Quantum Probability and Asymptotic Spectral Analysis of Growing Graphs

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1. Asymptotic Spectral Analysis of Graphs
2. Quantum Probability
3. Quantum Probabilistic Approach to Spectral Graph Theory
4. Graph Products and Concepts of Independence
5. Method of Quantum Decomposition
6. Method of Quantum Decomposition for Growing Graphs

presentation: Roma, November 14, 2013
1. Asymptotic Spectral Analysis of Graphs
1.1. Graphs and Adjacency Matrices

Definition (graph)

A graph is a pair $G = (V, E)$, where $V$ is the set of vertices and $E$ the set of edges. We write $x \sim y$ (adjacent) if they are connected by an edge.

Definition (adjacency matrix)

The adjacency matrix of a graph $G = (V, E)$ is defined by

$$A = [A_{xy}]_{x,y \in V} \quad A_{xy} = \begin{cases} 1, & x \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

▶ The adjacency matrix possesses all the information of a graph.
1.2. Spectra of Graphs

**Definition (spectrum and spectral distribution)**

Let $G$ be a finite graph. The **spectrum (eigenvalues)** of $G$ is the list of eigenvalues of the adjacency matrix $A$.

$$\text{Spec}(G) = \begin{pmatrix} \cdots & \lambda_i & \cdots \\ \cdots & m_i & \cdots \end{pmatrix}$$

The **spectral (eigenvalue) distribution** of $G$ is defined by

$$\mu_G = \frac{1}{|V|} \sum_i m_i \delta_{\lambda_i}$$

- **Spec** $(G)$ is a fundamental invariant of finite graphs.
- (isospectral problem) Non-isomorphic graphs may have the same spectra.
- Adjacency matrix, Laplacian matrix, distance matrix, $Q$-matrix, ... etc.

▶ For **algebraic graph theory** or **spectral graph theory** see

Our Aim is to extend the spectral graph theory to cover

1. infinite graphs — $A$ is no longer a finite matrix and eigenvalues make no sense.
2. growing graphs — interests in the asymptotic properties (asymptotic combinatorics)

Then the adjacency matrix $A$ should be studied as

- an operator acting on a suitable Hilbert space:

$$ Af(x) = \sum_{y \in V} A_{xy} f(y), \quad f \in C_0(V) \subset \ell^2(V); $$

- a random variable in a suitable framework of probability theory
1.4. Illustration: Path Graphs $P_n$ ad $n \to \infty$

\[ A = \begin{bmatrix}
0 & 1 & & & \\
1 & 0 & 1 & & \\
1 & 0 & 1 & & \\
& & & \ddots & \\
1 & 0 & 1 & & \\
1 & 0 & 1 & & \\
\end{bmatrix} \]

\[ \text{Spec} (P_n) = \begin{pmatrix}
\cdots & 2 \cos \frac{k \pi}{n+1} & \cdots \\
\cdots & 1 & \cdots \\
1 \leq k \leq n
\end{pmatrix} \]
2. Quantum Probability

\[= \text{Non-Commutative Probability}\]

A random variable $X$ on a probability space $(\Omega, \mathcal{F}, P)$ satisfying the property:

$$P(X = +1) = P(X = -1) = \frac{1}{2}$$

More essential is the probability distribution of $X$:

$$\mu_X = \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_{+1}$$
2.1. Quantum Probabilistic Model for Coin-toss

Traditional Model for Coin-toss

A random variable $X$ on a probability space $(\Omega, \mathcal{F}, P)$ satisfying the property:

$$P(X = +1) = P(X = -1) = \frac{1}{2}$$

More essential is the probability distribution of $X$:

$$\mu_X = \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_{+1}$$

Moment sequence is one of the most fundamental characteristics

Moment sequence $\{M_m\} \iff$ Probability distributions $\mu$

Up to *determinate moment problem*

For a coin-toss we have

$$M_m(\mu_X) = \int_{-\infty}^{+\infty} x^m \mu_X(dx) = \begin{cases} 1, & \text{if } m \text{ is even}, \\ 0, & \text{otherwise}. \end{cases}$$
Let us observe:

\[
A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad e_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

It is straightforward to see that

\[
\langle e_0, A^m e_0 \rangle = \begin{cases} 
1, & \text{if } m \text{ is even}, \\
0, & \text{otherwise}, 
\end{cases} = M_m(\mu_X) = \int_{-\infty}^{+\infty} x^m \mu_X(dx).
\]

In other words, we have another model of coin-toss by means of the matrix \( A \).

Matrices form a non-commutative algebra so we say

Quantum probabilistic (or non-commutative) model for cointoss

\[
\mathcal{A} = \ast\text{-algebra generated by } A; \quad \varphi(a) = \langle e_0, ae_0 \rangle, \quad a \in \mathcal{A},
\]

We call \( \mathcal{A} \) an algebraic realization of the random variable \( X \).
2.2. Axioms: Quantum Probability

**Definition (Algebraic probability space)**

An *algebraic probability space* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a $\ast$-algebra over $\mathbb{C}$ with multiplication unit $1_{\mathcal{A}}$, and a state $\varphi : \mathcal{A} \to \mathbb{C}$, i.e.,

(i) $\varphi$ is linear;  
(ii) $\varphi(a^\ast a) \geq 0$;  
(iii) $\varphi(1_{\mathcal{A}}) = 1$.

Each $a \in \mathcal{A}$ is called an *(algebraic) random variable*.

**Definition (Spectral distribution)**

For a real random variable $a = a^\ast \in \mathcal{A}$ there exists a probability measure $\mu$ on $\mathbb{R} = (-\infty, +\infty)$ such that

$$\varphi(a^m) = \int_{-\infty}^{+\infty} x^m \mu(dx) \equiv M_m(\mu), \quad m = 1, 2, \ldots.$$ 

This $\mu$ is called the *spectral distribution* of $a$ (with respect to the state $\varphi$).

