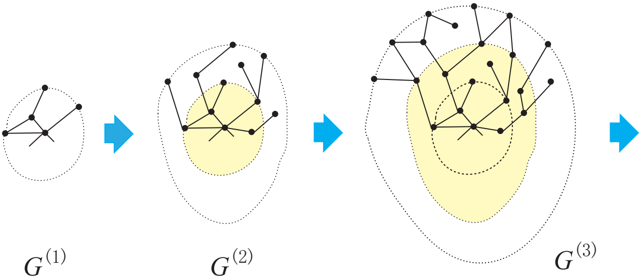
- $\mathrm{Spec}\,(G)$ is a fundamental invariant of finite graphs.

[1] N. Biggs: Algebraic Graph Theory, Cambridge UP, 1993.

[2] D. M. Cvetković, M. Doob and H. Sachs: Spectra of Graphs, Academic Press, 1979.

1.2. Growing Graphs

$G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$: a growing graph ($\nu$: growing parameter)



Problem (asymptotic spectral analysis)

Describe the *asymptotic behaviour* of $\mathbf{Spec}(G^{(\nu)})$ as $\nu \to \infty$.

- Potential application: Modeling a growing complex network in the real world
  $\Longrightarrow$ "spectral model" or "spectral reconstruction" of networks (in progress)

1.3. Formulation of Problem

1. *Adjacency algebra* $\mathcal{A}(G)$, that is, the $*$-algebra generated by A.
2. Equipped with a state $\varphi(\cdot) = \langle\cdot\rangle$, $\mathcal{A}(G)$ becomes an *algebraic probability space*.
3. The adjacency matrix A as a *real (algebraic) random variable* of $(\mathcal{A}, \varphi)$.

- In this talk we consider the normalized trace:

$$\langle a\rangle_{\mathrm{tr}} = \frac{1}{|V|}\,\mathrm{Tr}\,(a) = \frac{1}{|V|}\sum_{x\in V}\langle\delta_x\,, a\delta_x\rangle, \qquad a \in \mathcal{A}(G)$$

- The *spectral distribution* μ of A is determined (uniquely because G is finite) by

$$\langle A^m\rangle_{\mathrm{tr}} = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \dots.$$

- μ coincides with the *eigenvalue distribution* of A:

$$\mu = \frac{1}{|V|}\sum_i m_i\delta_{\lambda_i}\,, \qquad \mathrm{Spec}\,(G) = \begin{pmatrix} \cdots & \lambda_i & \cdots \\ \cdots & m_i & \cdots \end{pmatrix}$$

Main Problem

Let $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ be a growing graph and let $\varphi_\nu(\cdot) = \langle \cdot \rangle_\nu$ be a state on $\mathcal{A}(G^{(\nu)})$. Find a probability distribution μ on $\mathbb{R}$ satisfying

$$\left\langle \left(\frac{A^{(\nu)} - \langle A^{(\nu)} \rangle_\nu}{\Sigma_\nu(A^{(\nu)})} \right)^m \right\rangle_\nu \longrightarrow \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \dots,$$

where the left-hand side is normalized with

$$\langle A^{(\nu)} \rangle_\nu : \text{mean},$$
$$\Sigma_\nu^2(A^{(\nu)}) = \langle (A^{(\nu)} - \langle A^{(\nu)} \rangle_\nu)^2 \rangle_\nu : \text{variance}.$$

The above μ is called the *asymptotic spectral distribution* of $G^{(\nu)}$ in the states $\langle \cdot \rangle_\nu$.

1. If the limit of LHS exists, so does μ by Hamburger's theorem.
2. Uniqueness of μ does not hold in general due to the indeterminate moment problem.

1.4. Quantum Probabilistic Approach

1. Quantum decomposition of adjacency matrix
   $\Longrightarrow$ One-mode interacting Fock spaces
   $\Longrightarrow$ Orthogonal polynomials and classical analysis

   homogeneous trees, Hamming graphs, Johnson graphs, odd graphs, other distance regular graphs, Cayley graphs of S_∞, spidernets, ...

