量子確率論の最近の話題から
～スペクトルグラフ理論とフォック空間あたり～

尾畑 伸明

東北大学情報科学研究科

ISM, July 6, 2015

0.1. Some Backgrounds

1900 Max Planck found “quanta” sparking the “old quantum theory”

1925 Matrix mechanics by Heisenberg $\Rightarrow$ uncertainty principle (1927)

1926 Wave mechanics by Schrödinger $\Rightarrow$ probabilistic interpretation by Born

1932 J. von Neumann: “Mathematische Grundlagen der Quantenmechanik”

1933 A. Kolmogorov: “Grundbegriffe der Wahrscheinlichkeitsrechnung”

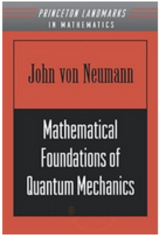
A germ of new probability theory — von Neumann

random variable $X \Longleftrightarrow$ selfadjoint operator (observable) A

Lebesgue measure $\Longleftrightarrow$ trace

density function $f(x)dx \Longleftrightarrow$ density operator (state) ρ

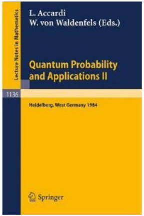
$$\mathbf{E}[X] = \int_{-\infty}^{+\infty} x f(x)dx \Longleftrightarrow \mathrm{Tr}\,(\rho A)$$



But the quantum counterparts of probabilistic concepts such as random variable, noise, stochastic process, conditional probability, independence, dependence, Markovianity,... were not yet established.

0.2. Some Achievements of Quantum Probability

LNM (1984)

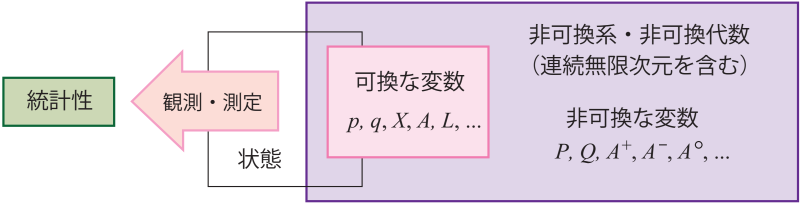


— "Quantum Probability" appeared in the late 1970s.
— It is a generalization of probability theory in such a way that random variables are not assumed to commute.
— Developing quantum version of probabilistic concepts for applications to quantum theory and its probabilistic interpretation.
— An alternative name is *non-commutative probability theory*.
— Some researchers in the first generation:

Accardi (I), Belavkin (Rus), Hudson (UK), Meyer (F), Parthasarathy (India), von Waldenfels (D), ..., Gudder (US), Bożejko (P), Voiculescu (Rou),...

1. Quantum open system and quantum dissipation
2. Quantum stochastic approach to unitary dilation problem
3. Stochastic limit theory for micro-macro relations
4. Constructive models of quantum observation processes
5. Quantum stochastic filtering and feedback control theory
6. Quantum information
7. Providing new mathematical aspects and methods (based on non-commutative nature) for classical subjects

0.3. Working Hypothesis or ...



1. Quantization: classical variables $p, q \Longrightarrow$ non-commuting operators P, Q
2. Quantm stochastic calculus (Hudson-Parthasarathy, 1984)
   quantum decomposition of Brownian $B(t) = A(t) + A^*(t)$
   the Itô formula $(dB)^2 = dt$ is a consequence of CCR $[dA, dA^*] = dt$.
3. Gassianization of probability distribution (Accardi-Bozejko, 1998)
   quantum decomposition of a random variable $X = A^+ + A^\circ + A^-$
4. Quantum field and stochastic analysis: non-commutative + infinite dimension
5. Quantum walks: classical random walk $p + q = 1 \Longrightarrow P + Q = U$
6. Spectra of graphs: adjacency matrix, Laplacian matrix, ... quantum decomposition
7. Algebraic combinatorics: Association scheme, Terwillinger algebra, ...

Plan

1. Quantum Probability
2. Quantum Probabilistic Approach to Spectral Graph Theory
3. Graph Products and Concepts of Independence
4. Asymptotic Spectral Distributions of Growing Graphs
5. Quantum White Noise Calculus
6. Implementation Problems

1. Quantum Probability
   = Noncommutative Probability
   = Algebraic Probability

- K. R. Parthasarathy: "An Introduction to Quantum Stochastic Calculus," Birkhäuser, 1992.
- P.-A. Meyer: "Quantum Probability for Probabilists," Lect. Notes in Math. Vol. 1538, Springer, 1993.
- L. Accardi, Y. G. Lu and I. Volovich: "Quantum Theory and Its Stochastic Limit," Springer, 2002.

Traditional Model for Coin-toss

A random variable X on a probability space $(\Omega, \mathcal{F}, P)$ satisfying the property:

$$P(X = +1) = P(X = -1) = \frac{1}{2}$$

More essential is the probability distribution of X:

$$\mu_X = \frac{1}{2}\,\delta_{-1} + \frac{1}{2}\,\delta_{+1}$$

Moment sequence is one of the most fundamental characteristics

Moment sequence $\{M_m\}$ $\iff$ Probability distributions μ

(up to determinate moment problem)

For a coin-toss we have

$$M_m(\mu_X) = \int_{-\infty}^{+\infty} x^m \mu_X(dx) = \begin{cases} 1, & \text{if } m \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}$$

1.1. Let's Start with Coin-toss (cont)

1. Set

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad e_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

2. It is straightforward to see that

$$\langle e_0, A^m e_0 \rangle = \left\{ \begin{array}{ll} 1, & \text{if } m \text{ is even,} \\ 0, & \text{otherwise,} \end{array} \right\} = M_m(\mu_X) = \int_{-\infty}^{+\infty} x^m \mu_X(dx).$$

3. In other words, we have another model of coin-toss by means of $(\mathcal{A}, \varphi)$, where

$$\mathcal{A} = *\text{-algebra generated by } A; \quad \varphi(a) = \langle e_0, a e_0 \rangle, \quad a \in \mathcal{A},$$

4. We call A an *algebraic realization* of the random variable X.

Non-commutative structure emerges – quantum decomposition

$$(\text{coin toss } X) = A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = A^+ + A^-.$$

1.2. Axioms: Quantum Probability

Definition (Algebraic probability space)

An *algebraic probability space* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a $*$-algebra over $\mathbb{C}$ with multiplication unit $1_{\mathcal{A}}$, and a state $\varphi : \mathcal{A} \to \mathbb{C}$, i.e.,

$$\text{(i) } \varphi \text{ is linear;} \quad \text{(ii) } \varphi(a^* a) \geq 0; \quad \text{(iii) } \varphi(1_{\mathcal{A}}) = 1.$$

Each $a \in \mathcal{A}$ is called an *(algebraic) random variable*.

Definition (Spectral distribution)

For a real random variable $a = a^* \in \mathcal{A}$ there exists a probability measure $\mu = \mu_a$ on $\mathbb{R} = (-\infty, +\infty)$ such that

$$\varphi(a^m) = \int_{-\infty}^{+\infty} x^m \mu(dx) \equiv M_m(\mu), \quad m = 1, 2, \dots.$$

This μ is called the *spectral distribution* of a (with respect to the state φ).

- Existence of μ by Hamburger's theorem using Hanckel determinants.
- In general, μ is not uniquely determined (indeterminate moment problem).

