Random walks, Quantum Walks, and Free Meixner Laws

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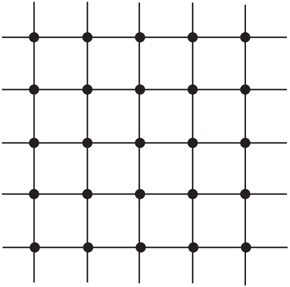
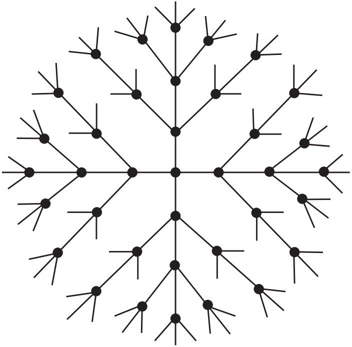
Lattices vs Trees





additive group $\mathbf{Z}^n$	free group F_n
commutative independence	free independence
many cycles	no cycles
binomial coefficients	Catalan numbers
Normal distribution	Wigner semi-circle law

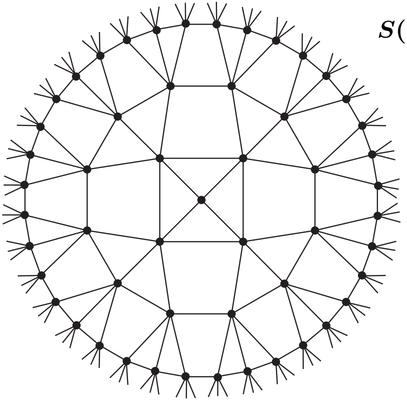
Spidernet = Tree + Large cycles

$S(a,b,c)$

$a = \deg(o)$

$b = \deg(x)$ for $x \neq o$

$c = \omega_+(x)$ for $x \neq o$



$S(4,6,3)$

Spectral analysis [Igarashi–O. Banach Center Publ. **73** (2006)]

<u>Method</u>: Stratification $\Longrightarrow$ Quantum decomposition of the adjacency matrix

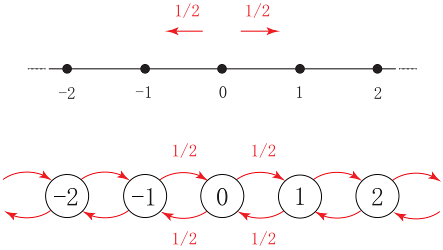
$\Longrightarrow$ Orthogonal polynomials $\Longrightarrow$ Free Meixner law

Plan

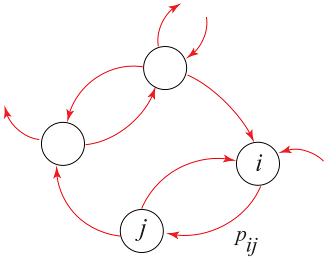
[0]

1. Random Walks
2. One-Mode Interacting Fock Spaces
3. Free Meixner Laws
4. Quantum Walks

1. Random Walks



Random Walk on 1-dim Integer Lattice



Transition Diagram

Markov chain

Digraph + Weights on edges = Transition Matrix $\boldsymbol{P} = [\boldsymbol{p_{ij}}]$

1.1. Recurrence of Markov Chains

$\{X_n\}$: a (time homogeneous) Markov chain on a state space $S = \{i, j, \dots\}$
$p(i,j) = P(X_1 = j | X_0 = i)$: One-step transition probability
$P = [p(i,j)]$: transition matrix
Then the n-step transition probability is given by

$$p_n(i,j) = P(X_n = j | X_0 = i) = P^n(i,j)$$

Definition

A state $i \in S$ is called *recurrent* if

$$P(T_i < \infty | X_0 = i) = 1,$$

where $T_i = \inf\{n \geq 1\,;\, X_n = i\}$ is the first hitting time of i.

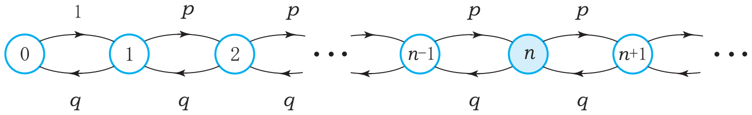
Theorem (Standard)

A state $i \in S$ is recurrent if and only if

$$\sum_{n=1}^{\infty} P^n(i,i) = +\infty.$$

1.2. Random Walks on a Half Line

$\{X_n\}$: random walk on $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$



$P(\text{right move}) = p$, $\quad q = 1 - p = P(\text{left move})$, $\quad 0 < p < 1$.
0 is a reflection barrier

Transition matrix:

$$P = \begin{bmatrix} 0 & 1 & & & & \\ q & 0 & p & & & \\ & \ddots & \ddots & \ddots & & \\ & & q & 0 & p & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}, \qquad \Phi_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

We have

$$p_n(i, j) = P^n(i, j) = \langle \Phi_i, P^n \Phi_j \rangle$$

1.3. Integral Representation of Transition Probability

$\{X_n\}$: random walk on $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$
$P(\text{right move}) = p$, $\quad q = 1 - p = P(\text{left move})$, $\quad 0 < p < 1$.
0 is a reflection barrier

Theorem (Karlin-McGregor formula)

$$p_n(i,j) = P(X_n = j | X_0 = i) = \frac{1}{\pi(i)} \int_{-\infty}^{+\infty} x^n Q_i(x) Q_j(x) \mu(dx),$$