- Existence of $\mu$ by Hamburger’s theorem using Hanckel determinants.
- In general, $\mu$ is not uniquely determined (indeterminate moment problem).
2.3. Concept of Independence and Central Limit Theorem (classical)

Classical independence

If two random variables \( X, Y \) are independent, we have

\[
E[XY Y XX] =
\]
2.3. Concept of Independence and Central Limit Theorem (classical)

Classical independence

If two random variables $X, Y$ are independent, we have

2.3. Concept of Independence and Central Limit Theorem (classical)

### Classical independence

If two random variables $X, Y$ are independent, we have


### Classical central limit theorem (classical CLT)

If $X_1, X_2, \ldots$ are independent, identically distributed, normalized (mean zero, variance one) random variables, then $\frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n$ obeys the standard normal law (Gaussian distribution) in the limit:

$$\lim_{N \to \infty} P \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \leq a \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx,$$

or equivalently, for all bounded continuous functions $f$,

$$\lim_{N \to \infty} E \left[ f \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \right) \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-x^2/2} dx.$$
If $X_1, X_2, \ldots$ are independent, identically distributed, normalized (mean zero, variance one) random variables having moments of all orders, we have

$$\lim_{N \to \infty} E \left[ \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \right)^m \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^m e^{-x^2/2} dx, \quad m = 1, 2, \ldots.$$ 

For the proof the factorization rule is essential

$$E[X_1X_3X_1X_2X_1X_3X_1] = E[X_1^4X_2X_3^2] = E[X_1^4]E[X_2]E[X_3^2]$$

If $X_1, X_2, X_2, \ldots$ are non-commutative, what can we do for

$$E[X_1X_3X_1X_2X_1X_3X_1] \neq E[X_1^4X_2X_3^2] \neq E[X_1^4]E[X_2]E[X_3^2]$$

As a result, there are many different factorization rules $\iff$ various concepts of independence.
Consider an algebraic probability space and \((\mathcal{A}, \varphi)\) and \(a \in \mathcal{A}_1, b \in \mathcal{A}_2\).

<table>
<thead>
<tr>
<th>Product</th>
<th>commutative</th>
<th>free</th>
<th>Boolean</th>
<th>monotone</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\varphi(aba))</td>
<td>(\varphi(a^2)\varphi(b))</td>
<td>(\varphi(a^2)\varphi(b))</td>
<td>(\varphi(a)^2\varphi(b))</td>
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<tr>
<td>CLM</td>
<td>Gaussian</td>
<td>Wigner</td>
<td>Bernoulli</td>
<td>arcsine</td>
</tr>
</tbody>
</table>


## 2.4. Four Concepts of Independence and Quantum CLTs

Consider an algebraic probability space and \((\mathcal{A}, \varphi)\) and \(a \in \mathcal{A}_1, b \in \mathcal{A}_2\).

\[
\begin{array}{|c|c|c|c|c|}
\hline
& \text{commutative} & \text{free} & \text{Boolean} & \text{monotone} \\
\hline
\varphi(aba) & \varphi(a^2)\varphi(b) & \varphi(a^2)\varphi(b) & \varphi(a^2)\varphi(b) & \varphi(a^2)\varphi(b) \\
\hline
\varphi(bab) & \varphi(a)\varphi(b^2) & \varphi(a)\varphi(b^2) & \varphi(a)\varphi(b)^2 & \varphi(a)\varphi(b)^2 \\
\hline
\varphi(abab) & \varphi(a^2)\varphi(b^2) & \varphi(a)^2\varphi(b^2) + \varphi(a^2)\varphi(b)^2 & \varphi(a)^2\varphi(b)^2 & \varphi(a^2)\varphi(b)^2 \\
\hline
\text{CLM} & \text{Gaussian} & \text{Wigner} & \text{Bernoulli} & \text{arcsine} \\
\hline
\end{array}
\]

CLM (Central Limit Measure) \(\mu\) is obtained from a sequence of normalized random variables \(\{a_n = a_n^*\}\) which are \{commutative/free/Boolean/monotone\} independent:

\[
\lim_{N \to \infty} \varphi \left[ \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n \right)^m \right] = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \ldots
\]

2.5. Monotone Independence

**Definition (monotone independence)**

Let \((\mathcal{A}, \varphi)\) be an algebraic probability space and \(\{\mathcal{A}_n\}\) a family of \(*\)-subalgebras. We say that \(\{\mathcal{A}_n\}\) is **monotone independent** if for \(a_1 \in \mathcal{A}_{n_1}, \ldots, a_m \in \mathcal{A}_{n_m}\) we have

\[
\varphi(a_1 \cdots a_m) = \varphi(a_i)\varphi(a_1 \cdots \tilde{a}_i \cdots a_m)
\]

(\(\tilde{a}_i\) stands for omission) holds when \(n_{i-1} < n_i\) and \(n_i > n_{i+1}\) happen for \(i \in \{1, 2, \ldots, n\}\).

Computing \(\varphi(a_1 a_2 a_3 \cdots a_m)\) where \(a_1 \in \mathcal{A}_2, a_2 \in \mathcal{A}_1, a_3 \in \mathcal{A}_4, \ldots\)

\[
\varphi(214343664435) = \varphi(4)\varphi(4)\varphi(66)\varphi(21334435) = \varphi(4)\varphi(4)\varphi(66)\varphi(44)\varphi(213335) = \cdots = \varphi(4)\varphi(4)\varphi(66)\varphi(44)\varphi(2)\varphi(5)\varphi(333)\varphi(1)
\]
Monotone Central Limit Theorem (Muraki 2001)

Let \((\mathcal{A}, \varphi)\) be an algebraic probability space and let \(\{a_n\}_{n=1}^{\infty}\) be a sequence of algebraic random variables satisfying the following conditions:

(i) \(a_n\) are real, i.e., \(a_n = a_n^*\);

(ii) \(a_n\) are normalized, i.e., \(\varphi(a_n) = 0, \varphi(a_n^2) = 1\);

(iii) \(a_n\) have uniformly bounded mixed moments, i.e., for each \(m \geq 1\) there exists \(C_m \geq 0\) such that \(|\varphi(a_{n_1} \ldots a_{n_m})| \leq C_m\) for any choice of \(n_1, \ldots, n_m\).

(iv) \(\{a_n\}\) are monotone independent.

Then,

\[
\lim_{N \to \infty} \varphi \left( \left\{ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} a_i \right\}^m \right) = \frac{1}{\pi} \int_{-\sqrt{2}}^{+\sqrt{2}} \frac{x^m}{\sqrt{2 - x^2}} dx, \quad m = 1, 2, \ldots.
\]

The probability measure in the right hand side is the normalized *arcsine law*. 
Monotone Central Limit Theorem (Muraki 2001)

Let \((\mathcal{A}, \varphi)\) be an algebraic probability space and let \(\{a_n\}_{n=1}^{\infty}\) be a sequence of algebraic random variables satisfying the following conditions:

(i) \(a_n\) are real, i.e., \(a_n = a_n^*\);

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Then,

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\]

The probability measure in the right hand side is the normalized \emph{arcsine law}.