1.3. Comparison with Classical Probability

	Classical Probability	Quantum Probability
probability space	$(\Omega, \mathcal{F}, P)$	$(\mathcal{A}, \varphi)$
random variable	$X : \Omega \to \mathbb{R}$	$a = a^* \in \mathcal{A}$
expectation	$\mathbf{E}[X] = \int_\Omega X(\omega)\, P(d\omega)$	$\varphi(a)$
moments	$\mathbf{E}[X^m]$	$\varphi(a^m)$
distribution	$\mu_X((-\infty, x]) = P(X \le x)$	NA
	$\mathbf{E}[X^m] = \int_{-\infty}^{+\infty} x^m \mu_X(dx)$	$\varphi(a^m) = \int_{-\infty}^{+\infty} x^m \mu_a(dx)$
independence	$\mathbf{E}[X^m Y^n] = \mathbf{E}[X^m]\mathbf{E}[Y^n]$	*
LLN	$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_k$	*
CLT	$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k$	*

1.4. A Non-classical Independence: Monotone Independence

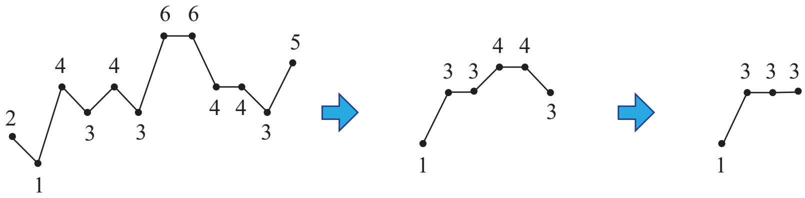
Definition (monotone independence)

Let $(\mathcal{A}, \varphi)$ be an algebraic probability space and $\{\mathcal{A}_n\}$ a family of $*$-subalgebras. We say that $\{\mathcal{A}_n\}$ is *monotone independent* if for $a_1 \in \mathcal{A}_{n_1}, \dots, a_m \in \mathcal{A}_{n_m}$ we have

$$\varphi(a_1 \cdots a_m) = \varphi(a_i)\varphi(a_1 \cdots \check{a}_i \cdots a_m) \qquad (\check{a}_i \text{ stands for omission})$$

holds when $n_{i-1} < n_i$ and $n_i > n_{i+1}$ happen for $i \in \{1, 2, \dots, m\}$.

Illustration: $a_1 \in \mathcal{A}_2, a_2 \in \mathcal{A}_1, a_3 \in \mathcal{A}_4, \dots$



$$\varphi(214343664435) = \varphi(2)\varphi(4)\varphi(4)\varphi(66)\varphi(133443)\varphi(5)$$
$$= \varphi(2)\varphi(4)\varphi(4)\varphi(66)\varphi(44)\varphi(1333)\varphi(5)$$
$$= \varphi(2)\varphi(4)\varphi(4)\varphi(66)\varphi(44)\varphi(333)\varphi(1)$$

Monotone Central Limit Theorem (Muraki 2001)

Let $(\mathcal{A}, \varphi)$ be an algebraic probability space and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of algebraic random variables satisfying the following conditions:

(i) a_n are real, i.e., $a_n = a_n^*$;

(ii) a_n are normalized, i.e., $\varphi(a_n) = 0$, $\varphi(a_n^2) = 1$;

(iii) a_n have uniformly bounded mixed moments, i.e., for each $m \geq 1$ there exists $C_m \geq 0$ such that $|\varphi(a_{n_1} \dots a_{n_m})| \leq C_m$ for any choice of $n_1, \dots, n_m$.

(iv) $\{a_n\}$ are monotone independent.

Then,

$$\lim_{N\to\infty} \varphi\left(\left\{\frac{1}{\sqrt{N}}\sum_{i=1}^{N} a_i\right\}^m\right) = \frac{1}{\pi}\int_{-\sqrt{2}}^{+\sqrt{2}} \frac{x^m}{\sqrt{2-x^2}}\,dx, \qquad m = 1, 2, \dots.$$

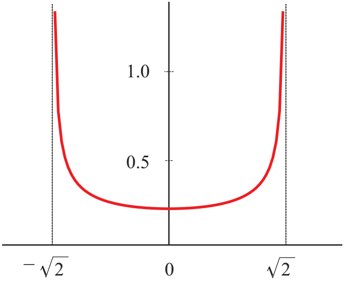
The probability measure in the right hand side is the normalized *arcsine law*.

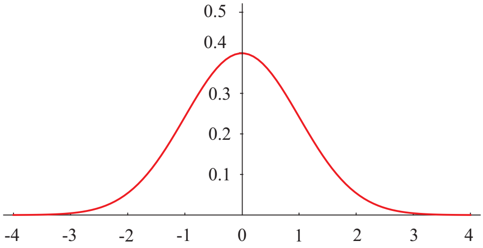
Proof is to show

$$\lim_{N\to\infty} \varphi\left(\left\{\frac{1}{\sqrt{N}}\sum_{i=1}^{N} a_i\right\}^{2m}\right) = \frac{(2m)!}{2^m m! m!} = \frac{1}{\pi}\int_{-\sqrt{2}}^{+\sqrt{2}} \frac{x^m}{\sqrt{2-x^2}}\,dx$$

Arcsine law vs normal (Gaussian) law

$$\frac{1}{\pi\sqrt{2-x^2}} \qquad \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$





Classical CLT in moment form

If $X_1, X_2, \dots$ are independent, identically distributed, normalized (mean zero, variance one) random variables having moments of all orders, we have

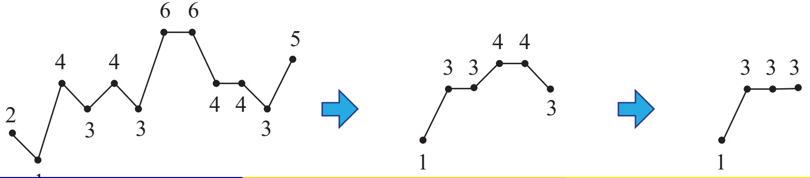
$$\lim_{N\to\infty} \mathbf{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} X_n\right)^m\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^m e^{-x^2/2}dx, \quad m = 1, 2, \dots .$$

- For the proof the factorization rule is essential:

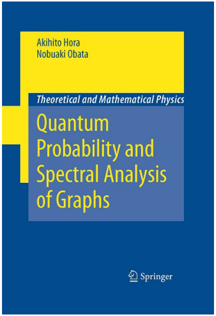
$$\begin{aligned}&\mathbf{E}[X_2X_1X_4X_3X_4X_3X_6X_6X_4X_4X_3X_5] \\ &\qquad = \mathbf{E}[X_1]\mathbf{E}[X_2]\mathbf{E}[X_3^3]\mathbf{E}[X_4^4]\mathbf{E}[X_5]\mathbf{E}[X_6^2]\end{aligned}$$

- Cf. monotone independence

$$\varphi(214343664435) = \varphi(2)\varphi(4)\varphi(4)\varphi(66)\varphi(44)\varphi(333)\varphi(1).$$



2. Quantum Probabilistic Approach to Spectral Graph Theory



A. Hora and N. Obata:
Quantum Probability and Spectral Analysis of Graphs,
Springer, 2007.

2.1. Adjacency Matrices and Adjacency Algebras

Definition (graph)

A (finite or infinite) *graph* is a pair $G = (V, E)$, where V is the set of *vertices* and E the set of *edges*. We write $x \sim y$ (adjacent) if they are connected by an edge.

Definition (adjacency matrix)

The *adjacency matrix* $A = [A_{xy}]$ is defined by $A_{xy} = \begin{cases} 1, & x \sim y, \\ 0, & \text{otherwise.} \end{cases}$

Assumption (1) (connected) Any pair of distinct vertices are connected by a walk.
(2) (locally finite) $\deg_G(x) = (\text{degree of } x) < \infty$ for all $x \in V$.