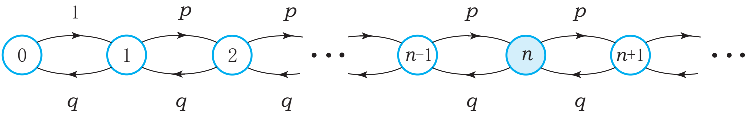
where

1. μ *is the* *Kesten distribution* *with parameters* q, pq
2. $\{Q_n\}$ *be the polynomials associated with* P^t *(transposed* P*).*
3. $\pi(0) = 1$, $\pi(n) = \dfrac{p^{n-1}}{q^n}$ *for* $n = 1, 2, \dots$

In particular,

$$p_n(0,0) = \int_{-\infty}^{+\infty} x^n \mu(dx), \qquad n = 0, 1, 2, \dots .$$

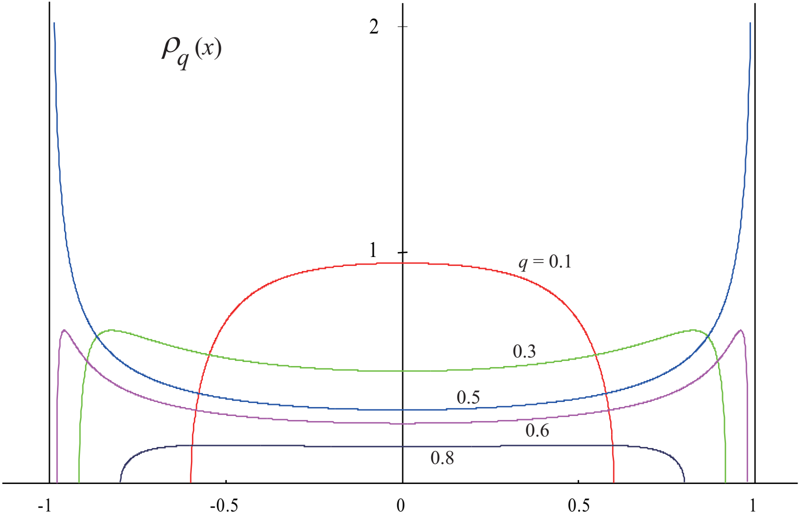
1.4. Kesten Distribution



$\Longrightarrow$ by standard method (one-mode interacting Fock spaces, Stieltjies transform, etc..)
$\Longrightarrow$ we obtain the explicit expression of the Kesten distribution with parameters q, pq:

$$\mu_q(dx) = \begin{cases} \rho_q(x)dx, & \text{if } 0 < q \le 1/2, \\ \rho_q(x)dx + \dfrac{2q-1}{2q}(\delta_{-1} + \delta_{+1}), & \text{if } 1/2 < q < 1, \end{cases}$$

$$\rho_q(x) = \begin{cases} \dfrac{\sqrt{4q(1-q) - x^2}}{2\pi q(1-x^2)}, & \text{if } |x| < 2\sqrt{q(1-q)}, \\ 0, & \text{otherwise.} \end{cases}$$



$\rho_q(x)$
2
1
$q = 0.1$
0.3
0.5
0.6
0.8
-1
-0.5
0
0.5
1

1.5. Recurrence

For recurrence we need to check

$$\sum_{n=0}^{\infty} p_n(0,0) = \sum_{n=0}^{\infty} p_{2n}(0,0) = \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} x^{2n} \mu_q(dx)$$

(Case I) $0 < q < 1/2$. Since $\mathrm{supp}\,\rho_q \subset (-1,1)$,

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{2n}(0,0) &= \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} x^{2n} \mu_q(dx) \\
&= \sum_{n=0}^{\infty} \int_{-2\sqrt{q(1-q)}}^{2\sqrt{q(1-q)}} x^{2n} \frac{\sqrt{4q(1-q)-x^2}}{2\pi q(1-x^2)}\, dx \\
&= \int_{-2\sqrt{q(1-q)}}^{2\sqrt{q(1-q)}} \frac{\sqrt{4q(1-q)-x^2}}{2\pi q(1-x^2)^2}\, dx < \infty.
\end{aligned}$$

Therefore, the origin $\mathbf{0}$ is not recurrent.

(Case II) $1/2 < q < 1$. Using the delta measure contained in $\mu_q(dx)$,

$$\sum_{n=0}^{\infty} p_{2n}(0,0) = \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} x^{2n} \mu_q(dx) \geq \sum_{n=0}^{\infty} \frac{2q-1}{q} = +\infty.$$

Therefore, 0 is recurrent.

(Case III) $q = 1/2$. We perform an explicit calculation. Note that

$$\rho_{1/2}(x) = \frac{1}{\pi\sqrt{1-x^2}}, \quad |x| < 1,$$

is reduced to the arcsine law. By means of the moment sequence of the arcsine law (known) we have

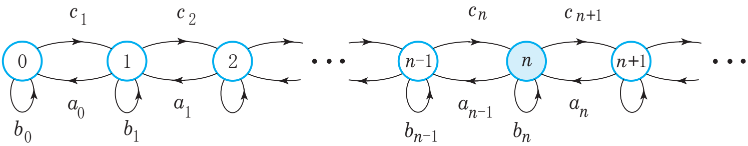
$$\sum_{n=0}^{\infty} p_{2n}(0,0) = \sum_{n=0}^{\infty} \frac{1}{\pi} \int_{-1}^{+1} \frac{x^{2n}}{\sqrt{1-x^2}}\, dx = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{4^n}$$
$$= \lim_{z \to 1/4-0} \frac{1}{\sqrt{1-4z}} = +\infty.$$

Therefore, the origin 0 is recurrent.