Proof is to show

\[
\lim_{N \to \infty} \varphi \left( \left\{ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} a_i \right\}^{2m} \right) = \frac{(2m)!}{2^m m! m!} = \frac{1}{\pi} \int_{-\sqrt{2}}^{+\sqrt{2}} \frac{x^m}{\sqrt{2 - x^2}} \, dx
\]
Arcsine Law as a Central Limit Measure for Monotone Independence

Arcsine law vs normal (Gaussian) law

\[ \frac{1}{\pi \sqrt{2 - x^2}} \quad \text{Arcsine law} \]

\[ \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{normal (Gaussian) law} \]
3. Quantum Probabilistic Approach to Spectral Graph Theory

A. Hora and N. Obata: 
3.1. Quantum Probabilistic Approach

We deal with the adjacency algebra $\mathcal{A}(G)$ as an algebraic probability space equipped with a state $\langle \cdot \rangle$, and $A$ as a real random variable.

**Main Problem (I)**

Given a graph $G = (V, E)$ and a state $\langle \cdot \rangle$ on $\mathcal{A}(G)$, find a probability measure $\mu$ on $\mathbb{R}$ satisfying

$$\langle A^m \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \ldots.$$  

$\mu$ is called the *spectral distribution* of $A$ in the state $\langle \cdot \rangle$.

**Main Problem (II)**

Given a growing graph $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ and a state $\langle \cdot \rangle_\nu$ on $\mathcal{A}(G^{(\nu)})$, find a probability measure $\mu$ on $\mathbb{R}$ satisfying

$$\langle (A^{(\nu)})^m \rangle_\nu \approx \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \ldots.$$  

$\mu$ is called the *asymptotic spectral distribution* of $G^{(\nu)}$ in the state $\langle \cdot \rangle_\nu$. 
3.2. States on $\mathcal{A}(G)$

For a finite graph $G$ we set 

$$
\langle a \rangle_{\text{tr}} = |V| \text{Tr} (a) = |V| \sum_{x \in V} \langle \delta_x, a \delta_x \rangle, \quad a \in \mathcal{A}(G)
$$

Then $(\mathcal{A}(G), \langle \cdot \rangle_{\text{tr}})$ becomes an algebraic probability space.

▶ The spectral distribution $\mu$ of $A$, determined (uniquely) by 

$$
\langle A^m \rangle_{\text{tr}} = \int_{-\infty}^{\infty} x^m \mu(dx), \quad m = 1, 2, \ldots
$$

coincides with the eigenvalue distribution of $A$:

$$
\mu = \frac{1}{|V|} \sum_{i} m_i \delta_{\lambda_i}, \quad \text{Spec}(G) = \text{Spec}(A) = (\cdots \lambda_i \cdots m_i \cdots) 
$$
3.2. States on $\mathcal{A}(G)$

(l) Trace

For a finite graph $G$ we set

$$\langle a \rangle_{tr} = \frac{1}{|V|} \text{Tr} (a) = \frac{1}{|V|} \sum_{x \in V} \langle \delta_x, a \delta_x \rangle, \quad a \in \mathcal{A}(G)$$

Then $(\mathcal{A}(G), \langle \cdot \rangle_{tr})$ becomes an algebraic probability space.

- The spectral distribution $\mu$ of $A$, determined (uniquely) by

$$\langle A^m \rangle_{tr} = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \ldots,$$

coincides with the eigenvalue distribution of $A$:

$$\mu = \frac{1}{|V|} \sum_i m_i \delta_{\lambda_i}, \quad \text{Spec} (G) = \text{Spec} (A) = \left( \begin{array}{ccc} \vdots & \lambda_i & \vdots \\ \vdots & m_i & \vdots \end{array} \right)$$
(II) Vacuum State

The vacuum state at \( o \in V \) is the vector state defined by

\[
\langle a \rangle_o = \langle \delta_o, a\delta_o \rangle, \quad a \in A(G).
\]

For the adjacency matrix \( A \) we have

\[
\langle A^m \rangle_o = \langle \delta_o, A^m\delta_o \rangle = |\{m\text{-step walks from } o \text{ to } o\}| = \int_{-\infty}^{+\infty} x^m \mu(dx)
\]

(III) Deformed Vacuum State

The \( Q \)-matrix of \( G \) is defined by

\[
Q = Q_q = [q^{\partial(x,y)}]_{x,y \in V} \quad -1 \leq q \leq 1.
\]

Then the deformed vacuum state is defined by

\[
\langle a \rangle_q = \langle Q\delta_o, a\delta_o \rangle, \quad a \in A(G).
\]

1. \( \langle \cdot \rangle_q \) gives rise to a one-parameter deformation of vacuum states.
2. In general, \( \langle \cdot \rangle_q \) is not necessarily positive (i.e., not a state), but one may hope it is positive for \( q \) close to 0.
3. (Open question) To determine the range of \( q \) for which \( Q_q \) is positive definite.
4. Graph Products and Concepts of Independence
4.1. Independence and Graph Structures (I) Cartesian product

**Definition**

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two graphs. For \((x, y), (x', y') \in V_1 \times V_2\) we write \((x, y) \sim (x', y')\) if one of the following conditions is satisfied:

(i) \( x = x' \) and \( y \sim y' \); (ii) \( x \sim x' \) and \( y = y' \).

Then \( V_1 \times V_2 \) becomes a graph in such a way that \((x, y), (x', y') \in V_1 \times V_2\) are adjacent if \((x, y) \sim (x', y')\). This graph is called the **Cartesian product** or **direct product** of \( G_1 \) and \( G_2 \), and is denoted by \( G_1 \times G_2 \).

**Example \((C_4 \times C_3)\)**

![Graph diagram](image)
Theorem

Let $G = G_1 \times G_2$ be the cartesian product of two graphs $G_1$ and $G_2$. Then the adjacency matrix of $G$ admits a decomposition:

$$A = A_1 \otimes I_2 + I_1 \otimes A_2$$

as an operator acting on $\ell^2(V) = \ell^2(V_1 \times V_2) \cong \ell^2(V_1) \otimes \ell^2(V_2)$. Moreover, the right hand side is a sum of commutative independent random variables with respect to the vacuum state.

▶ The asymptotic spectral distribution is the Gaussian distribution by applying the commutative central limit theorem.