Definition (adjacency algebra)

Let $G = (V, E)$ be a graph. The $*$-algebra generated by the adjacency matrix A is called the *adjacency algebra* of G and is denoted by $\mathcal{A}(G)$. In fact, $\mathcal{A}(G)$ is the set of polynomials of A.

$\Longrightarrow$ Equipped with a state φ, $\mathcal{A}$ becomes an *algebraic probability space*

$\Longrightarrow$ Quantum probabilistic spectral analysis of graphs

2.2. States on $\mathcal{A}(G)$: Adjacency Algebras as Algebraic Probability Spaces

(i) *Trace* (when G is a finite graph)

$$\langle a \rangle_{\mathrm{tr}} = \frac{1}{|V|} \mathrm{Tr}\,(a) = \frac{1}{|V|} \sum_{x \in V} \langle \delta_x \,, a\delta_x \rangle$$

$\star$ $\langle A^m \rangle_{\mathrm{tr}} = \int_{-\infty}^{+\infty} x^m \mu(dx) \quad \Rightarrow \quad \mu$ is the eigenvalue distribution of A.

(ii) *Vacuum state* (at a fixed origin $o \in V$)

$$\langle a \rangle_o = \langle \delta_o \,, a\delta_o \rangle$$

$\star$ $\langle A^m \rangle_o = \langle \delta_o, A^m \delta_o \rangle = |\{m\text{-step walks from } o \text{ to } o\}| = \int_{-\infty}^{+\infty} x^m \mu(dx)$

(iii) *Deformed vacuum state by Q-matrix*

$$\langle a \rangle_q = \langle Q\delta_o \,, a\delta_o \rangle, \quad Q = [q^{\partial(x,y)}], \quad -1 \le q \le 1$$

- $Q\delta_o$ does not necessarily belong to $\ell^2(V)$, but $\langle a \rangle_q$ is well-defined since a is locally finite.
- Interesting to determine the domain of $q \in [-1, 1]$ for which $\langle \cdot \rangle_q$ is positive [see Bożejko (1989), Obata (2007, 2010)]

2.3. Main Problem and Quantum Probabilistic Approaches

Main Problem

Given a graph $G = (V, E)$ (resp. a growing graph) and a state $\langle \cdot \rangle$ on $\mathcal{A}(G)$, find a probability measure μ on $\mathbb{R}$ satisfying

$$\langle A^m \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx) \quad \left(\text{resp. } \langle A^m \rangle \approx \int_{-\infty}^{+\infty} x^m \mu(dx) \right), \quad m = 1, 2, \dots .$$

μ is called the *(asymptotic) spectral distribution* of A in the state $\langle \cdot \rangle$.

Quantum Probabilistic Approaches — Use of Non-Commutativity

1. Use of various independence and associated CLTs
2. Quantum decomposition

   closely related to orthogonal polynomials ($\approx$ one-mode ingteracting Fock spaces)

$$A = A^+ + A^- + A^\circ \qquad \text{(non-commuting quantum components)}$$

3. Partition statistics and moment-cumulant formulas (Various convolution products)

$$a_1 \sim \mu_1 \text{ and } a_2 \sim \mu_2 \text{ are independent} \quad \Longrightarrow \quad a_1 + a_2 \sim \exists\, \mu = \mu_1 * \mu_2$$

3. Graph Products and Concepts of Independence

3.1. Independence and Graph Structures (I) Cartesian product

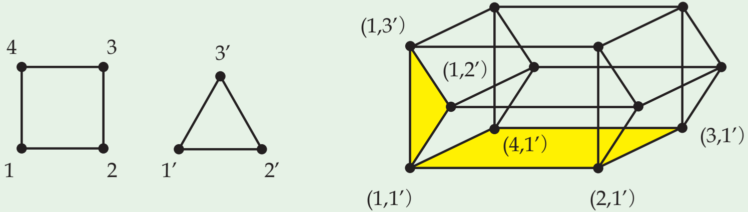
Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. For $(x, y), (x', y') \in V_1 \times V_2$ we write $(x, y) \sim (x', y')$ if one of the following conditions is satisfied:

(i) $x = x'$ *and* $y \sim y'$*;* *(ii)* $x \sim x'$ *and* $y = y'$*.*

Then $V_1 \times V_2$ becomes a graph in such a way that $(x, y), (x', y') \in V_1 \times V_2$ are adjacent if $(x, y) \sim (x', y')$. This graph is called the *Cartesian product* or *direct product* of G_1 and G_2, and is denoted by $G_1 \times G_2$.

Example ($C_4 \times C_3$)



C_4 C_3 $C_4 \times C_3$

Theorem

Let $G = G_1 \times G_2$ be the cartesian product of two graphs G_1 and G_2. Then the adjacency matrix of G admits a decomposition:

$$A = A_1 \otimes I_2 + I_1 \otimes A_2$$

as an operator acting on $\ell^2(V) = \ell^2(V_1 \times V_2) \cong \ell^2(V_1) \otimes \ell^2(V_2)$. Moreover, the right hand side is a sum of commutative independent random variables with respect to the vacuum state.

- The asymptotic spectral distribution is the Gaussian distribution by applying the commutative central limit theorem.

Example (Hamming graphs)

The Hamming graph is the Cartesian product of complete graphs:

$$H(d, N) \cong K_N \times \cdots \times K_N \quad (d\text{-fold cartesian power of complete graphs})$$

$$A_{d,N} = \sum_{i=1}^{n} \overbrace{I \otimes \cdots \otimes I}^{i-1} \otimes B \otimes \overbrace{I \otimes \cdots \otimes I}^{n-i}$$

where B is the adjacency matrix of K_N.

- Hora (1998) obtained the limit distribution by a direct calculation of eigenvalues.

3.2. Independence and Graph Structures (II) Comb product

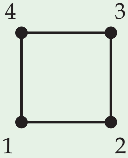
Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We fix a vertex $o_2 \in V_2$. For $(x, y), (x', y') \in V_1 \times V_2$ we write $(x, y) \sim (x', y')$ if one of the following conditions is satisfied:

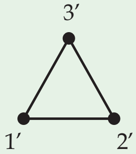
(i) $x = x'$ *and* $y \sim y'$; *(ii)* $x \sim x'$ *and* $y = y' = o_2$.

Then $V_1 \times V_2$ becomes a graph, denoted by $G_1 \rhd_{o_2} G_2$, and is called the *comb product* or the *hierarchical product*.

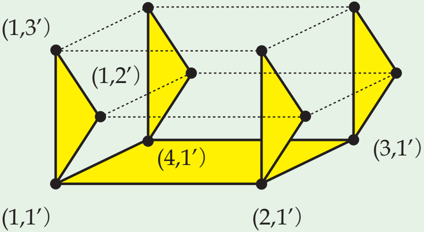
Example ($C_4 \rhd C_3$)



C_4



C_3



$C_4 \rhd C_3$

Theorem

As an operator on $C_0(V_1) \otimes C_0(V_2)$ *the adjacency matrix of* $G_1 \rhd_{o_2} G_2$ *is given by*

$$A = A_1 \otimes P_2 + I_1 \otimes A_2$$

where $P_2 : C_0(V_2) \to C_0(V_2)$ *is the projection onto the space spanned by* δ_{o_2} *and* I_1 *is the identity matrix acting on* $C_0(V_1)$.