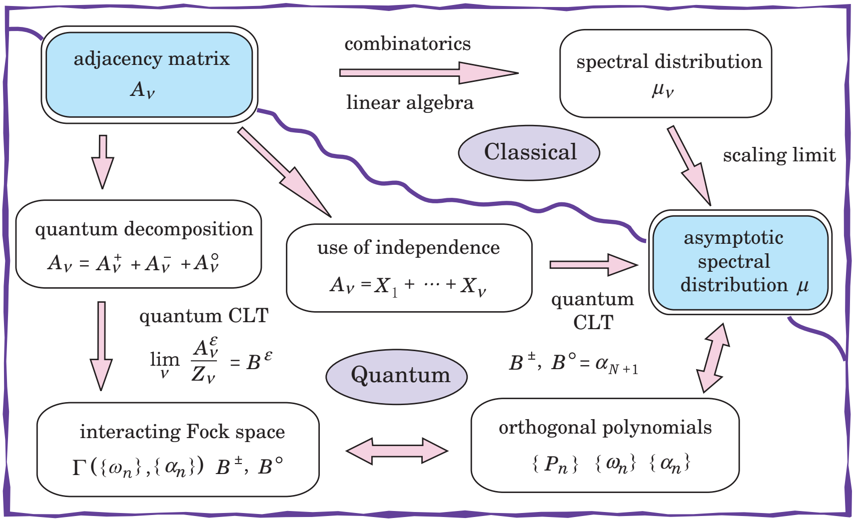
2. Graph product structure
   $\Longrightarrow$ Adjacency matrix as a sum of "independent" random variables
   $\Longrightarrow$ quantum central limit theorem

   direct product, free product, comb product (hierarchical product of Godsil–McKay, 1978), star product

[1] A. Hora and N. Obata: *Quantum Probability and Spectral Analysis of Graphs*, Springer, 2007.

[2] Refinements, generalizations, applications during the last few years: digraphs, weighted graphs, ..., random walks, quantum walks, ...

1.4. Quantum Probabilistic Approach (illustration)



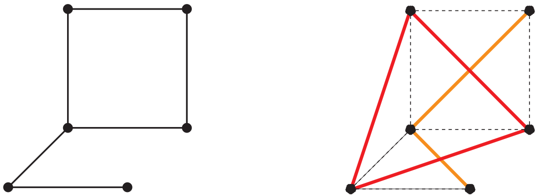
2. Distance k-Graphs of Direct Product Graphs

Definition (Distance k-graph)

Let $G = (V, E)$ be a graph. For $k \geq 1$ the *distance k-graph* of G is a graph

$$G^{[k]} = (V, E^{[k]}), \quad E^{[k]} = \{\{x, y\}\,;\, x, y \in V,\ \partial_G(x, y) = k\},$$

where $\partial_G(x, y)$ is the graph distance.

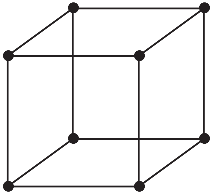


- The adjacency matrix of $G^{[k]}$ coincides with the k-th distance matrix of G defined by

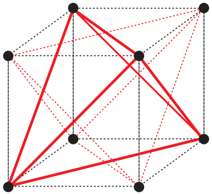
$$D_k = [(D_k)_{xy}]_{x,y\in V} \qquad (D_k)_{xy} = \begin{cases} 1, & \partial_G(x, y) = k, \\ 0, & \text{otherwise.} \end{cases}$$

2.2. Distance k-Graphs of Direct Product Graphs

$G = (V, E)$: a finite graph with $|V| \geq 2$
$G^N = G \times \cdots \times G$: N-fold direct power $(N \geq 1)$
$G^{[N,k]}$: the distance k-graph of G^N $(1 \leq k \leq N)$
$A^{[N,k]}$: the adjacency matrix of $G^{[N,k]}$



$K_2 \times K_2 \times K_2$



$(K_2 \times K_2 \times K_2)^{[2]}$

Our Question

For fixed $k \geq 1$ find the asymptotic spectral distribution of $A^{[N,k]}$ as $N \to \infty$ in the normalized trace.