Example (Hamming graphs)

The Hamming graph is the Cartesian product of complete graphs:

$$H(d, N) \cong K_N \times \cdots \times K_N \quad (d\text{-fold cartesian power of complete graphs})$$

$$A_d, N = \sum_{i=1}^{n} \underbrace{I \otimes \cdots \otimes I}_{i-1} \otimes B \otimes \underbrace{I \otimes \cdots \otimes I}_{n-i}$$

where $B$ is the adjacency matrix of $K_N$.

▶ Hora (1998) obtained the limit distribution by a direct calculation of eigenvalues.
4.2. Independence and Graph Structures (II) Comb product

**Definition**

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We fix a vertex $o_2 \in V_2$. For $(x, y), (x', y') \in V_1 \times V_2$ we write $(x, y) \sim (x', y')$ if one of the following conditions is satisfied:

(i) $x = x'$ and $y \sim y'$;  
(ii) $x \sim x'$ and $y = y' = o_2$.

Then $V_1 \times V_2$ becomes a graph, denoted by $G_1 \triangleright_{o_2} G_2$, and is called the **comb product** or the **hierarchical product**.

**Example ($C_4 \triangleright C_3$)**

![Diagram of $C_4 \triangleright C_3$](image)
Theorem

As an operator on $C_0(V_1) \otimes C_0(V_2)$ the adjacency matrix of $G_1 \circ_{o_2} G_2$ is given by

$$A = A_1 \otimes P_2 + I_1 \otimes A_2$$

where $P_2 : C_0(V_2) \to C_0(V_2)$ is the projection onto the space spanned by $\delta_{o_2}$ and $I_1$ is the identity matrix acting on $C_0(V_1)$.

Theorem (Accardi–Ben Ghobal–O. IDAQP(2004))

The adjacency matrix of the comb product $G^{(1)} \circ_{o_2} G^{(2)} \circ_{o_3} \cdots \circ_{o_n} G^{(n)}$ admits a decomposition of the form:

$$A^{(1)} \circ A^{(2)} \circ \cdots \circ A^{(n)}$$

$$= \sum_{i=1}^{n} I^{(1)} \otimes \cdots \otimes I^{(i-1)} \otimes A^{(i)} \otimes P^{(i+1)} \otimes \cdots \otimes P^{(n)},$$

where $P^{(i)}$ the projection from $\ell^2(V^{(i)})$ onto the one-dimensional subspace spanned by $\delta_{o_i}$. Moreover, the right-hand side is a sum of monotone independent random variables with respect to $\psi \otimes \delta_{o_2} \otimes \cdots \otimes \delta_{o_n}$, where $\psi$ is an arbitrary state on $\mathcal{B}(\ell^2(V^{(1)}))$. 
4.3. Independence and Graph Structures (III) Star product

**Definition**

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with distinguished vertices $o_1 \in V_1$ and $o_2 \in V_2$. Define a subset of $V_1 \times V_2$ by

$$V_1 \star V_2 = \{(x, o_2) ; x \in V_1\} \cup \{(o_1, y) ; y \in V_2\}$$

The induced subgraph of $G_1 \times G_2$ spanned by $V_1 \star V_2$ is called the **star product** of $G_1$ and $G_2$ (with contact vertices $o_1$ and $o_2$), and is denoted by $G_1 \star G_2 = G_1 \circ_{o_1 \star o_2} G_2$.

**Example ($C_4 \star C_3$)**

![Diagram of star product of two graphs]

Nobuaki Obata (GSIS, Tohoku University) Quantum Probability and Asymptotic Spectral Analysis Roma, November 14, 2013
Theorem (O. IIS(2004))

As an operator on $C_0(V_1) \otimes C_0(V_2)$ the adjacency matrix of $G_1 \star G_2$ is given by

$$A = A^{(1)} \otimes P^{(2)} + P^{(1)} \otimes A^{(2)}, \quad P^{(i)} = |\Phi^{(i)}_0\rangle\langle\Phi^{(i)}_0|.$$ 

where $P^{(i)} : C_0(V_i) \to C_0(V_i)$ is the projection onto the space spanned by $\delta_{o_i}$.

Theorem (O. IIS(2004))

The adjacency matrix of the star product $G^{(1)} \star G^{(2)} \star \cdots \star G^{(n)}$ admits a decomposition of the form:

$$A^{(1)} \star A^{(2)} \star \cdots \star A^{(n)} = \sum_{i=1}^{n} P^{(1)} \otimes \cdots \otimes P^{(i-1)} \otimes A^{(i)} \otimes P^{(i+1)} \otimes \cdots \otimes P^{(n)},$$

where $P^{(i)}$ is the projection from $\ell^2(V^{(i)})$ onto the one-dimensional subspace spanned by $\delta_{o_i}$. Moreover, the right-hand side is a sum of Boolean independent random variables with respect to $\delta_{o_1} \otimes \delta_{o_2} \otimes \cdots \otimes \delta_{o_n}$. 

Asymptotic spectral distribution of graph products $G^{(n)} = G \# G \# \ldots \# G$

<table>
<thead>
<tr>
<th>independence</th>
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</tr>
<tr>
<td>examples</td>
<td>comb graph</td>
<td>star graph</td>
<td>integer lattice</td>
<td>homogeneous tree</td>
</tr>
</tbody>
</table>

![Graph examples](image-url)
5. Method of Quantum Decomposition
5.1. A Simple Prototype: Coin-toss

Recall that the coin-toss is modelled by $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $e_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

A prototype of quantum decomposition

$$A = A^+ + A^- = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\langle e_0, A^m e_0 \rangle = \langle e_0, (A^+ + A^-)^m e_0 \rangle = \sum_{\epsilon_1, \ldots, \epsilon_m \in \{\pm\}} \langle e_0, A^{\epsilon_m} \cdots A^{\epsilon_1} e_0 \rangle.$$ 

⇒ Relations among non-commutative quantum components $A^\pm$ are useful.

$$\langle e_0, A^m e_0 \rangle = \begin{cases} 1, & \text{if } m \text{ is even}, \\ 0, & \text{otherwise}. \end{cases}$$
5.2. Quantum Decomposition of the Adjacency Matrix

Fix an origin $o \in V$ of $G = (V, E)$.
5.2. Quantum Decomposition of the Adjacency Matrix

Fix an origin \( o \in V \) of \( G = (V, E) \).