Theorem (Accardi–Ben Ghobal–O. IDAQP(2004))

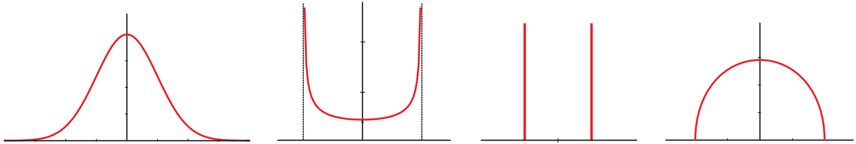
The adjacency matrix of the comb product $\mathcal{G}^{(1)} \rhd_{o_2} \mathcal{G}^{(2)} \rhd_{o_3} \cdots \rhd_{o_n} \mathcal{G}^{(n)}$ *admits a decomposition of the form:*

$$A^{(1)} \rhd A^{(2)} \rhd \cdots \rhd A^{(n)}$$
$$= \sum_{i=1}^{n} \overbrace{I^{(1)} \otimes \cdots \otimes I^{(i-1)}}^{i-1} \otimes A^{(i)} \otimes \overbrace{P^{(i+1)} \otimes \cdots \otimes P^{(n)}}^{n-i},$$

where $P^{(i)}$ *the projection from* $\ell^2(V^{(i)})$ *onto the one-dimensional subspace spanned by* δ_{o_i}. *Moreover, the right-hand side is a sum of monotone independent random variables with respect to* $\psi \otimes \delta_{o_2} \otimes \cdots \otimes \delta_{o_n}$, *where* ψ *is an arbitrary state on* $\mathcal{B}(\ell^2(V^{(1)}))$.

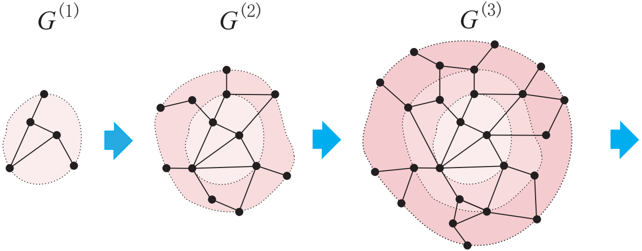
3.3. Four Concepts of Independence and Beyond

independence	commutative	monotone	Boolean	free
CLM	Gaussian	arcsine	Bernoulli	Wigner
graph product	cartesian	comb	star	free
examples	integer lattice	comb graph	star graph	homogeneous tree



1. Asymptotic spectral distribution of a graph product follows from QCLT.
   $$G^{(n)} = G\#G\#\dots\#G \text{ as } n \to \infty.$$
2. The above four concepts of independence look fundamental (Speicher, Muraki, Franz, ...) while many other notions have been proposed.
3. Further generalization (graph products or graph compositions).
4. Digraphs and beyond.

4. Asymptotic Spectral Distributions of Growing Graphs



Fix an origin $o \in V$ of $G = (V, E)$.

Stratification (Distance Partition)

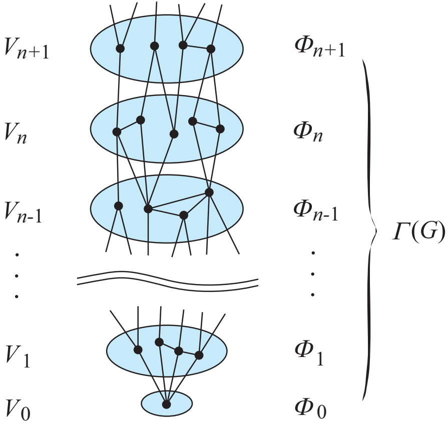
$$V = \bigcup_{n=0}^{\infty} V_n$$

$$V_n = \{x \in V \,;\, \partial(o, x) = n\}$$

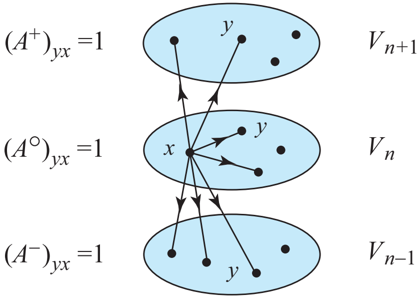
Associated Hilbert space $\Gamma(G) \subset \ell^2(V)$

$$\Gamma(G) = \sum_{n=0}^{\infty} \oplus \mathbf{C}\Phi_n$$

$$\Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} \delta_x$$



4.1. Quantum Decomposition of the Adjacency Matrix (cont)



$$A = A^+ + A^- + A^\circ$$
$$(A^+)^* = A^-, \quad (A^\circ)^* = A^\circ$$

Cases so far studied in detail

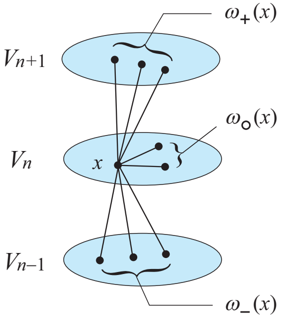
In general, $\Gamma(G) = \sum_{n=0}^{\infty} \oplus \mathbf{C}\Phi_n$ is *not invariant* under the actions of A^ϵ.

1. $\Gamma(G)$ is invariant under A^ϵ — e.g., distance-regular graphs
2. $\Gamma(G)$ is asymptotically invariant under A^ϵ.

For $x \in V_n$ we define

$$\omega_\epsilon(x) = |\{y \in V_{n+\epsilon}\,;\, y \sim x\}|, \quad \epsilon = +, -, \circ$$

Then, $\Gamma(G)$ is invariant under A^ϵ if and only if
$(*)$ $\omega_\epsilon(x)$ is *constant* on each V_n .



Typical examples: distance-regular graphs
(in this case the constant in $(*)$ is independent of the choice of $o \in V$) e.g., homogeneous trees, Hamming graphs, Johnson graphs, odd graphs, ...

Theorem

If $\Gamma(G)$ is invariant under A^+, A^-, A°, there exists a pair of sequences $\{\alpha_n\}$ and $\{\omega_n\}$ such that

$$A^+\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1}, \quad A^-\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1}, \quad A^\circ\Phi_n = \alpha_{n+1}\Phi_n.$$

In other words, $\Gamma_{\{\omega_n\},\{\alpha_n\}} = (\Gamma(G), \{\Phi_n\}, A^+, A^-, A^\circ)$ is an interacting Fock space associated with Jacobi parameters $(\{\omega_n\}, \{\alpha_n\})$.

4.3. Computing the Spectral Distribution

1. By the interacting Fock space structure $(\Gamma(G), A^+, A^-, A^\circ)$:

$$A\Phi_n = (A^+ + A^\circ + A^-)\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1} + \alpha_{n+1}\Phi_n + \sqrt{\omega_n}\,\Phi_{n-1}.$$

2. Associated with a probability distribution μ on $\mathbb{R}$, the orthogonal polynomials $\{P_0(x) = 1, \dots, P_n(x) = x^n + \cdots\}$ verify the three term recurrence relation:

$$xP_n = P_{n+1} + \alpha_{n+1}P_n + \omega_n P_{n-1}\,, \quad P_0 = 1, \quad P_1 = x - \alpha_1,$$

where $(\{\omega_n\}, \{\alpha_n\})$ is called the *Jacobi parameters* of μ.

3. An isometry $U : \Gamma(G) \to L^2(\mathbb{R}, \mu)$ is defined by $\Phi_n \mapsto \|P_n\|^{-1}P_n$.
4. Then $UAU^* = x$ (multiplication operator) and

$$\langle \Phi_0, A^m\Phi_0\rangle = \langle U\Phi_0, UA^mU^*U\Phi_0\rangle = \langle P_0, x^m P_0\rangle_\mu = \int_{-\infty}^{+\infty} x^m \mu(dx).$$

Theorem (Graph structure $\Longrightarrow$ $(\{\omega_n\}, \{\alpha_n\})$ $\Longrightarrow$ spectral distribution)

If $\Gamma(G)$ is invariant under A^+, A^-, A°, the vacuum spectral distribution μ defined by

$$\langle \Phi_0, A^m\Phi_0\rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \dots,$$

is a probability distribution on $\mathbb{R}$ that has the Jacobi parameters $(\{\omega_n\}, \{\alpha_n\})$.