2. One-Mode Interacting Fock Spaces
A Basic Concept of Quantum Probability

A. Hora and N.O.: Quantum Probability and Spectral Analysis of Graphs, Springer, 2007.

2.1. One-Mode Interacting Fock Spaces (in a slightly generalized form)



$\mathfrak{T}$: The class of finite or infinite tridiagonal matrices of the form:

$$\boldsymbol{T} = \begin{bmatrix} b_0 & a_0 & & & & \\ c_1 & b_1 & a_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & c_n & b_n & a_n & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}, \qquad \begin{array}{l} a_n, b_n, c_n \in \mathbb{R}, \\ a_n \neq 0, \ \ c_n \neq 0. \end{array}$$

The associated interacting Fock space: $\Gamma_T = (\Gamma, \{\Phi_n\}, B^+, B^-, B^\circ)$

$$B^+\Phi_n = c_{n+1}\Phi_{n+1}\,, \quad B^-\Phi_n = a_{n-1}\Phi_{n-1}\,, \quad B^\circ\Phi_n = b_n\Phi_n$$

$$T = B^+ + B^- + B^\circ \quad \text{(quantum decomposition)}$$

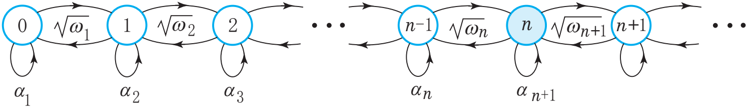
Remark: Jacobi Matrix

A tridiagonal matrix $\boldsymbol{T} \in \mathfrak{T}$ is called a Jacobi matrix if it is real, symmetric with positive off-diagonal entries, i.e., it is of the form:

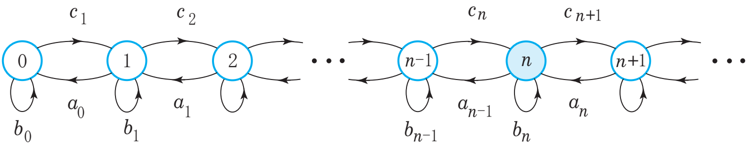
$$\boldsymbol{T} = \begin{bmatrix} \alpha_1 & \sqrt{\omega_1} & & & & \\ \sqrt{\omega_1} & \alpha_2 & \sqrt{\omega_2} & & & \\ & \sqrt{\omega_2} & \alpha_3 & \sqrt{\omega_3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \sqrt{\omega_{n-1}} & \alpha_n & \sqrt{\omega_n} \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}, \quad \omega_n > 0, \quad \alpha_n \in \mathbb{R}.$$

In this case the pair of two sequences $(\{\omega_n\}, \{\alpha_n\})$ is called Jacobi parameters.

▶ $\Gamma_T = (\Gamma, \{\Phi_n\}, B^+, B^-, B^\circ)$ associated with Jacobi parameters $(\{\omega_n\}, \{\alpha_n\})$ is an interacting Fock space in an original sense (e.g., Accardi–Bożejko (1998)).



2.2. Spectral Properties



- positivity condition:

$$\omega_n \equiv a_{n-1}c_n > 0, \qquad \alpha_n \equiv b_{n-1} \in \mathbb{R}, \qquad \text{for } n = 1, 2, \dots$$

$\Longrightarrow$ $\exists$ a probability distribution μ on $\mathbb{R}$ with Jacobi parameter $(\{\omega_n\}, \{\alpha_n\})$, i.e., the orthogonal polynomials $\{P_n\}$ associated with μ satisfy

$$xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x)$$
$$P_0(x) = 1, \quad P_{-1}(x) = 0.$$

Theorem

μ is the distribution of T in the vacuum state, i.e.,

$$(T^m)_{00} = \langle \Phi_0, T^m \Phi_0 \rangle = \langle \Phi_0, (B^+ + B^- + B^\circ)^m \Phi_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx)$$

2.3. How to Obtain μ from $(\{\omega_n\}, \{\alpha_n\})$

$(\{\omega_n\}, \{\alpha_n\}) \quad \Longrightarrow \quad \mu \quad$ up to the determinate moment problem

Stieltjes transform

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \frac{\omega_3}{z-\alpha_4} - \cdots,$$

where the right-hand side is convergent in $\{\mathrm{Im}\, z \neq 0\}$ if the moment problem is determinate, e.g., if $\omega_n = O((n \log n)^2)$ (Carleman's test).

Stieltjes inversion formula

$$\begin{aligned} &\frac{1}{2}\{F(t) + F(t-0)\} - \frac{1}{2}\{F(s) + F(s-0)\} \\ &\qquad = -\frac{1}{\pi} \lim_{y \to +0} \int_s^t \mathrm{Im}\, G_\mu(x+iy)dx, \quad s < t, \\ &\rho(x) = -\frac{1}{\pi} \lim_{y \to +0} \mathrm{Im}\, G_\mu(x+iy) \end{aligned}$$

where $F(x) = \mu((-\infty, x])$ is the (right-continuous) distribution function of μ and $\rho(x)$ is the absolutely continuous part.

Let $\Gamma_T = (\Gamma, \{\Phi_n\}, B^+, B^-, B^\circ)$, $T \in \mathfrak{T}$ with positivity condition

$$(T^m)_{00} = \langle \Phi_0, T^m \Phi_0 \rangle = \langle \Phi_0, (B^+ + B^- + B^\circ)^m \Phi_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx)$$

where μ is the distribution of T in the vacuum state.