**Stratification (Distance Partition)**

\[
V = \bigcup_{n=0}^{\infty} V_n
\]

\( V_n = \{ x \in V ; \partial(o, x) = n \} \)
5.2. Quantum Decomposition of the Adjacency Matrix

Fix an origin \( o \in V \) of \( G = (V, E) \).

**Stratification (Distance Partition)**

\[
V = \bigcup_{n=0}^{\infty} V_n
\]

\[
V_n = \{ x \in V ; \partial(o, x) = n \}
\]

**Associated Hilbert space** \( \Gamma(G) \subset \ell^2(V) \)

\[
\Gamma(G) = \sum_{n=0}^{\infty} \oplus C\Phi_n
\]

\[
\Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} \delta_x
\]
5.2. Quantum Decomposition of the Adjacency Matrix (cont)

\[ (A^+)_{yx} = 1 \quad \text{for } y \in V_{n+1} \]

\[ (A^\circ)_{yx} = 1 \quad \text{for } y \in V_n \]

\[ (A^-)_{yx} = 1 \quad \text{for } y \in V_{n-1} \]

\[ A = A^+ + A^- + A^\circ \]

\[ (A^+)^* = A^- \quad \text{and} \quad (A^\circ)^* = A^\circ \]
5.2. Quantum Decomposition of the Adjacency Matrix (cont)

\[(A^+)_{yx} = 1\]

\[(A^\circ)_{yx} = 1\]

\[(A^-)_{yx} = 1\]

\[\Gamma(G) = \sum_{n=0}^{\infty} \bigoplus C\Phi_n\] is not invariant under the actions of \(A^\epsilon\).

Cases so far studied in detail

1. \(\Gamma(G)\) is invariant under \(A^\epsilon\) — distance-regular graphs
2. \(\Gamma(G)\) is asymptotically invariant under \(A^\epsilon\).
5.3. When $\Gamma(G)$ is Invariant Under $A^\epsilon$

For $x \in V_n$ we define
$$\omega_\epsilon(x) = |\{y \in V_{n+\epsilon}; y \sim x\}|, \quad \epsilon = +, -, \circ$$

Then, $\Gamma(G)$ is invariant under $A^\epsilon$ if and only if

(\ast) $\omega_\epsilon(x)$ is constant on each $V_n$.

Typical examples: distance-regular graphs
(in this case the constant in (\ast) is independent of the choice of $o \in V$) e.g., homogeneous trees, Hamming graphs, Johnson graphs, odd graphs, ...
5.3. When $\Gamma(G)$ is Invariant Under $A^\epsilon$

For $x \in V_n$ we define

$$\omega_\epsilon(x) = |\{y \in V_{n+\epsilon} ; y \sim x\}|, \quad \epsilon = +, -, o$$

Then, $\Gamma(G)$ is invariant under $A^\epsilon$ if and only if

($*$) $\omega_\epsilon(x)$ is constant on each $V_n$.

Typical examples: distance-regular graphs

(in this case the constant in ($*$) is independent of the choice of $o \in V$) e.g., homogeneous trees, Hamming graphs, Johnson graphs, odd graphs, ...

**Theorem**

If $\Gamma(G)$ is invariant under $A^+, A^-, A^o$, there exists a pair of sequences $\{\alpha_n\}$ and $\{\omega_n\}$ such that

$$A^+ \Phi_n = \sqrt{\omega_{n+1}} \Phi_{n+1}, \quad A^- \Phi_n = \sqrt{\omega_n} \Phi_{n-1}, \quad A^o \Phi_n = \alpha_{n+1} \Phi_n.$$ 

In other words, $\Gamma_{\{\omega_n\},\{\alpha_n\}} = (\Gamma(G), \{\Phi_n\}, A^+, A^-, A^o)$ is an interacting Fock space associated with Jacobi parameters ($\{\omega_n\}, \{\alpha_n\}$).
5.4. Computing the Spectral Distribution

By the interacting Fock space structure \((\Gamma(G), A^+, A^-, A^\circ)\):

\[
A \Phi_n = (A^+ + A^\circ + A^-) \Phi_n = \sqrt{\omega_{n+1}} \Phi_{n+1} + \alpha_{n+1} \Phi_n + \sqrt{\omega_n} \Phi_{n-1}.
\]

Associated with a probability distribution \(\mu\) on \(\mathbb{R}\), the orthogonal polynomials \(\{P_0(x) = 1, \ldots, P_n(x) = x^n + \cdots\}\) verify the three term recurrence relation:

\[
x P_n = P_{n+1} + \alpha_{n+1} P_n + \omega_n P_{n-1}, \quad P_0 = 1, \quad P_1 = x - \alpha_1,
\]

where \((\{\omega_n\}, \{\alpha_n\})\) is called the Jacobi parameters of \(\mu\).
5.4. Computing the Spectral Distribution

By the interacting Fock space structure \((\Gamma(G), A^+, A^-, A^\circ)\):

\[ A\Phi_n = (A^+ + A^\circ + A^-)\Phi_n = \sqrt{\omega_{n+1}}\Phi_{n+1} + \alpha_{n+1}\Phi_n + \sqrt{\omega_n}\Phi_{n-1}. \]

Associated with a probability distribution \(\mu\) on \(\mathbb{R}\), the orthogonal polynomials \(\{P_0(x) = 1, \ldots, P_n(x) = x^n + \cdots\}\) verify the three term recurrence relation:

\[ xP_n = P_{n+1} + \alpha_{n+1}P_n + \omega_nP_{n-1}, \quad P_0 = 1, \quad P_1 = x - \alpha_1, \]

where \((\{\omega_n\}, \{\alpha_n\})\) is called the Jacobi parameters of \(\mu\).

An isometry \(U : \Gamma(G) \to L^2(\mathbb{R}, \mu)\) is defined by \(\Phi_n \mapsto \|P_n\|^{-1}P_n\).

Then \(UAU^* = x\) (multiplication operator) and

\[ \langle \Phi_0, A^m\Phi_0 \rangle = \langle U\Phi_0, U A^mU^*U\Phi_0 \rangle = \langle P_0, x^mP_0\rangle_\mu = \int_{-\infty}^{+\infty} x^m\mu(dx). \]
5.4. Computing the Spectral Distribution

By the interacting Fock space structure \( (\Gamma(G), A^+, A^-, A^\circ) \):

\[
A\Phi_n = (A^+ + A^\circ + A^-)\Phi_n = \sqrt{\omega_{n+1}} \Phi_{n+1} + \alpha_{n+1} \Phi_n + \sqrt{\omega_n} \Phi_{n-1}.
\]

Associated with a probability distribution \( \mu \) on \( \mathbb{R} \), the orthogonal polynomials \( \{P_0(x) = 1, \ldots, P_n(x) = x^n + \cdots\} \) verify the three term recurrence relation:

\[
xP_n = P_{n+1} + \alpha_{n+1} P_n + \omega_n P_{n-1}, \quad P_0 = 1, \quad P_1 = x - \alpha_1,
\]

where \( (\{\omega_n\}, \{\alpha_n\}) \) is called the Jacobi parameters of \( \mu \).