4.4. How to know μ from the Jacobi Parameters $(\{\omega_n\}, \{\alpha_n\})$

We need to find a probability distribution μ for which the orthogonal polynomials satisfy

$$xP_n = P_{n+1} + \alpha_{n+1}P_n + \omega_n P_{n-1}\,, \quad P_0 = 1, \quad P_1 = x - \alpha_1,$$

Cauchy–Stieltjes transform

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \frac{\omega_3}{z-\alpha_4} - \cdots$$

$$= \cfrac{1}{z-\alpha_1 - \cfrac{\omega_1}{z-\alpha_2 - \cfrac{\omega_2}{z-\alpha_3 - \cfrac{\omega_3}{z-\alpha_4 - \cdots}}}}$$

where the right-hand side is convergent in $\{\mathrm{Im}\, z \neq 0\}$ if the moment problem is determinate, e.g., if $\omega_n = O((n\log n)^2)$ (Carleman's test).

4.4. How to know μ from the Jacobi Parameters (cont)

- Cauchy–Stieltjes transform

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \frac{\omega_3}{z-\alpha_4} - \cdots$$

Stieltjes inversion formula

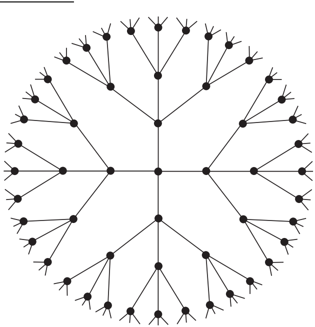
The (right-continuous) distribution function $F(x) = \mu((-\infty, x])$ and the absolutely continuous part of μ is given by

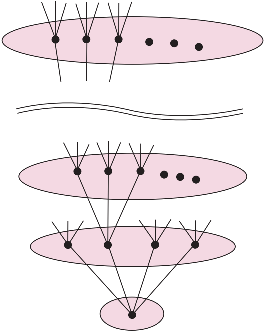
$$\frac{1}{2}\{F(t) + F(t-0)\} - \frac{1}{2}\{F(s) + F(s-0)\} = -\frac{1}{\pi} \lim_{y \to +0} \int_s^t \operatorname{Im} G_\mu(x+iy)\,dx, \quad s < t,$$

$$\rho(x) = -\frac{1}{\pi} \lim_{y \to +0} \operatorname{Im} G_\mu(x+iy)$$

4.5. Illustration: Homogeneous tree T_κ ($\kappa \geq 2$)

Stratification of T_4





(1) Quantum decomposition: $A = A^+ + A^-$

$$A^+\Phi_0 = \sqrt{\kappa}\,\Phi_1, \quad A^+\Phi_n = \sqrt{\kappa-1}\,\Phi_{n+1} \quad (n \geq 1)$$

$$A^-\Phi_0 = 0, \quad A^-\Phi_1 = \sqrt{\kappa}\,\Phi_0, \quad A^-\Phi_n = \sqrt{\kappa-1}\,\Phi_{n-1} \quad (n \geq 2)$$

(2) Jacobi parameters: $\{\omega_1 = \kappa,\ \omega_2 = \omega_3 = \cdots = \kappa - 1\}$, $\{\alpha_n \equiv 0\}$

(3) Cauchy–Stieltjes transform: $(\omega_1 = \kappa,\ \omega_2 = \omega_3 = \cdots = \kappa - 1)$

$$\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = G_\mu(z) = \frac{1}{z -} \frac{\omega_1}{z -} \frac{\omega_2}{z -} \frac{\omega_3}{z -} \frac{\omega_4}{z -} \frac{\omega_5}{z -} \cdots$$

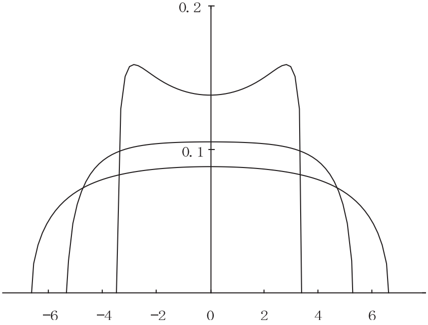
$$= \frac{(\kappa-2)z - \kappa\sqrt{z^2 - 4(\kappa-1)}}{2(\kappa^2 - z^2)}$$

(4) Spectral distribution: $\mu(dx) = \rho_\kappa(x)dx$

$$\rho_\kappa(x) = \frac{\kappa}{2\pi} \frac{\sqrt{4(\kappa-1) - x^2}}{\kappa^2 - x^2}$$

$$|x| \le 2\sqrt{\kappa - 1}$$

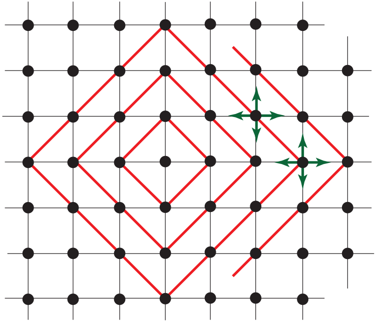
Kesten Measures (1959)



$\rho_4, \rho_8, \rho_{12}$

(5) Wigner's semicircle law (free CLT)

$$\lim_{\kappa\to\infty} \sqrt{\kappa}\,\rho_\kappa(\sqrt{\kappa}\,x) = \frac{1}{2\pi}\sqrt{4 - x^2}$$



1. Asymptotic invariance of $\Gamma(\mathbb{Z}^N)$ under A^ϵ:

$$A^+\Phi_n = \sqrt{2N}\sqrt{n+1}\,\Phi_{n+1} + O(1),$$

$$A^-\Phi_n = \sqrt{2N}\sqrt{n}\,\Phi_{n-1} + O(N^{-1/2}).$$

2. Normalized adjacency matrices:

$$\frac{A^\epsilon}{\sqrt{\kappa(N)}} = \frac{A^\epsilon}{\sqrt{2N}} \to B^\epsilon$$

3. The interacting Fock space in the limit

$$B^+\Psi_n = \sqrt{n+1}\,\Psi_{n+1},$$

$$B^-\Phi_n = \sqrt{n}\,\Psi_{n-1}, \quad B^\circ = 0.$$

Boson Fock space!

4. The asymptotic spectral distribution is the standard Gaussian distribution:

$$\lim_{N\to\infty}\left\langle \Phi_0, \left(\frac{A_N}{\sqrt{2N}}\right)^m \Phi_0 \right\rangle = \langle \Psi_0, (B^+ + B^-)^m \Psi_0 \rangle$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^m e^{-x^2/2}\,dx.$$

4.7. Growing Regular Graphs

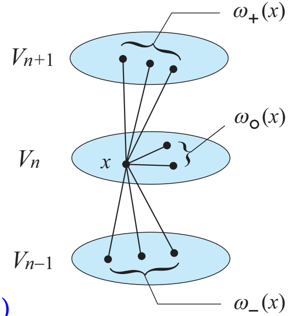
Here $\Gamma(G)$ is not necessarily invariant but asymptotically invariant under A^ϵ.