2.3 $A^{[N,k]}$ for $k = 1$

$G = (V, E)$: a finite graph with $|V| \geq 2$, D_l: the l-th distance matrix of G
$G^N = G \times \cdots \times G$: N-fold direct power ($N \geq 1$)
$G^{[N,k]}$: the distance k-graph of G^N ($1 \leq k \leq N$)
$A^{[N,k]}$: the adjacency matrix of $G^{[N,k]}$

For $k = 1$ we have $G^{[N,1]} = G^N$ and

$$A^{[N,1]} = \sum_{i=1}^{N} 1 \otimes \cdots \otimes D_1 \otimes \cdots \otimes 1 \qquad (D_1 \text{ at } i\text{-th position}),$$

where D_1 is the adjacency matrix (1st distance matrix) of G

Theorem (Commutative (classical) central limit theorem)

$$\frac{A^{[N,1]}}{N^{1/2}} \xrightarrow{m} \left(\frac{2|E|}{|V|}\right)^{1/2} g \qquad \textit{(convergence in moments)},$$

where g is a real algebraic random variable obeying the normal law $N(0,1)$.

We only need to note that

$$\varphi_{\mathrm{tr}}(D_1) = 0, \qquad \varphi_{\mathrm{tr}}(D_1^2) = \frac{2|E|}{|V|} = (\text{mean degree of } G)$$

$G^N = K_2 \times \cdots \times K_2$: N-dimensional hypercube
$G^{[N,k]}$: the distance k-graph of G^N $(1 \le k \le N)$

Theorem (Kurihara–Hibino (IDAQP, 2011))

For $k = 2$ we have

$$\lim_{N\to\infty} \varphi_{\mathrm{tr}}\left(\left(\frac{A^{[N,2]}}{\binom{N}{2}^{1/2}}\right)^m\right) = \int_{-\frac{1}{\sqrt{2}}}^{\infty} x^m \frac{e^{-(\sqrt{2}x+1)/2}}{\sqrt{\pi(\sqrt{2}x+1)}}\,dx, \quad m = 1, 2, \dots,$$

where the probability distribution in the right-hand side is the normalized χ_1^2-distribution. In our notation we have

$$\frac{A^{[N,2]}}{N} \xrightarrow{m} \frac{1}{2}\tilde{H}_2(g).$$

- Proof is by *quantum decomposition*. (The case of $k \ge 3$ is not covered.)
- The distribution of $\dfrac{\tilde{H}_2(g)}{\sqrt{2}} = \dfrac{g^2-1}{\sqrt{2}}$ is the normalized χ_1^2-distribution.

2.4. Special Case: Distance k-Graphs of Hypercubes (cont)

Theorem (O. (BCP, 2012))

For a general $k \ge 1$ we have

$$\lim_{N\to\infty} \varphi_{\mathrm{tr}}\Big(\Big(\frac{A^{[N,k]}}{N^{k/2}}\Big)^m\Big) = \int_{-\infty}^{+\infty} \Big\{\frac{1}{k!}\,\tilde{H}_k(x)\Big\}^m \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx, \quad m = 1, 2, \dots.$$

where $\tilde{H}_k(x)$ are the "normalized" Hermite polynomials (OP wrt $N(0,1)$):

$$\tilde{H}_0(x) = 1, \quad \tilde{H}_1(x) = x, \quad x\tilde{H}_k(x) = \tilde{H}_{k+1}(x) + k\tilde{H}_{k-1}(x).$$

In our notation we have

$$\frac{A^{[N,k]}}{N^{k/2}} \xrightarrow{m} \frac{1}{k!}\,\tilde{H}_k(g).$$

Theorem (O. (BCP, 2012))

For a general $k \geq 1$ we have

$$\lim_{N\to\infty} \varphi_{\mathrm{tr}}\left(\left(\frac{A^{[N,k]}}{N^{k/2}}\right)^m\right) = \int_{-\infty}^{+\infty} \left\{\frac{1}{k!}\,\tilde{H}_k(x)\right\}^m \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx, \quad m = 1, 2, \dots.$$

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In our notation we have

$$\frac{A^{[N,k]}}{N^{k/2}} \xrightarrow{m} \frac{1}{k!}\,\tilde{H}_k(g).$$

1. $A^{[1]}A^{[k]} = (k+1)A^{[k+1]} + (N-k+1)A^{[k-1]}$
2. $A^{[N,k]} = K_k^{(N)}(A^{[N,1]})/k!$, where $K_k^{(N)}(x)$ are *Krautchouk polynomials* modified so as to be OP wrt $\beta_N = \sum_j \binom{N}{j}\frac{1}{2^j}\delta_{-N+2j}$
3. $\beta_N \sim N(0,N)$ and $N^{-k/2}K_k^{(N)}(\sqrt{N}\,x) \sim \tilde{H}_k(x)$