Theorem (essentially due to Karlin-McGregor (1959))

Let $T \in \mathfrak{T}$ satisfy the positivity condition and μ a probability distribution on $\mathbb{R}$ with Jacobi parameter $(\{\omega_n\}, \{\alpha_n\})$. Let $\Gamma_T = (\mathcal{H}, \{\Phi_n\}, B^+, B^-, B^\circ)$ the associated interacting Fock space and $\{Q_n(x)\}$ the associated polynomials. It then holds that

$$(T^m)_{ij} = \langle \Phi_i, T^m \Phi_j \rangle = \frac{a_0 a_1 \dots a_{j-1}}{c_1 c_2 \dots c_j} \int_{-\infty}^{+\infty} x^m Q_i(x) Q_j(x) \mu(dx).$$

- Note: Here T is not necessarily a Jacobi matrix, but the proof of Karlin–McGregor (1959) is valid to our case.
- Application: Computing the transition probability $P(X_n = i | X_0 = j)$ for a nearest-neighbor random walk on $\{0, 1, 2, \dots\}$.

where the polynomials associated with a tridiagonal matrix are defined as follows:

$$T = \begin{bmatrix} b_0 & a_0 & & & & \\ c_1 & b_1 & a_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & c_n & b_n & a_n & \\ & & & \ddots & \ddots & \ddots \end{bmatrix} \in \mathfrak{T}$$

we define a sequence of polynomials $Q_0(x), Q_1(x), Q_2(x), \dots$ by

$$\begin{cases} xQ_n(x) = a_n Q_{n+1}(x) + b_n Q_n(x) + c_n Q_{n-1}(x) \\ Q_0(x) = 1, \\ Q_{-1}(x) = 0 \end{cases}$$

or equivalently, $$x \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \end{bmatrix} = T \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \\ \vdots \end{bmatrix}$$

3. Free Meixner Laws

3.1. Free Meixner Laws

The *free Meixner law* with parameter $p > 0$, $q \geq 0$, $a \in \mathbb{R}$ is a probability distribution $\mu = \mu_{p,q,a}$ specified uniquely by

$$\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z} - \frac{p}{z-a} - \frac{q}{z-a} - \frac{q}{z-a} - \frac{q}{z-a} - \cdots$$

$$= \frac{(2q-p)z + pa - p\sqrt{(z-a)^2 - 4q}}{2(q-p)z^2 + 2paz + 2p^2}$$

$$\mu(dx) = \rho(x)dx + w_1\delta_{c_1} + w_2\delta_{c_2}\,,$$

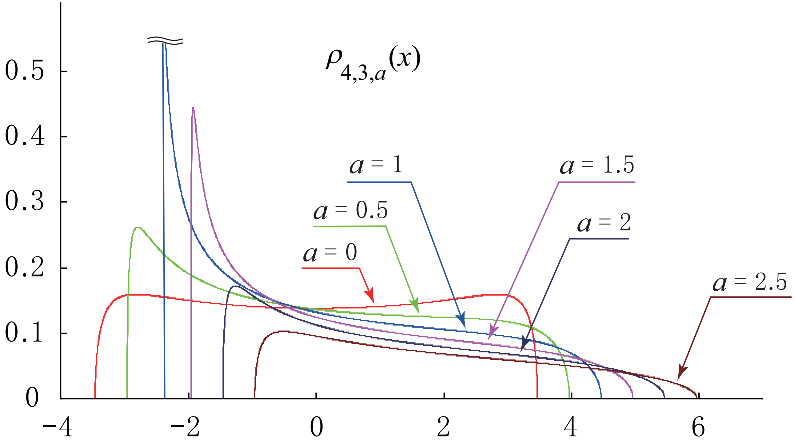
$$\rho(x) = \frac{p}{2\pi}\,\frac{\sqrt{4q-(x-a)^2}}{(q-p)x^2 + pax + p^2}, \quad |x-a| \leq 2\sqrt{q},$$

c_1, c_2, w_1, w_2 are known explicitly, see e.g., [Hora-O. Book, 2007]

Special cases

1. A free Meixner law with $a = 0$ is the Kesten measure with parameter p, q.
2. A free Meixner law with $a = 0$ and $p = q = 1$ is the (normalized) Wigner semicircle law.

Free Meixner laws $\quad \mu_{4,3,a}(dx) = \rho_{4,3,a}(x)dx + w_1\delta_{c_1} + w_2\delta_{c_2}$



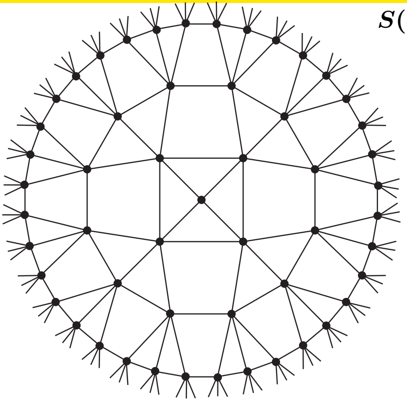
3.2. Spectral Distributions of Spidernets

$G = S(a, b, c)$

$a = \deg(o)$

$b = \deg(x)$ for $x \neq o$

$c = \omega_+(x)$ for $x \neq o$



$S(4, 6, 3)$

A: adjacency matrix of G, i.e.,

$$(A)_{xy} = \begin{cases} 1, & x \sim y \\ 0, & \text{otherwise} \end{cases}$$

Theorem (Igarashi–O. (2006))

Let A be the adjacency matrix of a spidernet $S(a, b, c)$. Then we have

$$\langle \delta_o, A^m \delta_o \rangle = \int_{-\infty}^{+\infty} x^m \mu_{a,c,b-1-c}(dx), \qquad m = 1, 2, \ldots,$$

where $\mu_{a,c,b-1-c}$ is the free Meixner law with parameter $a, c, b-1-c$.