An isometry \( U : \Gamma(G) \to L^2(\mathbb{R}, \mu) \) is defined by \( \Phi_n \mapsto \|P_n\|^{-1}P_n \).

Then \( UAU^* = x \) (multiplication operator) and

\[
\langle \Phi_0, A^m \Phi_0 \rangle = \langle U\Phi_0, U A^m U^* U \Phi_0 \rangle = \langle P_0, x^m P_0 \rangle_\mu = \int_{-\infty}^{+\infty} x^m \mu(dx).
\]

Theorem (Graph structure \( \Gamma(G) \) \( \mapsto \) \( \{\omega_n\}, \{\alpha_n\} \) \( \mapsto \) spectral distribution)

If \( \Gamma(G) \) is invariant under \( A^+, A^-, A^\circ \), the vacuum spectral distribution \( \mu \) defined by

\[
\langle \Phi_0, A^m \Phi_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \ldots,
\]

is a probability distribution on \( \mathbb{R} \) that has the Jacobi parameters \( (\{\omega_n\}, \{\alpha_n\}) \).
5.5. How to know $\mu$ from the Jacobi Parameters ($\{\omega_n\}$, $\{\alpha_n\}$)

We need to find a probability distribution $\mu$ for which the orthogonal polynomials satisfy

$$xP_n = P_{n+1} + \alpha_{n+1}P_n + \omega_nP_{n-1}, \quad P_0 = 1, \quad P_1 = x - \alpha_1,$$
5.5. How to know $\mu$ from the Jacobi Parameters ($\{\omega_n\}, \{\alpha_n\}$)

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$$xP_n = P_{n+1} + \alpha_{n+1}P_n + \omega_nP_{n-1}, \quad P_0 = 1, \quad P_1 = x - \alpha_1,$$

**Cauchy–Stieltjes transform**

$$G_{\mu}(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z - \alpha_1} - \frac{\omega_1}{z - \alpha_2} - \frac{\omega_2}{z - \alpha_3} - \frac{\omega_3}{z - \alpha_4} - \cdots$$

where the right-hand side is convergent in $\{\text{Im } z \neq 0\}$ if the moment problem is determinate, e.g., if $\omega_n = O((n \log n)^2)$ (Carleman’s test).
5.5. How to know $\mu$ from the Jacobi Parameters (cont)

Cauchy–Stieltjes transform

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \frac{\omega_3}{z-\alpha_4} - \cdots$$

Stieltjes inversion formula

The (right-continuous) distribution function $F(x) = \mu((\mathbb{R}, x])$ and the absolutely continuous part of $\mu$ is given by

$$\frac{1}{2}\{F(t) + F(t-0)\} - \frac{1}{2}\{F(s) + F(s-0)\}$$

$$= -\frac{1}{\pi} \lim_{y\to+0} \int_s^t \operatorname{Im} G_\mu(x+iy) dx, \quad s < t,$$

$$\rho(x) = -\frac{1}{\pi} \lim_{y\to+0} \operatorname{Im} G_\mu(x+iy)$$
5.5. Illustration: Homogeneous tree $\mathcal{T}_\kappa$ ($\kappa \geq 2$)

Stratification of $\mathcal{T}_4$
5.5. Illustration: Homogeneous tree $\mathcal{T}_\kappa$ ($\kappa \geq 2$)

Stratification of $\mathcal{T}_4$

(1) Quantum decomposition: $A = A^+ + A^-$

$$A^+ \Phi_0 = \sqrt{\kappa} \Phi_1, \quad A^+ \Phi_n = \sqrt{\kappa - 1} \Phi_{n+1} \quad (n \geq 1)$$

$$A^- \Phi_0 = 0, \quad A^- \Phi_1 = \sqrt{\kappa} \Phi_0, \quad A^- \Phi_n = \sqrt{\kappa - 1} \Phi_{n-1} \quad (n \geq 2)$$
5.5. Illustration: Homogeneous tree $\mathcal{T}_\kappa (\kappa \geq 2)$

Stratification of $\mathcal{T}_4$

\[ A = A^+ + A^- \]

\[ A^+ \Phi_0 = \sqrt{\kappa} \Phi_1, \quad A^+ \Phi_n = \sqrt{\kappa - 1} \Phi_{n+1} \quad (n \geq 1) \]
\[ A^- \Phi_0 = 0, \quad A^- \Phi_1 = \sqrt{\kappa} \Phi_0, \quad A^- \Phi_n = \sqrt{\kappa - 1} \Phi_{n-1} \quad (n \geq 2) \]

(2) Jacobi parameters: $\{ \omega_1 = \kappa, \omega_2 = \omega_3 = \cdots = \kappa - 1 \}, \{ \alpha_n \equiv 0 \}$
(3) Cauchy–Stieltjes transform: \((\omega_1 = \kappa, \omega_2 = \omega_3 = \cdots = \kappa - 1)\)

\[
\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = G_\mu(z) = \frac{1}{z} - \frac{\omega_1}{z} - \frac{\omega_2}{z} - \frac{\omega_3}{z} - \frac{\omega_4}{z} - \frac{\omega_5}{z} - \cdots
\]

\[
= \frac{(\kappa - 2)z - \kappa \sqrt{z^2 - 4(\kappa - 1)}}{2(\kappa^2 - z^2)}
\]
(3) Cauchy–Stieltjes transform: \( \omega_1 = \kappa, \omega_2 = \omega_3 = \cdots = \kappa - 1 \)

\[
\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z - x} = G_\mu(z) = \frac{1}{z} - \frac{\omega_1}{z} - \frac{\omega_2}{z} - \frac{\omega_3}{z} - \frac{\omega_4}{z} - \frac{\omega_5}{z} - \cdots
\]

\[
= \frac{(\kappa - 2)z - \kappa \sqrt{z^2 - 4(\kappa - 1)}}{2(\kappa^2 - z^2)}
\]

(4) Spectral distribution: \( \mu(dx) = \rho_\kappa(x)dx \)

\[
\rho_\kappa(x) = \frac{\kappa}{2\pi} \frac{\sqrt{4(\kappa - 1) - x^2}}{\kappa^2 - x^2}
\]

\[|x| \leq 2\sqrt{\kappa - 1}\]

Kesten Measures (1959)
6. Method of Quantum Decomposition for Growing Graphs
6.1. Growing Regular Graphs $G^{(\nu)} = (V^{(\nu)}, A^{(\nu)})$

Here $\Gamma(G^{(\nu)})$ is not necessarily invariant but asymptotically invariant under $A^{\epsilon}_{\nu}$.