3. Asymptotic Spectral Distribution of $G^{[N,k]}$ as $N \to \infty$

Joint work with

Yuji Hibino (Saga University, Japan) and
Hun Hee Lee (Chungbuk National University, Korea)

3.1. Main Result

$G = (V, E)$: a finite graph with $|V| \geq 2$
$G^N = G \times \cdots \times G$: N-fold direct power ($N \geq 1$)
$G^{[N,k]}$: the distance k-graph of G^N ($1 \leq k \leq N$)
$A^{[N,k]}$: the adjacency matrix of $G^{[N,k]}$
regarded as a real random variable of $(\mathcal{A}(G^{[N,k]}), \varphi_{\mathrm{tr}})$

Theorem (Hibino-Lee-O. (2012))

For any $k \geq 1$ we have

$$\frac{A^{[N,k]}}{N^{k/2}} \xrightarrow{m} \left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!} \tilde{H}_k(g),$$

where g is a real algebraic random variable $\sim N(0, 1)$. The limit distribution does not depend on the detailed structure of G!

$\tilde{H}_k(x)$: the "normalized" Hermite polynomials (OP wrt $N(0, 1)$) verifying

$$\tilde{H}_0(x) = 1, \quad \tilde{H}_1(x) = x, \quad x\tilde{H}_n(x) = \tilde{H}_{n+1}(x) + n\tilde{H}_{n-1}(x)$$

▶ For $k \geq 3$ the uniqueness of the limit distribution is not known. Probably does not hold, cf. [Berg (Ann. Prob. 1988)].

3.2. Idea of the Proof

$G = (V, E)$: a finite graph with $|V| \geq 2$ with the l-th distance matrix D_l
$G^N = G \times \cdots \times G$: N-fold direct power $(N \geq 1)$
$G^{[N,k]}$: the distance k-graph of G^N $(1 \leq k \leq N)$
$A^{[N,k]}$: the adjacency matrix of $G^{[N,k]}$

Lemma

$$A^{[N,k]} = B^{[N,k]} + C(N,k),$$

where

$$B^{[N,k]} = \sum 1 \otimes \cdots \otimes D_1 \otimes \cdots \otimes D_1 \otimes \cdots \otimes 1 \quad (D_1 \text{ appears } k \text{ times})$$

$$C(N,k) = \sum_{\lambda \in \Lambda(k) \setminus \{\lambda_0\}} C(\lambda), \qquad \lambda_0 = (k, 0, 0, \dots),$$

$$C(\lambda) = \sum 1 \otimes \cdots \otimes D_1 \otimes \cdots \otimes D_l \otimes \cdots \otimes 1,$$

D_l *appears* j_l *times according to* $\lambda = (j_1, j_2, \dots) \in \Lambda(k)$

Strategy:

$$\frac{B^{[N,k]}}{N^{k/2}} \xrightarrow{m} \left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!} \tilde{H}_k(g), \qquad \frac{C(N,k)}{N^{k/2}} \xrightarrow{m} 0 \qquad \Longrightarrow \text{Result}$$

3.3. Algebraic Convergence Lemma (A Technical Tool)

Definition (Convergence in moments)

For $a_n = a_n^*$ in $(\mathcal{A}_n, \varphi_n)$ and $a = a^*$ in $(\mathcal{A}, \varphi)$ we say that

$$a_n \xrightarrow{m} a \quad \iff \quad \lim_{n\to\infty} \varphi_n(a_n^m) = \varphi(a^m), \qquad m = 1, 2, \dots .$$

- For any polynomial $p(x)$ we have

$$a_n \xrightarrow{m} a \quad \Longrightarrow \quad p(a_n) \xrightarrow{m} p(a).$$

- However, it does not hold in general that

$$a_n \xrightarrow{m} a, \quad b_n \xrightarrow{m} b \quad \Longrightarrow \quad p(a_n, b_n) \xrightarrow{m} p(a, b)$$

for a non-commutative polynomial $p(x, y)$.