4. Quantum Walks

N. Konno, N.O. and E. Segawa: Localization of the Grover walks on spidernets and free Meixner laws, arXiv:1206.4422 (June 2012)

4.1. Grover Walks on Graphs

$G = (V, E)$: a graph
$A(G) = \{(u, v) \in V \times V \,;\, u \sim v\}$ (half-edge)
$\mathcal{H}(G) = \ell^2(A(G))$: state space of Grover walk
$\{\delta_{(u,v)} \,;\, (u, v) \in A(G)\}$: canonical basis of $\mathcal{H}(G)$

1. *Coin flip operator* C is defined by

$$C\delta_{(u,v)} = \sum_{w\sim u} (H^{(u)})_{vw}\delta_{(u,w)},$$

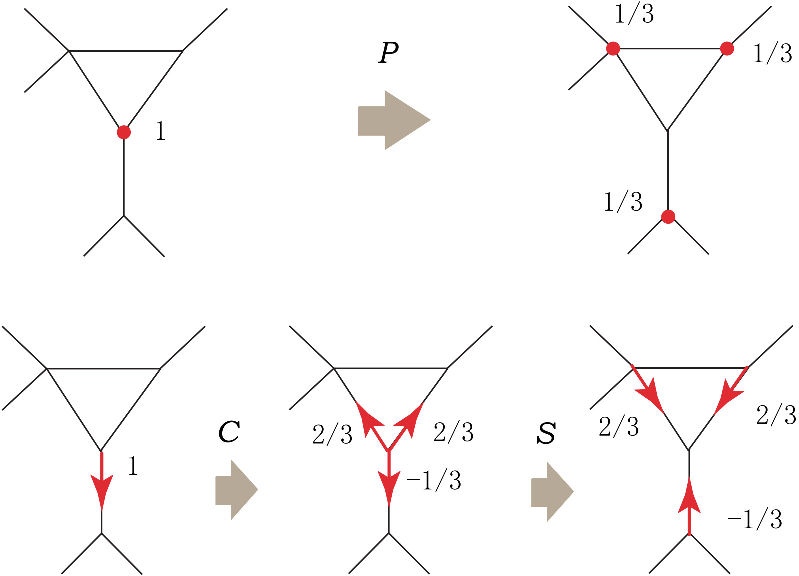
   where $H^{(u)}$ is the Grover matrix, i.e.,

$$(H^{(u)})_{vw} = \frac{2}{\deg(u)} - \delta_{wv},$$

2. *Shift operator* S is defined by

$$S\delta_{(u,v)} = \delta_{(v,u)}$$

3. the time evolution of the quantum walk is given by $U = SC$.
4. $\{\Phi_n = U^n\Phi_0\}$ is the *Grover walk* with initial state Φ_0.


1
P
1/3
1/3
1/3
1
C
2/3
2/3
−1/3
S
2/3
2/3
−1/3

4.2. Spidernets $S(a, b, c)$

$U = SC$: Grover walk on the spidernet $G = S(a, b, c)$

$a = \deg(o)$

$b = \deg(x)$ for $x \neq o$

$c = \omega_+(x)$ for $x \neq o$

One-step transition probabilities at $x \neq o$:

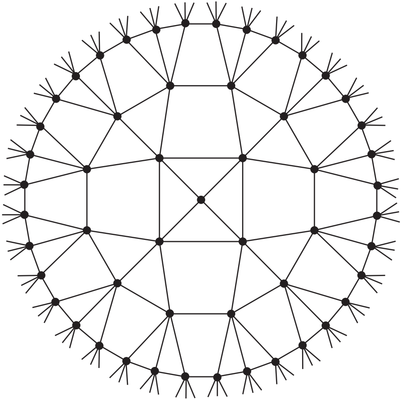
$$p = \frac{c}{b}$$

$$q = \frac{1}{b}$$

$$r = \frac{b - c - 1}{b}$$

$$p > 0, \quad q > 0, \quad r \geq 0,$$

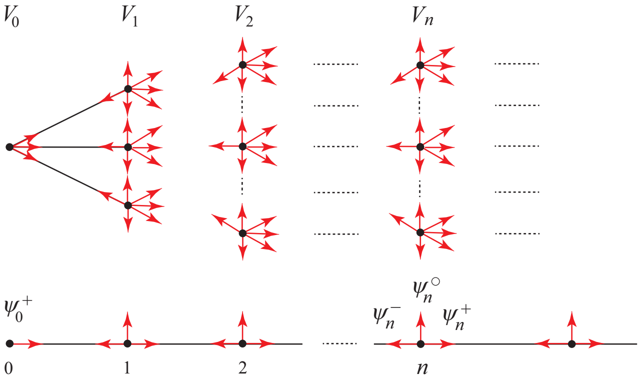
$$p + q + r = 1.$$



$S(4, 6, 3)$

Stratification:

$$V = \bigcup_{n=0}^{\infty} V_n \,, \qquad V_n = \{u \in V \,;\, \partial(u, o) = n\}$$



$$\mathcal{H}(G) = \mathcal{H}_0^+ \oplus \sum_{n=1}^{\infty} \oplus (\mathcal{H}_n^+ \oplus \mathcal{H}_n^\circ \oplus \mathcal{H}_n^-)$$

$$\downarrow$$

$$\mathcal{H}(\mathbf{Z}_+) = \mathbb{C}\psi_0^+ \oplus \sum_{n=1}^{\infty} \oplus (\mathbb{C}\psi_n^+ \oplus \mathbb{C}\psi_n^\circ \oplus \mathbb{C}\psi_n^-)$$

$$\psi_n^+ = \frac{1}{\sqrt{ac^n}} \sum_{u \in V_n} \sum_{\substack{v \in V_{n+1} \\ v \sim u}} \delta_{(u,v)}\,, \qquad n \geq 0,$$

$$\psi_n^\circ = \frac{1}{\sqrt{a(b-c-1)c^{n-1}}} \sum_{u \in V_n} \sum_{\substack{v \in V_n \\ v \sim u}} \delta_{(u,v)}\,, \quad n \geq 1,$$