Statistics of $\omega_\epsilon(x)$

$$M(\omega_\epsilon|V_n) = \frac{1}{|V_n|}\sum_{x\in V_n}|\omega_\epsilon(x)|$$

$$\Sigma^2(\omega_\epsilon|V_n) = \frac{1}{|V_n|}\sum_{x\in V_n}\left\{|\omega_\epsilon(x)| - M(\omega_\epsilon|V_n)\right\}^2$$

$$L(\omega_\epsilon|V_n) = \max\{|\omega_\epsilon(x)|\,;\, x\in V_n\}.$$



Conditions for a growing regular graph $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$

(A1) $\lim_\nu \deg(G^{(\nu)}) = \infty$.

(A2) for each $n = 1, 2, \dots,$

$$\exists \lim_\nu M(\omega_-|V_n^{(\nu)}) = \omega_n < \infty, \quad \lim_\nu \Sigma^2(\omega_-|V_n^{(\nu)}) = 0, \quad \sup_\nu L(\omega_-|V_n^{(\nu)}) < \infty.$$

(A3) for each $n = 0, 1, 2, \dots,$

$$\exists \lim_\nu \frac{M(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} = \alpha_{n+1} < \infty, \quad \lim_\nu \frac{\Sigma^2(\omega_\circ|V_n^{(\nu)})}{\kappa(\nu)} = 0, \quad \sup_\nu \frac{L(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty.$$

Theorem (QCLT in a general form)

Let $\{G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})\}$ *be a growing regular graph satisfying*

(A1) $\lim_\nu \kappa(\nu) = \infty$, *where* $\kappa(\nu) = \deg(G^{(\nu)})$.

(A2) *for each* $n = 1, 2, \dots,$

$$\exists \lim_\nu M(\omega_-|V_n^{(\nu)}) = \omega_n < \infty, \quad \lim_\nu \Sigma^2(\omega_-|V_n^{(\nu)}) = 0, \quad \sup_\nu L(\omega_-|V_n^{(\nu)}) < \infty.$$

(A3) *for each* $n = 0, 1, 2, \dots,$

$$\exists \lim_\nu \frac{M(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} = \alpha_{n+1} < \infty, \quad \lim_\nu \frac{\Sigma^2(\omega_\circ|V_n^{(\nu)})}{\kappa(\nu)} = 0, \quad \sup_\nu \frac{L(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty.$$

Let $(\Gamma, \{\Psi_n\}, B^+, B^-, B^\circ)$ *be the interacting Fock space associated with the Jacobi parameters* $(\{\omega_n\}, \{\alpha_n\})$. *Then*

$$\lim_\nu \frac{A_\nu^\epsilon}{\sqrt{\kappa(\nu)}} = B^\epsilon \quad \textit{(stochastically)}$$

In particular, the asymptotic spectral distribution of the normalized A_ν *in the vacuum state is a probability distribution determined by* $(\{\omega_n\}, \{\alpha_n\})$.

A. Hora and N. Obata: *Asymptotic spectral analysis of growing regular graphs*, Trans. Amer. Math. Soc. **360** (2008), 899–923.

Some results: Asymptotic spectral distributions

graphs	IFS	vacuum state	deformed vacuum state
Hamming graphs $H(d, N)$	$\omega_n = n$ (Boson)	Gaussian $(N/d \to 0)$ Poisson $(N/d \to \lambda^{-1} > 0)$	Gaussian or Poisson
Johnson graphs $J(v, d)$	$\omega_n = n^2$	exponential $(2d/v \to 1)$ geometric $(2d/v \to p \in (0, 1))$	'Poissonization' of exponential distribution
odd graphs O_k	$\omega_{2n-1} = n$ $\omega_{2n} = n$	two-sided Rayleigh	?
homogeneous trees $\mathcal{T}_\kappa$	$\omega_n = 1$ (free)	Wigner semicircle	free Poisson
integer lattices $\mathbb{Z}^N$	$\omega_n = n$ (Boson)	Gaussian	Gaussian
symmetric groups $\mathfrak{S}_n$ (Coxeter)	$\omega_n = n$ (Boson)	Gaussian	Gaussian
Coxeter groups (Fendler)	$\omega_n = 1$ (free)	Wigner semicircle	free Poisson
Spidernets $S(a, b, c)$	$\omega_1 = 1$ $\omega_2 = \cdots = q$	free Meixner law	(free Meixner law)

5. Quantum White Noise Calculus

- U. C. Ji and N. Obata: "Transforms in Quantum White Noise Calculus,"
  a monograph to appear, World Scientific, 2015.
- N. Obata: "White Noise Calculus and Fock Space,"
  Lect. Notes in Math. Vol. 1544, Springer, 1994.

5.1. Probability Theory Encountering Quantum Theory

Boson quantum field (canonical commutation relation, CCR)

$\{a(f), a^*(g)\,;\, f, g \in H = L^2(T, dt)\}$ is called a *Boson quantum field* over T if

$$[a(f), a(g)] = [a^*(f), a^*(g)] = 0, \qquad [a(f), a^*(g)] = \langle f, g \rangle_H I$$

Fock representation

$$\Gamma(H) = \left\{ \phi = (f_n)\,;\, f_n \in H^{\hat{\otimes} n},\ \|\phi\|^2 = \sum_{n=0}^{\infty} n! |f_n|^2 < \infty \right\}: \text{ Fock space}$$

$$A(f) : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, n\langle f, \xi \rangle \xi^{\otimes(n-1)}, 0, 0, \dots)$$

$$A^*(f) : (0, \dots, 0, \xi^{\otimes n}, 0, \dots) \mapsto (0, \dots, 0, 0, \xi^{\otimes n} \widehat{\otimes} f, 0, \dots)$$

$$A(f) = \int_T f(t) a_t \, dt, \qquad A^*(f) = \int_T f(t) a_t^* \, dt$$

$\Longrightarrow \{a_t, a_t^*\,;\, t \in T\}$: *quantum white noise* (field over T)

$$[a_s, a_t] = [a_s^*, a_t^*] = 0, \qquad [a_s, a_t^*] = \delta(s - t)\, I$$

Gaussian space $(E_{\mathbb{R}}^*, \mu)$

$E_{\mathbb{R}} \subset H_{\mathbb{R}} = L^2(T, dt; \mathbb{R}) \subset E_{\mathbb{R}}^*$: Gelfand triple

μ: a probability measure on $E_{\mathbb{R}}^*$ uniquely specified by the characteristic function:

$$\exp\left\{-\frac{1}{2}|\xi|_H^2\right\} = \int_{E_{\mathbb{R}}^*} e^{i\langle x,\xi\rangle}\mu(dx), \qquad \xi \in E_{\mathbb{R}}.$$

Wiener–Itô–Segal isomorphism

$$\Gamma(H) \cong L^2(E^*, \mu)$$

$$\phi = (f_n) \leftrightarrow \phi(x) = \sum_{n=0}^{\infty} \langle :x^{\otimes n}:, f_n\rangle$$

$$(0, 1_{[0,t]}, 0, \dots) \leftrightarrow B_t = \langle x, 1_{[0,t]}\rangle \quad \text{Brownian motion}$$

Quantum decomposition and quantum Brownian motion

$$B_t = A(1_{[0,t]}) + A^*(1_{[0,t]}) = \int_0^t a_s\, ds + \int_0^t a_s^*\, ds$$

5.2. Quantum White Noise Calculus: A Standpoint

$\{ B_t \}$ **Quantum Brownian Motion**

$\{ P_t \}$ **Quantum Poisson Process**

$\{ B_t \}$ **Brownian Motion**

Ito Calculus (1940s)
SDE $\{ dB_t \}$

$dB_t = dA_t + dA_t^*$ →

Hudson-Parthasarathy Calculus (1984)
QSDE $\{ dA_t, dA_t^*, d\Lambda_t \}$

$$W_t = \frac{dB_t}{dt}$$

white noise

$$a_t = \frac{dA_t}{dt} \qquad a_t^* = \frac{dA_t^*}{dt}$$

quantum white noise

Hida Calculus (1975)

$W_t = a_t + a_t^*$ →

Quantum White Noise Calculus

Integral and Differential Calculus for white noise functions

$$F(W_t \,; t \in T)$$

Integral and Differential Calculus for white noise operators

$$\Xi(a_s, a_t^* \,; s, t \in T)$$

5.3. White Noise Distributions

T: a topological space (time interal, space-time manifold, even a discrete space,...)