3.3. Algebraic Convergence Lemma (A Technical Tool)

Lemma (Algebraic convergence lemma)

Let $a_n = a_n^*, z_{1n} = z_{1n}^*, \dots, z_{kn} = z_{kn}^*$ *be real random variables in* $(\mathcal{A}_n, \varphi_n)$*,* $n = 1, 2, \dots$ *. Assume the following conditions are satisfied:*

(i) *There exist a real random variable* $a = a^* \in \mathcal{A}$ *and* $\zeta_1, \dots, \zeta_k \in \mathbb{R}$ *such that*

$$a_n \xrightarrow{m} a, \qquad z_{in} \xrightarrow{m} \zeta_i 1, \quad i = 1, 2, \dots, k;$$

(ii) $\{a_n, z_{1n}, \dots, z_{kn}\}$ *have uniformly bounded mixed moments in the sense that*

$$C_m = \sup_n \max \left\{ |\varphi(a_n^{\alpha_1} z_{1n}^{\beta_1} \cdots z_{kn}^{\gamma_1} \cdots a_n^{\alpha_i} z_{1n}^{\beta_i} \cdots z_{kn}^{\gamma_i} \cdots)| \,; \begin{array}{l} \alpha_i, \beta_i, \gamma_i \ge 0 \text{ are integers} \\ \sum_i (\alpha_i + \beta_i + \cdots + \gamma_i) = m \end{array} \right\} < \infty;$$

(iii) φ_n *is a tracial state for* $n = 1, 2, \dots$ *.*

Then, for any non-commutative polynomial $p(x, y_1, \dots, y_k)$ *we have*

$$p(a_n, z_{1n}, \dots, z_{kn}) \xrightarrow{m} p(a, \zeta_1 1, \dots, \zeta_k 1).$$

3.4. Convergence (I): $\frac{B^{[N,k]}}{N^{k/2}} \xrightarrow{m} \left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!}\, \tilde{H}_k(g)$

$$\tilde{B}^{[N,k]} = \sum 1 \otimes \cdots \otimes \tilde{D}_1 \otimes \cdots \otimes \tilde{D}_1 \otimes \cdots \otimes 1, \quad \tilde{D}_1 = \left(\frac{2|E|}{|V|}\right)^{-1/2} D_1$$

3.4. Convergence (I): $\frac{B^{[N,k]}}{N^{k/2}} \xrightarrow{m} \left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!} \tilde{H}_k(g)$

$$\tilde{B}^{[N,k]} = \sum 1 \otimes \cdots \otimes \tilde{D}_1 \otimes \cdots \otimes \tilde{D}_1 \otimes \cdots \otimes 1, \quad \tilde{D}_1 = \left(\frac{2|E|}{|V|}\right)^{-1/2} D_1$$

---

1. $\dfrac{\tilde{B}^{[1]}}{N^{1/2}} \xrightarrow{m} g = \tilde{H}_1(g)$ (commutative CLT)
2. $(k+1)\tilde{B}^{[k+1]} = \tilde{B}^{[1]}\tilde{B}^{[k]} - (N-k+1)\tilde{B}^{[k-1]} - F_{N,k}$
3. For each $k = 1, 2, \dots$ there exists a polynomial $p_k(x, y_1, \dots, y_{k-1})$ such that

$$\frac{\tilde{B}^{[k]}}{N^{k/2}} = p_k\left(\frac{\tilde{B}^{[1]}}{N^{1/2}}, \frac{F_1(z)}{N}, \frac{F_2(z)}{N^{3/2}}, \dots, \frac{F_{k-1}(z)}{N^{k/2}}\right) + Z_{k,N},$$

   Moreover, where the arguments are mutually commutative and

$$\frac{F_i(z)}{N^{(k+1)/2}} \xrightarrow{m} 0, \quad i = 1, 2, \dots, k-1, \qquad Z_{k,N} \xrightarrow{m} 0$$

4. By induction

$$(k+1)!\,\frac{B^{[k+1]}}{N^{(k+1)/2}} \xrightarrow{m} g\tilde{H}_k(g) - k\tilde{H}_{k-1}(g) = \tilde{H}_{k+1}(g)$$