$$\psi_n^- = \frac{1}{\sqrt{ac^{n-1}}} \sum_{u \in V_n} \sum_{\substack{v \in V_{n-1} \\ v \sim u}} \delta_{(u,v)}\,, \qquad n \geq 1.$$

$$C\psi_n^+ = \begin{cases} \psi_0^+\,, & n = 0, \\ (2p-1)\psi_n^+ + 2\sqrt{pr}\,\psi_n^\circ + 2\sqrt{pq}\,\psi_n^-\,, & n \geq 1, \end{cases}$$

$$C\psi_n^\circ = 2\sqrt{pr}\,\psi_n^+ + (2r-1)\psi_n^\circ + 2\sqrt{qr}\,\psi_n^-\,, \qquad n \geq 1,$$

$$C\psi_n^- = 2\sqrt{pq}\,\psi_n^+ + 2\sqrt{qr}\,\psi_n^\circ + (2q-1)\psi_n^-\,, \qquad n \geq 1.$$

$$S\psi_n^+ = \psi_{n+1}^-\,, \quad n \geq 0; \quad S\psi_n^\circ = \psi_n^\circ\,, \quad n \geq 1; \quad S\psi_n^- = \psi_{n-1}^+\,, \quad n \geq 1.$$

- $\mathcal{H}(\mathbf{Z}_+) = \mathbb{C}\psi_0^+ \oplus \sum_{n=1}^\infty \oplus(\mathbb{C}\psi_n^+ \oplus \mathbb{C}\psi_n^\circ \oplus \mathbb{C}\psi_n^-)$ is invariant under $U = SC$.
- The unitary operator $U = SC$ restricted to $\mathcal{H}(\mathbf{Z}_+)$ is called a (p,q)-*quantum walk* on $\mathbf{Z}_+$.

4.4. Main Problem

Probability amplitude (≈ quantum counterpart of transition probability)

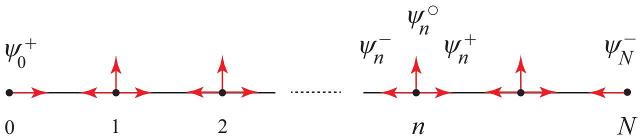
Let U be the (p,q)-quantum walk U on $\mathbf{Z}_+$. We are interested in

$$\langle \psi_0^+, U^n \psi_0^+ \rangle, \qquad n = 0, 1, 2, \dots,$$

and its asymptotics as $n \to \infty$.

We employ "cutoff" of the (p,q)-quantum walk on $\mathbf{Z}_+$ thanks to the fact that

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \langle \psi_0^+, U_N^n \psi_0^+ \rangle, \qquad n < N. \tag{4.1}$$

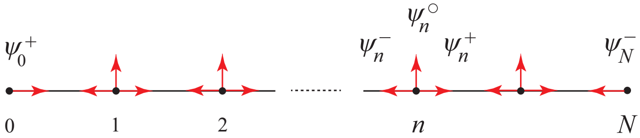


$$\mathcal{H}(N) = \mathbb{C}\psi_0^+ \oplus \sum_{n=1}^{N-1} \oplus (\mathbb{C}\psi_n^+ \oplus \mathbb{C}\psi_n^\circ \oplus \mathbb{C}\psi_n^-) \oplus \mathbb{C}\psi_N^-$$

$U_N = S_N C_N$, where $C = C_N$ and $S = S_N$ as before except $C\psi_N^- = \psi_N^-$

4.5. Spectral Analysis of U_N

$U = U_N$ acts on $\mathcal{H}(N)$:



For further reduction we define

$$\Psi_0 = \psi_0^+,$$

$$\Psi_n = \sqrt{p}\,\psi_n^+ + \sqrt{r}\,\psi_n^\circ + \sqrt{q}\,\psi_n^-, \quad 1 \le n \le N-1,$$

$$\Psi_N = \psi_N^-$$

$$\Gamma(N) = \sum_{n=0}^{N} \mathbb{C}\Psi_n \subset \mathcal{H}(N)$$

$$\Pi : \mathcal{H}(N) \to \Gamma(N) \quad \text{orthogonal projection}$$

$$T = \Pi U \Pi \qquad \text{as an operator on } \Gamma(N)$$

It is shown that $\mathrm{Tr}\,U = (2r-1)(N-1)$.

Matrix expression of T with respect to the orthnormal basis $\{\Psi_j \,;\, 0 \le j \le N\}$:

$$T = \begin{bmatrix} 0 & \sqrt{q} & & & & & \\ \sqrt{q} & r & \sqrt{pq} & & & & \\ & \sqrt{pq} & r & \sqrt{pq} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \sqrt{pq} & r & \sqrt{pq} & \\ & & & & \sqrt{pq} & r & \sqrt{p} \\ & & & & & \sqrt{p} & 0 \end{bmatrix}.$$

Notice: In what follows we asume that $r > 0$. The case of $r = 0$ is similar.

1. Every eigenvalue of T is simple.
2. $\mathrm{Spec}(T) \subset [-1, 1]$.
3. $1 \in \mathrm{Spec}(T)$ and $-1 \notin \mathrm{Spec}(T)$.

Thus, the eigenvalues of T are arranged in such a way that

$$\lambda_0 = 1 = \cos\theta_0, \quad \lambda_1 = \cos\theta_1\,, \quad \lambda_2 = \cos\theta_2\,, \quad \dots, \quad \lambda_N = \cos\theta_N\,,$$
$$0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_N < \pi.$$

Let $\{\Omega_j \,;\, 0 \le j \le N\}$ be an orthonormal basis of $\Gamma(N)$ such that $T\Omega_j = \lambda_j \Omega_j$.