Statistics of $\omega_\epsilon(x)$

\[
M(\omega_\epsilon|V_n) = \frac{1}{|V_n|} \sum_{x \in V_n} |\omega_\epsilon(x)|
\]

\[
\Sigma^2(\omega_\epsilon|V_n) = \frac{1}{|V_n|} \sum_{x \in V_n} \left\{ |\omega_\epsilon(x)| - M(\omega_\epsilon|V_n) \right\}^2
\]

\[
L(\omega_\epsilon|V_n) = \max\{|\omega_\epsilon(x)|; x \in V_n\}.
\]
6.1. Growing Regular Graphs $G^{(\nu)} = (V^{(\nu)}, A^{(\nu)})$

Here $\Gamma(G^{(\nu)})$ is not necessarily invariant but asymptotically invariant under $A^{\epsilon}_{\nu}$.

Statistics of $\omega_\epsilon(x)$

$$M(\omega_\epsilon|V_n) = \frac{1}{|V_n|} \sum_{x \in V_n} |\omega_\epsilon(x)|$$

$$\Sigma^2(\omega_\epsilon|V_n) = \frac{1}{|V_n|} \sum_{x \in V_n} \{ |\omega_\epsilon(x)| - M(\omega_\epsilon|V_n) \}^2$$

$$L(\omega_\epsilon|V_n) = \max\{|\omega_\epsilon(x)|; x \in V_n\}.$$ 

**Conditions for a growing regular graph $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$**

(A1) $\lim_{\nu} \deg(G^{(\nu)}) = \infty.$

(A2) for each $n = 1, 2, \ldots,$

$$\exists \lim_{\nu} M(\omega_-|V_n^{(\nu)}) = \omega_n < \infty, \quad \lim_{\nu} \Sigma^2(\omega_-|V_n^{(\nu)}) = 0, \quad \sup_{\nu} L(\omega_-|V_n^{(\nu)}) < \infty.$$ 

(A3) for each $n = 0, 1, 2, \ldots,$

$$\exists \lim_{\nu} \frac{M(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} = \alpha_{n+1} < \infty, \quad \lim_{\nu} \frac{\Sigma^2(\omega_\circ|V_n^{(\nu)})}{\kappa(\nu)} = 0, \quad \sup_{\nu} \frac{L(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty.$$
Let \( \{ G^{(\nu)} = (V^{(\nu)}, E^{(\nu)}) \} \) be a growing regular graph satisfying

(A1) \( \lim_{\nu} \kappa(\nu) = \infty \), where \( \kappa(\nu) = \text{deg}(G^{(\nu)}) \).

(A2) for each \( n = 1, 2, \ldots \),

\[
\exists \lim_{\nu} M(\omega | V_n^{(\nu)}) = \omega_n < \infty, \quad \lim_{\nu} \Sigma^2(\omega | V_n^{(\nu)}) = 0, \quad \sup_{\nu} L(\omega | V_n^{(\nu)}) < \infty.
\]

(A3) for each \( n = 0, 1, 2, \ldots \),

\[
\exists \lim_{\nu} \frac{M(\omega_0 | V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} = \alpha_{n+1} < \infty, \quad \lim_{\nu} \frac{\Sigma^2(\omega_0 | V_n^{(\nu)})}{\kappa(\nu)} = 0, \quad \sup_{\nu} \frac{L(\omega_0 | V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty.
\]

Then the asymptotic spectral distribution of the normalized \( A_{\nu} \) in the vacuum state is a probability distribution \( \mu \) determined by \( (\{ \omega_n \}, \{ \alpha_n \}) \):

\[
\lim_{\nu} \left\langle \left( \frac{A_{\nu}}{\sqrt{\kappa(\nu)}} \right)^m \right\rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \ldots.
\]

Proof is to show that

\[
\lim_{\nu} \frac{A_{\nu}^\epsilon}{\sqrt{\kappa(\nu)}} = B^\epsilon \quad \text{(stochastically)}
\]

where \( (\Gamma, \{ \Psi_n \}, B^+, B^-, B^0) \) is the interacting Fock space associated with the Jacobi parameters \( (\{ \omega_n \}, \{ \alpha_n \}) \).
Outline of Proof

(1) \[ \frac{A^\varepsilon}{\sqrt{\kappa}} \Phi_n = \gamma_{n+\varepsilon}^n \Phi_{n+\varepsilon} + S_{n+\varepsilon}^\varepsilon, \quad \varepsilon \in \{+, -, \circ\}, \quad n = 0, 1, 2, \ldots. \]

\[ \gamma_n^+ = M(\omega_- |V_n|) \left( \frac{|V_n|}{\kappa |V_{n-1}|} \right)^{1/2}, \quad \gamma_n^- = M(\omega_+ |V_n|) \left( \frac{|V_n|}{\kappa |V_{n+1}|} \right)^{1/2}, \quad \gamma_n^{\circ} = \frac{M(\omega_0 |V_n|)}{\sqrt{\kappa}}. \]

(2) \[ |V_n| = \left\{ \prod_{k=1}^{n} M(\omega_- |V_k|) \right\}^{-1} \kappa^n + O(\kappa^{n-1}). \]

(3) \[ \lim_{\nu} \gamma_n^+ = \sqrt{\omega_n}, \quad \lim_{\nu} \gamma_n^- = \sqrt{\omega_{n+1}}, \quad \lim_{\nu} \gamma_n^{\circ} = \alpha_{n+1}. \]