3.5. Convergence (II): $\frac{A^{[N,k]}}{N^{k/2}} \xrightarrow{m} \left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!}\, \tilde{H}_k(g)$

$$\frac{A^{[N,k]}}{N^{k/2}} = \frac{B^{[N,k]}}{N^{k/2}} + \frac{C(N,k)}{N^{k/2}}$$

$$\frac{A^{[N,k]}}{N^{k/2}} = \frac{B^{[N,k]}}{N^{k/2}} + \frac{C(N,k)}{N^{k/2}}$$

---

1. We have shown that

$$\frac{B^{[N,k]}}{N^{k/2}} \xrightarrow{m} \left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!}\, \tilde{H}_k(g)$$

2. It is sufficient to show that

$$\frac{C(N,k)}{N^{k/2}} \xrightarrow{m} 0$$

3. $C(N,k)$ is a sum of

$$C(\lambda) = \sum 1 \otimes \cdots \otimes D_1 \otimes \cdots \otimes D_l \otimes \cdots \otimes 1, \quad \lambda \in \Lambda(k), \quad \lambda \neq (k,0,0,,\dots),$$

where D_l appears j_l times according to $\lambda = (j_1, j_2, \dots) \in \Lambda(k)$

4. Counting the number of terms of $C(\lambda)$ and noting uniformly bounded mixed moments, we can show

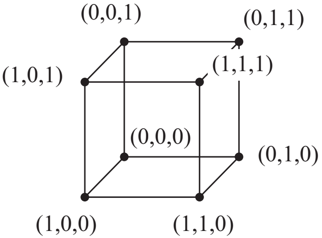
$$\frac{C(\lambda)}{N^{k/2}} \xrightarrow{m} 0, \quad \lambda \in \Lambda(k), \quad \lambda \neq (k,0,0,,\dots)$$

4. q-Deformation (in Case of Hypercubes)

Joint work with

Hun Hee Lee (Chungbuk National University, Korea)

4.1. N-Dimensional Hypercube



$G = K_2 = (V, E)$, $\quad V = \{0, 1\}$,
$D = D_1$: adjacency matrix
$G^N = G \times \cdots \times G$: N-dimensional hypercube

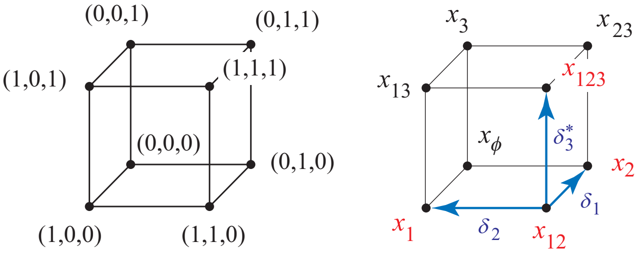
The adjacency matrix of $G^{[N,k]}$ is given by

$$A^{[N,k]} = \sum 1 \otimes \cdots \otimes D \otimes \cdots \otimes D \otimes \cdots \otimes 1 \quad (D \text{ appears } k\text{-times})$$

(This exact identity is valid only for $G = K_2$)

Our Result (revisited)

$$\frac{A^{[N,1]}}{N^{1/2}} \xrightarrow{m} g, \qquad \frac{A^{[N,k]}}{N^{k/2}} \xrightarrow{m} \frac{1}{k!}\,\tilde{H}_k(g)$$



- Adjacency matrix of G^N is given by

$$A^{[N,1]} = \sum_{i=1}^{N} (\delta_i + \delta_i^*) = \sum_{i=1}^{N} D[i],$$

$$\delta_i x_J = \begin{cases} x_{J \setminus \{i\}}, & i \in J; \\ 0, & \text{otherwise}, \end{cases} \qquad \delta_i^* x_J = \begin{cases} x_{J \cup \{i\}}, & i \notin J; \\ 0, & \text{otherwise} \end{cases}$$

- $D[i]D[j] - \epsilon(i,j)D[j]D[i] = 2\delta_{ij}$ with $\epsilon(i,j) = \begin{cases} 1, & i \neq j, \\ -1, & i = j. \end{cases}$