1. We have an orthogonal decomposition into U-invariant subspaces:

$$\mathcal{H}(N) = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_N \oplus \mathcal{M},$$
$$\mathcal{L}_0 = \mathbb{C}\Omega_0 \,, \qquad \mathcal{L}_j = \mathbb{C}\Omega_j + \mathbb{C}S\Omega_j \,, \quad 1 \le j \le N.$$

2. $U\restriction_{\mathcal{L}_j}$ with respect to the basis $\{\Omega_j, S\Omega_j\}$ is given by

$$\begin{bmatrix} 0 & -1 \\ 1 & 2\lambda_j \end{bmatrix},$$

   of which the eigenvalues are

$$\lambda_j \pm i\sqrt{1-\lambda_j^2} = e^{\pm i\theta_j}.$$

3. By $\mathcal{H}(N) = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_N \oplus \mathcal{M}$ we have

$$\mathrm{Tr}\,(U) = 2\mathrm{Tr}\,(T) - 1 + \mathrm{Tr}\,(U\restriction_{\mathcal{M}}) = 2r(N-1) - 1 + \mathrm{Tr}\,(U\restriction_{\mathcal{M}}).$$

4. Since $\mathrm{Tr}\,(U) = (2r-1)(N-1)$, we have $\mathrm{Tr}\,(U\restriction_{\mathcal{M}}) = -(N-2)$.
5. Since $\dim \mathcal{M} = (3N-1) - (2N+1) = N-2$, we see that $U\restriction_{\mathcal{M}} = -I$. Therefore the multiplicity of the eigenvalue -1 coincides with $\dim \mathcal{M} = N-2$.

Theorem (Spectra of U_N for $r > 0$)

(1) The eigenvalues of $U = U_N$ are

$$1, \quad e^{\pm i\theta_j} \quad (1 \le j \le N), \quad -1.$$

(2) All the eigenvalues except -1 are multiplicity free and the multiplicity of the eigenvalue -1 is $N-2$.

(3) We set

$$\Omega_j^{\pm} = \frac{1}{\sqrt{2}\,\sin\theta_j}\,(\Omega_j - e^{\pm i\theta_j} S\Omega_j), \quad 1 \le j \le N.$$

Then $\Omega_j^{\pm}$ is a normalized eigenvector of U with eigenvalue $e^{\pm i\theta_j}$, i.e.,

$$\Omega_j^{\pm} \in \mathcal{L}_j\,, \quad \|\Omega_j^{\pm}\| = 1, \quad U\Omega_j^{\pm} = e^{\pm i\theta_j}\Omega_j^{\pm}.$$

Theorem (Spectra of U_N for $r = 0$)

(1) The eigenvalues of U are

$$1, \quad e^{\pm i\theta_j} \quad (1 \le j \le N-1), \quad -1.$$

(2) All the eigenvalues except -1 are multiplicity free and the multiplicity of the eigenvalue -1 is N.

(3) We set

$$\Omega_j^{\pm} = \frac{1}{\sqrt{2}\,\sin\theta_j}\,(\Omega_j - e^{\pm i\theta_j} S\Omega_j), \quad 1 \le j \le N-1.$$

Then $\Omega_j^{\pm}$ is a normalized eigenvector of U with eigenvalue $e^{\pm i\theta_j}$, i.e.,

$$\Omega_j^{\pm} \in \mathcal{L}_j\,, \quad \|\Omega_j^{\pm}\| = 1, \quad U\Omega_j^{\pm} = e^{\pm i\theta_j}\Omega_j^{\pm}.$$

Lemma

Let $p > 0$, $q > 0$, $r = 1 - p - q \geq 0$ be constant numbers and U the (p, q)-quantum walk on $\mathbf{Z}_+$. Then for $n = 0, 1, 2, \dots$ it holds that

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \sum_{j=0}^{N} |\langle \Psi_0, \Omega_j \rangle|^2 \cos n\theta_j ,$$

where $N > n$ and Ω_j is the eigenvector of T_N with eigenvalue $\cos\theta_j$.

Proof By expansion in terms of eigenvectors.

$$\begin{aligned}
\langle \Psi_0, U^n \Psi_0 \rangle &= |\langle \Omega_0, \Psi_0 \rangle|^2 + \sum_{j=1}^{N} |\langle \Omega_j^+, \Psi_0 \rangle|^2 e^{in\theta_j} + \sum_{j=1}^{N} |\langle \Omega_j^-, \Psi_0 \rangle|^2 e^{-in\theta_j} \\
&= |\langle \Omega_0, \Psi_0 \rangle|^2 + \sum_{j=1}^{N} |\langle \Omega_j, \Psi_0 \rangle|^2 \frac{e^{in\theta_j} + e^{-in\theta_j}}{2} \\
&= |\langle \Omega_0, \Psi_0 \rangle|^2 + \sum_{j=1}^{N} |\langle \Omega_j, \Psi_0 \rangle|^2 \cos n\theta_j \, .
\end{aligned}$$

Noting that $\theta_0 = 0$ we come to the desired expression.

Lemma

The probability distribution associated with the T_N is given by

$$\mu_N = \sum_{j=0}^{N} |\langle \Omega_j, \Psi_0 \rangle|^2 \delta_{\lambda_j}$$

Moreover, μ_N converges weakly to the free Meixner law with parameters q, pq, r.

Proof The first part is standard.

The second part is verified by observing

$$T_N = \begin{bmatrix} 0 & \sqrt{q} & & & & & \\ \sqrt{q} & r & \sqrt{pq} & & & & \\ & \sqrt{pq} & r & \sqrt{pq} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \sqrt{pq} & r & \sqrt{pq} & \\ & & & & \sqrt{pq} & r & \sqrt{p} \\ & & & & & \sqrt{p} & 0 \end{bmatrix}$$

and the fact that convergence in moments + uniqueness of moment problem $\Longrightarrow$ weak convergence.