Gelfand (nuclear) triple for $H = L^2(T)$

$$E \subset H = L^2(T) \subset E^*, \qquad E = \mathop{\mathrm{proj\,lim}}_{p\to\infty} E_p\,, \quad E^* = \mathop{\mathrm{ind\,lim}}_{p\to\infty} E_{-p}\,,$$

where E_p is a dense subspace of H and is a Hilbert space for itself.

Example: $E = \mathcal{S}(\mathbb{R}) = \mathrm{proj\,lim}_{p\to\infty} \mathcal{S}_p(\mathbb{R})$

The Boson Fock space over $H = L^2(T)$ is defined by

$$\Gamma(H) = \left\{ \phi = (f_n)\,;\, f_n \in H^{\widehat{\otimes} n}\,,\ \|\phi\|^2 = \sum_{n=0}^{\infty} n!|f_n|_0^2 < \infty \right\},$$

where $|f_n|_0$ is the usual L^2-norm of $H^{\widehat{\otimes} n} = L^2_{\mathrm{sym}}(T^n)$.

Gelfand (nuclear) triple for $\Gamma(H)$ [Kubo–Takenaka PJA 56A (1980)]

$$(E) \subset \Gamma(H) \subset (E)^*, \qquad (E) = \mathop{\mathrm{proj\,lim}}_{p\to\infty} \Gamma(E_p), \quad (E)^* = \mathop{\mathrm{ind\,lim}}_{p\to\infty} \Gamma(E_{-p})$$

5.4. White Noise Operators

Definition

A continuous operator from (E) into $(E)^*$ is called a *white noise operator*. The space of white noise operators is denoted by $\mathcal{L}((E),(E)^*)$ (bounded convergence topology).

The *annihilation* and *creation operator* at a point $t \in T$

$$a_t : (0,\dots,0,\xi^{\otimes n},0,\dots) \mapsto (0,\dots,0,n\xi(t)\xi^{\otimes(n-1)},0,0,\dots)$$
$$a_t^* : (0,\dots,0,\xi^{\otimes n},0,\dots) \mapsto (0,\dots,0,0,\xi^{\otimes n}\widehat{\otimes}\delta_t,0,\dots)$$

The pair $\{a_t, a_t^*\,;\, t \in T\}$ is called the *quantum white noise* on T.

Theorem

$a_t \in \mathcal{L}((E),(E))$ and $a_t^* \in \mathcal{L}((E)^*,(E)^*)$ for all $t \in \mathbb{R}$. Moreover, both maps $t \mapsto a_t \in \mathcal{L}((E),(E))$ and $t \mapsto a_t^* \in \mathcal{L}((E)^*,(E)^*)$ are operator-valued test functions, i.e., belongs to $E \otimes \mathcal{L}((E),(E))$ and $E \otimes \mathcal{L}((E)^*,(E)^*)$, respectively.

Quantum White Noise Calclus provides a distribution-theoretic framework for Fock space operators, which are expressible as $\Xi = \Xi(a_s, a_t^*\,;\, s,t, \in T)$.

5.5. Quantum White Noise Derivatives

Definition

For $\Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ and $\zeta \in E$ we define $D_\zeta^\pm \Xi \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ by

$$D_\zeta^+ \Xi = [a(\zeta), \Xi], \qquad D_\zeta^- \Xi = -[a^*(\zeta), \Xi].$$

These are called the *creation derivative* and *annihilation derivative* of Ξ, respectively.

1. $D_\zeta^\pm$ are rigorous realization of the idea: for $\Xi = \Xi(a_s, a_t^* \,;\, s, t \in T)$

$$D_t^+ \Xi = \frac{\delta \Xi}{\delta a_t^*}, \quad D_s^- \Xi = \frac{\delta \Xi}{\delta a_s}$$

2. For example, for $\Xi = \int_T \kappa(s) a_s^* ds = \Xi_{0,1}(\kappa)$

$$D_t^+ \int_T \kappa(s) a_s^* ds = \kappa(t)$$

$$\therefore \quad D_\zeta^+ \int_T \kappa(s) a_s^* ds = \int_T \kappa(t)\zeta(t) dt = \langle \kappa, \zeta \rangle.$$

5.6. Wick Derivations

Definition (Wick product)

For $\Xi_1, \Xi_2 \in \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ the *Wick product* $\Xi_1 \diamond \Xi_2$ is uniquely specified by

$$a_t \diamond \Xi = \Xi \diamond a_t = \Xi a_t \,, \qquad a_t^* \diamond \Xi = \Xi \diamond a_t^* = a_t^* \Xi.$$

Moreover,

$$(a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m}) \diamond \Xi = a_{s_1}^* \cdots a_{s_l}^* \Xi a_{t_1} \cdots a_{t_m}$$

Equipped with the Wick product, $\mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ becomes a commutative algebra.

Definition (Wick derivation)

A continuous linear map $\mathcal{D} : \mathcal{L}(\mathcal{W}, \mathcal{W}^*) \to \mathcal{L}(\mathcal{W}, \mathcal{W}^*)$ is called a *Wick derivation* if

$$\mathcal{D}(\Xi_1 \diamond \Xi_2) = (\mathcal{D}\Xi_1) \diamond \Xi_2 + \Xi_1 \diamond (\mathcal{D}\Xi_2)$$

Theorem (Ji-Obata: JMP 51 (2010))

The creation and annihilation derivatives $D_\zeta^\pm$ are Wick derivations for any $\zeta \in E$.

6. Implementation Problems

6.1. Quantum Aspect of Cameron–Martin Theorem

Find a function $\psi(x)$ (Ranon–Nikodym density) satisfying

$$\int_{E_{\mathbb{R}}^*} \phi(x+f)\mu(dx) = \int_{E_{\mathbb{R}}^*} \phi(x)\psi(x)\mu(dx). \qquad (1)$$

Answer is known: $\psi(x) = \phi_f(x) = e^{\langle x,f\rangle - \langle f,f\rangle/2}$ (Cameron–Martin theorem).

Define the translation operator and multiplication operator by

$$(T_f\phi)(x) = \phi(x-f), \qquad (M[\psi]\phi)(x) = \psi(x)\phi(x)$$

Then, (1) is equivalent to

$$\langle\langle M[T_{-f}\phi]\phi_0, \phi_0\rangle\rangle = \langle\langle M[\phi]\sqrt{\psi}, \sqrt{\psi}\rangle\rangle$$

On the other hand, for a unitary operator V we have

$$\langle\langle M[T_{-f}\phi]\phi_0, \phi_0\rangle\rangle = \langle\langle VM[T_{-f}\phi]V^*V\phi_0, V\phi_0\rangle\rangle$$

Hence, if we could find a unitary intertwining operator V such that

$$VM[T_{-f}\phi] = M[\phi]V, \qquad \phi \in L^2(E_{\mathbb{R}}^*, \mu), \qquad (2)$$

we obtain $\sqrt{\psi} = V\phi_0$ and the Girsanov transform $\mu \mapsto \tilde{\mu} = \psi(x)\mu$.