(4) \[ \frac{A^{\varepsilon_m}}{\sqrt{\kappa}} \cdots \frac{A^{\varepsilon_1}}{\sqrt{\kappa}} \Phi_n = \gamma_{n+\varepsilon_1} \gamma_{n+\varepsilon_1+\varepsilon_2} \cdots \gamma_{n+\varepsilon_1+\cdots+\varepsilon_m} \Phi_{n+\varepsilon_1+\cdots+\varepsilon_m} \]

\[ + \sum_{k=1}^{m} \gamma_{n+\varepsilon_1} \cdots \gamma_{n+\varepsilon_k-1} \frac{A^{\varepsilon_k}}{\sqrt{\kappa}} \cdots \frac{A^{\varepsilon_k+1}}{\sqrt{\kappa}} S_n^{\varepsilon_k} \]

(k - 1) times \quad (m - k) times

(5) Estimate the error terms and show that

\[ \lim_{\nu} \left\langle \Phi^{(\nu)}_j, \frac{A^{\varepsilon_m}}{\sqrt{\kappa(\nu)}} \cdots \frac{A^{\varepsilon_k}}{\sqrt{\kappa(\nu)}} S_n^{\varepsilon_k} \right\rangle = 0. \]
6.2. Illustration: $Z^N$ as $N \to \infty$

Asymptotic invariance of $\Gamma(Z^N)$ under $A \epsilon$:

$A^+ \Phi_n = \sqrt{2N} \sqrt{n} \Phi_n + O(1)$,

$A^- \Phi_n = \sqrt{2N} \Phi_n - \Phi_n^{-1} + O(N^{-1/2})$.

Normalized adjacency matrices:

$A^\epsilon \sqrt{\kappa(N)} = A^\epsilon \sqrt{2N}$.

The interacting Fock space in the limit as $N \to \infty$ is the Boson Fock space:

$B^+ \Psi_n = \sqrt{n} \Psi_n + 1 \Psi_n + 1$, $B^- \Phi_n = \sqrt{n} \Phi_n - \Phi_n^{-1}$, $B^\circ = 0$.

The asymptotic spectral distribution is the standard Gaussian distribution:

$\lim_{N \to \infty} \langle \Phi_0, (A_N \sqrt{2N})^m \Phi_0 \rangle = \langle \Psi_0, (B^+ + B^-)^m \Psi_0 \rangle = 1 / \sqrt{2\pi} \int_{-\infty}^{+\infty} x^m e^{-x^2/2} dx$. 

Nobuaki Obata (GSIS, Tohoku University)
6.2. Illustration: $Z^N$ as $N \to \infty$

1. Asymptotic invariance of $\Gamma(Z^N)$ under $A^\epsilon$:

$$A^+ \Phi_n = \sqrt{2N} \sqrt{n+1} \Phi_{n+1} + O(1),$$

$$A^- \Phi_n = \sqrt{2N} \sqrt{n} \Phi_{n-1} + O(N^{-1/2}).$$

2. Normalized adjacency matrices:

$$\frac{A^\epsilon}{\sqrt{\kappa(N)}} = \frac{A^\epsilon}{\sqrt{2N}}.$$

3. The interacting Fock space in the limit as $N \to \infty$ is the Boson Fock space:

$$B^+ \Psi_n = \sqrt{n+1} \Psi_{n+1}, \quad B^- \Phi_n = \sqrt{n} \Phi_{n-1}, \quad B^\circ = 0.$$

4. The asymptotic spectral distribution is the standard Gaussian distribution:

$$\lim_{N \to \infty} \left\langle \Phi_0, \left( \frac{A_N}{\sqrt{2N}} \right)^m \Phi_0 \right\rangle = \left\langle \Psi_0, (B^+ + B^-)^m \Psi_0 \right\rangle$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^m e^{-x^2/2} dx.$$
6.3. Some Results: Asymptotic Spectral Distributions

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<td>odd graphs $O_k$</td>
<td>$\omega_{2n-1} = n$ $\omega_{2n} = n$</td>
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<td>homogeneous trees $\mathcal{T}_k$</td>
<td>$\omega_n = 1$ (free)</td>
<td>Wigner semicircle</td>
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<tr>
<td>integer lattices $\mathbb{Z}^N$</td>
<td>$\omega_n = n$ (Boson)</td>
<td>Gaussian</td>
<td>Gaussian</td>
</tr>
<tr>
<td>symmetric groups $\mathfrak{S}_n$ (Coxeter)</td>
<td>$\omega_n = n$ (Boson)</td>
<td>Gaussian</td>
<td>Gaussian</td>
</tr>
<tr>
<td>Coxeter groups (Fendler)</td>
<td>$\omega_n = 1$ (free)</td>
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<td>Spidernets $S(a, b, c)$</td>
<td>$\omega_1 = 1$ $\omega_2 = \cdots = r$</td>
<td>free Meixner law</td>
<td>?</td>
</tr>
</tbody>
</table>
Quantum Probabilistic Approach to Spectral Analysis of Graphs

adjacency matrix $A_\nu$

quantum decomposition $A_\nu = A_\nu^+ + A_\nu^- + A_\nu^0$

quantum CLT $\lim \frac{A_\nu^\epsilon}{Z_\nu} = B^\epsilon$

interacting Fock space $\Gamma(\{\omega_n\}, \{\alpha_n\}) B^\pm, B^\circ$

use of independence $A_\nu = X_1 + \cdots + X_\nu$

orthogonal polynomials $\{P_n\}, \{\omega_n\}, \{\alpha_n\}$

combinatorics

linear algebra

spectral distribution $\mu_\nu$

Classical

scaling limit

Quantum

$\nu A\nu A\nu A\nu A\nu A\nu$

$\nu X X$

$B\nu B\nu$

$\alpha \omega$

$\Gamma(n\alpha) B\nu B\nu$

$B^\pm, B^\circ = \alpha_{N+1}$
Quantum probability is an extension of classical probability based on non-commutative algebras

1) algebraic probability space \((\mathcal{A}, \varphi)\) and random variables \(a \in \mathcal{A}\)
2) various concepts of independence
3) and associated quantum central limit theorems (QCLTs)

Quantum probabilistic techniques are useful for spectral analysis of graphs

1) quantum decomposition and use of one-mode interacting Fock spaces
2) concepts of independence and product structures
3) asymptotic spectral distribution as a result of QCLT

Some future directions

1) more on product structure of graphs and independence
2) extending the method of quantum decomposition beyond one-mode interacting Fock spaces (orthogonal polynomials in one variable)
3) digraphs (directed graphs)
4) applications to dynamics on graphs, e.g., coupled oscillators, random walks (or more generally Markov chains), quantum walks, etc.