Theorem (Integral representation)

Let U be the (p,q)-quantum walk on $\mathbf{Z}_+$ ($r > 0$ but $r = 0$ also). We have

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \int_{-1}^{1} \cos n\theta \, \mu(d\lambda), \qquad \lambda = \cos\theta,$$

where μ is the free Meixner distribution with parameters q, pq, r.

Proof

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \sum_{j=0}^{N} |\langle \Psi_0, \Omega_j \rangle|^2 \cos n\theta_j = \int_{-1}^{1} \cos n\theta \mu_N(d\lambda),$$

where $\cos\theta = \lambda$. This holds whenever $n < N$. Then letting $N \to \infty$, we obtain

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \int_{-1}^{1} \cos n\theta \, \mu(d\lambda),$$

where μ is the free Meixner distribution with parameters q, pq, r.

Recall (Igarashi-O.): for a random walk we have

$$\langle \delta_0, P^n \delta_0 \rangle = \int_{-1}^{1} \lambda^n \, \mu(d\lambda)$$

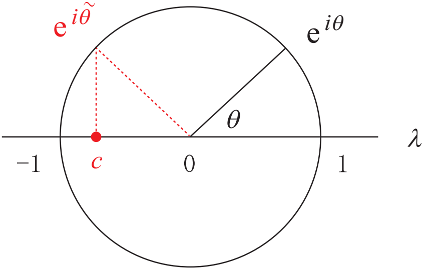
$\mu(dx)$: free Meixner distribution with parameters q, pq, r

1. For (p,q)-quantum walk on $\mathbf{Z}_+$ we have

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \int_{-1}^{1} \cos n\theta \, \mu(d\lambda), \qquad \lambda = \cos\theta,$$

2. For a lazy random walk we have

$$\langle \delta_0, P^n \delta_0 \rangle = \int_{-1}^{1} \lambda^n \, \mu(d\lambda)$$



- For a spidernet $S(a,b,c)$ we have $\mu(dx) = \rho(x)dx + w\delta_c$

Application: Initial Value Localization

- The free Meixner law with parameters q, pq, r is of the form

$$\mu(dx) = \rho(x)dx + w_1\delta_{c_1} + w_2\delta_{c_2}$$

(An explicit description is known.)

- For the spider net $S(a, b, c)$ it is sufficient to consider the case where

$$p + q + r = 1, \qquad p > q > 0, \qquad r > 0.$$

- In this case the free Meixner law is of the form:

$$\mu(dx) = \rho(x)dx + w\delta_c\,,$$

$$c = -\frac{q}{1-p}\,, \quad w = \max\left\{\frac{(1-p)^2 - pq}{(1-p)(1-p+q)}\,, 0\right\}$$

- Then, as $n \to \infty$ we have

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \int_{-1}^{1} \cos n\theta\, \mu(d\lambda) \sim w \cos n\tilde{\theta} \qquad \cos\tilde{\theta} = -\frac{q}{1-p}$$

- Hence, the localization occurs $\iff w > 0 \iff (1-p)^2 - pq > 0$.

Application: Initial Value Localization

Theorem (Konno-O.-Segawa (2012))

Let $\{X_n\}$ *be the "position process" of the Grover walk* $\{U^n\Psi_0\}$ *on the spidernet* $S(\kappa, \kappa+2, \kappa-1)$ *with* $\kappa \geq 2$*. Then we have*

$$P(X_n = o) = |\langle \psi_0^+, U^n \psi_0^+ \rangle|^2 \sim \begin{cases} 0, & \text{for } \kappa \geq 10, \\ \left(\dfrac{10-\kappa}{12}\right)^2 \cos^2(n\tilde{\theta}) & \text{for } 2 \leq \kappa < 10, \end{cases}$$

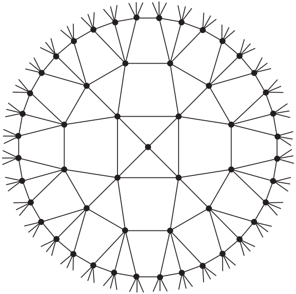
where $\tilde{\theta} = \arccos(-1/3)$*. Moreover, for* $2 \leq \kappa < 10$ *we have*

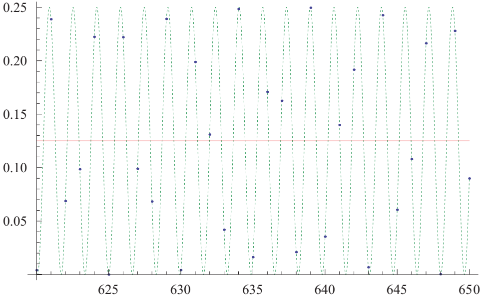
$$\overline{q}^{(\infty)}(o) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} P(X_n = o) = \frac{1}{2}\left(\frac{\kappa - 10}{12}\right)^2 > 0.$$

Namely, (initial point) localization occurs at position o*.*

[Chisaki *et al.* (2009)] No localization for Grover walks on trees (the initial state $= \psi_0^+$)
large $\kappa \iff$ density of large cycles is low (close to a tree)
small $\kappa \iff$ density of large cycles is high $\Longrightarrow$ emergence of localization

Example: Localization occurs for $S(4,6,3)$





$$P(X_n = o) \sim \left(\frac{10-4}{12}\right)^2 \cos^2(n\tilde{\theta}), \qquad \tilde{\theta} = \arccos(-1/3)$$

$$\overline{q}^{(\infty)}(o) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} P(X_n = o) = \frac{1}{2}\left(\frac{4-10}{12}\right)^2 = \frac{1}{8}\,.$$

Note: We have a good estimate for $P(X_n \in V_l)$ too.