A quantum aspect of Cameron-Martin theorem

$$\begin{array}{ccc} L^2(E_{\mathbb{R}}^*, \mu) & \xrightarrow{V} & L^2(E_{\mathbb{R}}^*, \mu) \\ M[T_{-f}\phi] \downarrow & & \downarrow M[\phi] \\ L^2(E_{\mathbb{R}}^*, \mu) & \xrightarrow[V]{} & L^2(E_{\mathbb{R}}^*, \mu) \end{array}$$

1. Sufficient to take $\phi(x) = \langle x, \xi \rangle$ and use $M[\langle x, \xi \rangle] = a(\xi) + a^*(\xi)$.
2. More generally, add quadratic functions in quantum white noises

Implementation problem

Given $\zeta_1, \eta_2 \in E$, $\eta_1, \zeta_2 \in E^*$, $S_1 \in \mathcal{L}(E, E)$, $S_2 \in \mathcal{L}(E^*, E^*)$ and $K \in \mathcal{L}((E)^*, (E)^*)$, find a white noise operator $V \in \mathcal{L}((E), (E)^*)$ satisfying

$$\begin{array}{ccc} (E) & \xrightarrow{V} & (E)^* \\ a^*(\zeta_1) + a(\eta_1) + \Lambda(S_1) \downarrow & & \downarrow a^*(\zeta_2) + a(\eta_2) + \Lambda(S_2) + K \\ (E) & \xrightarrow[V]{} & (E)^* \end{array}$$

$$\begin{array}{ccc} (E) & \xrightarrow{\ V\ } & (E)^* \\ {\scriptstyle a^*(\zeta_1)+a(\eta_1)+\Lambda(S_1)}\Big\downarrow & & \Big\downarrow{\scriptstyle a^*(\zeta_2)+a(\eta_2)+\Lambda(S_2)+K} \\ (E) & \xrightarrow[\ V\ ]{} & (E)^* \end{array}$$

1. Reduced to the linear differential equation of Wick type for V:

$$\mathcal{D}V = G \diamond V,$$
$$G = a^*(\zeta_2 - \zeta_1) + a(\eta_2 - \eta_1) + \Lambda(S_2 - S_1) + \tilde{K},$$
$$\mathcal{D} : \mathcal{L}((E),(E)^*) \to \mathcal{L}((E),(E)^*) \quad \text{a Wick derivation,}$$

   where $\diamond$ is the Wick (normal-ordered) product.

2. A solution is given in terms of Wick product:

$$V = F \diamond \mathrm{wexp}\ Y, \qquad \text{where } \mathcal{D}Y = G \text{ and } \mathcal{D}F = 0$$

3. Coming back to the operator product, verify $V^\dagger V \phi_0 = \phi_0$

6.3. Quantum Girsanov Transform

Theorem (Ji–Obata (2015))

Consider the implementation problem:

$$V(a^*(\zeta_1) + a(\eta_1) + \Lambda(S_1)) = (a^*(\zeta_2) + a(\eta_2) + \Lambda(S_2) + K)V,$$
$$K = \Delta_{\mathrm{G}}^*(A_3) + e^{\Delta_{\mathrm{G}}^*(A)} e^{a^*(\zeta)}\, \Delta_{\mathrm{G}}((S^{-1})^* B_3 S^{-1})\, e^{-a^*(\zeta)} e^{-\Delta_{\mathrm{G}}^*(A)} + k.$$

Under a set of relations for $\zeta_1, \eta_1, S_1, \dots, k$ (explicitly given), a solution is given by

$$V = C\, e^{\Delta_{\mathrm{G}}^*(A)}\, e^{a^*(\zeta)}\, \Gamma(S)\, e^{a(\eta)}\, e^{\Delta_{\mathrm{G}}(B)}, \quad C \in \mathbb{C}.$$

1. In particular,

$$\langle\langle V^\dagger V\phi_0 | (a^*(\zeta_1) + a(\eta_1) + \Lambda(S_1))^m \phi_0\rangle\rangle$$
$$= \langle\langle V\phi_0 | (a^*(\zeta_2) + a(\eta_2) + \Lambda(S_2) + K)^m V\phi_0\rangle\rangle, \quad m = 0, 1, 2, \dots,$$

2. Hence if $V^\dagger V\phi_0 = \phi_0$, distributions of

$$a^*(\zeta_1) + a(\eta_1) + \Lambda(S_1) \text{ in } \phi_0$$

coincides $a^*(\zeta_2) + a(\eta_2) + \Lambda(S_2) + K$ in $V\phi_0$

3. Cameron–Martin Theorem is reproduced by $\zeta_1 = \eta_1$, $\zeta_2 = \eta_2$, $S_1 = S_2 = 0$,

6.4. More Applications of Quantum White Noise Dervatives

▶ In a series of papers by U. C. Ji and N. Obata (2009–)

A white noise operator (Boson Fock space operator) : $\Xi = \Xi(a_s, a_t^* \,;\, s, t \in T)$.

Apply quantum white noise derivatives:

$$D_\zeta^- \Xi \approx \frac{\delta}{\delta a_s}\,\Xi \qquad D_\zeta^+ \Xi \approx \frac{\delta}{\delta a_t^*}\,\Xi$$

1. Implementation problem of canonical commutation reletions (CCR)
   $\Longrightarrow$ new derivation of Bogoliubov transform.

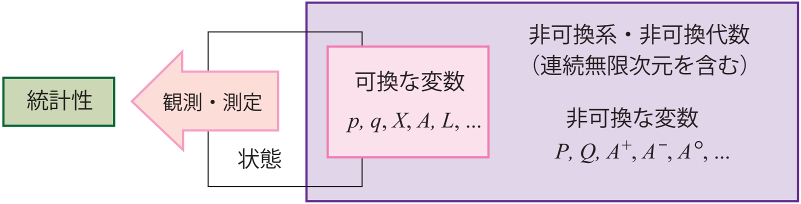
$$\begin{array}{ccc} (E) & \xrightarrow{\;U\;} & (E)^* \\ {\scriptstyle a(\zeta)}\downarrow & & \downarrow{\scriptstyle a(S\zeta)+a^*(T\zeta)} \\ (E) & \xrightarrow[\;U\;]{} & (E)^* \end{array} \qquad \begin{array}{ccc} (E) & \xrightarrow{\;U\;} & (E)^* \\ {\scriptstyle a^*(\zeta)}\downarrow & & \downarrow{\scriptstyle a^*(S\zeta)+a(T\zeta)} \\ (E) & \xrightarrow[\;U\;]{} & (E)^* \end{array}$$

2. Quantum stochastic integral representations of quantum martingales.
3. Calculation normal-ordered forms

$$\text{(an operator on Fock space)} = \sum \text{(creation operators)(annihilation operators)}$$

4. Quantum stochatic integrals as dual of quantum stochstic gradients

∞. Finally,.... Are We Approaching Toward a New Paradigm?



1. Quantization: classical variables $p, q \Longrightarrow$ non-commuting operators P, Q
2. Quantm stochastic calculus (Hudson-Parthasarathy, 1984)
   quantum decomposition of Brownian $B(t) = A(t) + A^*(t)$
   the Itô formula $(dB)^2 = dt$ is a consequence of CCR $[dA, dA^*] = dt$.
3. Gassianization of probability distribution (Accardi-Bozejko, 1998)
   quantum decomposition of a random variable $X = A^+ + A^\circ + A^-$
4. Quantum field and stochastic analysis: non-commutative + infinite dimension
5. Quantum walks: classical random walk $p + q = 1 \Longrightarrow P + Q = U$
6. Spectra of graphs: adjacency matrix, Laplacian matrix, ... quantum decomposition
7. Algebraic combinatorics: Association scheme, Terwillinger algebra, ...