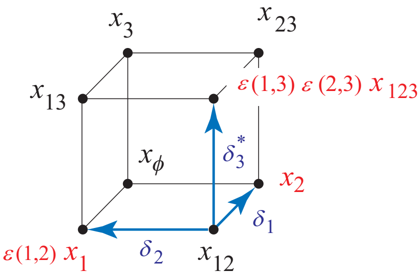
$\{\epsilon(i,j) = \epsilon(j,i) \in \{\pm 1\}\,;\ 1 \le i < j\}$: iid random variables; $\quad \epsilon(i,i) = 1$

$$\delta_i x_J = \begin{cases} \prod \epsilon(\cdot) x_{J\setminus\{i\}}, & i \in J; \\ 0, & \text{otherwise}, \end{cases} \qquad \delta_i^* x_J = \begin{cases} \prod \epsilon(\cdot) x_{J\cup\{i\}}, & i \notin J; \\ 0, & \text{otherwise}. \end{cases}$$

$D[i] = \delta_i + \delta_i^*$

$$A^{[N,1]} = \sum_{i=1}^{N} D[i]$$

adjacency matrix of a "weighted graph"



Theorem (q-CLT (Speicher 1992, Biane 1997))

$$\frac{A^{[N,1]}}{N^{1/2}} \xrightarrow{m} g_q \quad (q\text{-Gaussian}) \quad \textit{for almost surely in } \epsilon.$$

4.4. Main Result

A q-analogue of the adjacency matrix of the distance k-graph of the hypercube $G^N = K_2 \times \cdots \times K_2$ with weights

$$A^{[N,k]} = \frac{1}{k!} \sum_{\substack{i_1, \cdots, i_k \\ \neq}} D[i_1] \cdots D[i_k]$$

Theorem (Lee-O. (2012))

$$\frac{A^{[N,k]}}{N^{k/2}} \xrightarrow{m} \frac{1}{k!} \tilde{H}_k^q(g_q) \quad \textit{for almost surely in } \epsilon.$$

4.4. Main Result

A q-analogue of the adjacency matrix of the distance k-graph of the hypercube $G^N = K_2 \times \cdots \times K_2$ with weights

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Theorem (Lee-O. (2012))

$$\frac{A^{[N,k]}}{N^{k/2}} \xrightarrow{m} \frac{1}{k!} \tilde{H}_k^q(g_q) \quad \textit{for almost surely in } \epsilon.$$

1. For $X^{[N,k]} = \frac{k!}{N^{k/2}} A^{[N,k+1]}$ after taking $\mathbb{E}$ we have
$$X^{[N,k+1]} = X^{[N,1]} X^{[N,k]} - [k]_q X^{[N,k]} + Y_{N,k-1}, \quad k \geq 1.$$
2. Here $[k]_q$ is obtained from expectation of "correlated" random walk
$$W_i = \epsilon_{i,i_1} \cdots \epsilon_{i,i_k} + \epsilon_{i,i_2} \cdots \epsilon_{i,i_k} + \cdots + 1$$
3. After convergence estimation by induction on k we have the reccurence relation same as the q-Hermite polynomials:
$$x\tilde{H}_k^q(x) = \tilde{H}_{k+1}^q(x) + [k]_q \tilde{H}_{k-1}^q(x), \quad k \geq 1.$$

Summary

1. For $G^{[N,k]}$ (distance k-graph of the direct product G^N) we obtained:

$$\frac{A^{[N,k]}}{N^{k/2}} \xrightarrow{m} \frac{1}{k!}\,\tilde{H}_k(g)$$

2. Purely algebraic (combinatorial) proof based on the convergence lemma:

$$a_n \xrightarrow{m} a, z_{in} \xrightarrow{m} \zeta_i 1 \implies p(a_n, z_{1n}, \dots, z_{kn}) \xrightarrow{m} p(a, \zeta_1 1, \dots, \zeta_k 1).$$

3. A q-deformation for hypercubes (K_2^N) is introduced by means of "weights on edges" and the asymptosic spectral distribution is obtained.

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Some questions

1. More explicit expression of $A^{[N,k]}$ in terms of q-Krautchouk polynomials?
2. q-deformation for $G^{[N,k]}$: try first $G = K_r$? then a general graph G?
3. Direct product $\Longrightarrow$ Another product structures
4. Direct product of K_r (Hamming graph) $\Longrightarrow$ More genaral distance-regular graphs
5. Further generalization to association schemes (relation to Terwilliger algebras), ...