Counting Walks: A Quantum Probabilistic Viewpoint

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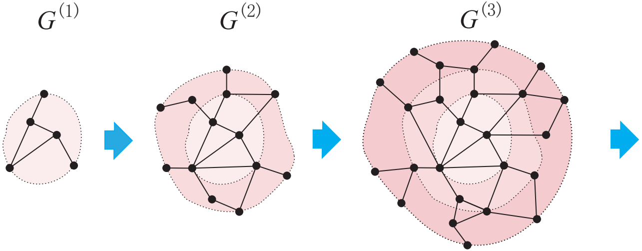
1. Motivations

1.1. Spectral Analysis of Growing Graphs



Network Science $\Longrightarrow$ *Mathematical Network Theory*

1. A.-L. Barabási and R. Albert: Emergence of scaling in random networks, Science **286** (1999), 509–512
2. D. J. Watts and S. H. Strogatz: Collective dynamics of 'small-world' networks, Nature **393** (1998) 440–442.
3. R. Durrett: *Random Graph Dynamics*, Cambridge UP, 2006.
4. F. Chung and L. Lu: *Complex Graphs and Networks*, AMS, 2006.
5. L. Lovasz: *Large Networks and Graph Limits*, AMS, 2012.

1.2. Quantum Probability

	Classical Probability	Quantum Probability
probability space	$(\Omega, \mathcal{F}, P)$	$(\mathcal{A}, \varphi)$
random variable	$X : \Omega \to \mathbb{R}$	$a = a^* \in \mathcal{A}$
expectation	$\mathbf{E}[X] = \int_\Omega X(\omega)\, P(d\omega)$	$\varphi(a)$
moments	$\mathbf{E}[X^m]$	$\varphi(a^m)$
distribution	$\mu_X((-\infty, x]) = P(X \le x)$ $\mathbf{E}[X^m] = \int_{-\infty}^{+\infty} x^m \mu_X(dx)$	NA $\varphi(a^m) = \int_{-\infty}^{+\infty} x^m \mu_a(dx)$
independence	$\mathbf{E}[X^m Y^n] = \mathbf{E}[X^m]\mathbf{E}[Y^n]$	many notions
CLT	$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k \sim N(0,1)$	many limit distributions

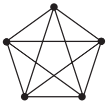
- *Our approach:* The adjacency matrix A of a graph G is studied as a real random variable of the algebraic probability space $(\mathcal{A}(G), \langle\cdot\rangle)$.

2. Graph Spectra

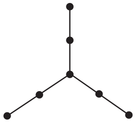
2.1. Graphs and Adjacency Matrix

Definition (graph)

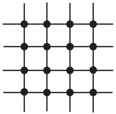
A *graph* is a pair $G = (V, E)$, where V is the set of *vertices* and E the set of *edges*. We write $x \sim y$ (adjacent) if they are connected by an edge.



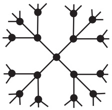
complete graph K_5



star graph



2-dim lattice



homogeneous tree T_4

Definition (adjacency matrix)

The *adjacency matrix* of a graph $G = (V, E)$ is defined by

$$A = [A_{xy}]_{x,y \in V} \qquad A_{xy} = \begin{cases} 1, & x \sim y, \\ 0, & \text{otherwise.} \end{cases}$$

▶ The adjacency matrix possesses all the information of a graph.

Definition (spectrum and spectral distribution)

Let G be a finite graph. The *spectrum (eigenvalues)* of G is the list of eigenvalues of the adjacency matrix A.

$$\mathbf{Spec}\,(G) = \begin{pmatrix} \cdots & \lambda_i & \cdots \\ \cdots & m_i & \cdots \end{pmatrix}$$

The *spectral (eigenvalue) distribution* of G is defined by

$$\mu_G = \frac{1}{|V|} \sum_i m_i \delta_{\lambda_i}$$

1. $\mathbf{Spec}\,(G)$ is a fundamental invariant of finite graphs.
2. (isospectral problem) Non-isomorphic graphs may have the same spectra.
3. Adjacency matrix, Laplacian matrix, distance matrix, Q-matrix, ... etc.

- For *algebraic graph theory* or *spectral graph theory* see

- N. Biggs: "Algebraic Graph Theory," Cambridge UP, 1993.
- D. M. Cvetković, M. Doob and H. Sachs: "Spectra of Graphs," Academic Press, 1979.

2.3. Quantum Probabilistic Approach

Definition (Algebraic probability space)

An *algebraic probability space* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a $*$-algebra over $\mathbb{C}$ with multiplication unit $1_{\mathcal{A}}$, and a state $\varphi : \mathcal{A} \to \mathbb{C}$, i.e.,

$$\text{(i) } \varphi \text{ is linear;} \quad \text{(ii) } \varphi(a^* a) \geq 0; \quad \text{(iii) } \varphi(1_{\mathcal{A}}) = 1.$$

Each $a \in \mathcal{A}$ is called an *(algebraic) random variable*.

Adjacency algebra with state

Let $G = (V, E)$ be a locally finite graph, i.e., $\deg x < \infty$ for all $x \in V$. The adjacency algebra $\mathcal{A}(G)$ is the (commutative) $*$-algebra generated by the adjacency matrix A. Equipped with a state $\langle \cdot \rangle$, $\mathcal{A}(G)$ becomes an algebraic probability space, and A a real random variable.

1. normalized trace: $\langle a \rangle_{\mathrm{tr}} = |V|^{-1} \operatorname{Tr} a$
2. vector state at $o \in V$ (often called vacuum state): $\langle a \rangle_o = \langle \delta_o, a\delta_o \rangle$
3. deformed vacuum state: $\langle a \rangle_q = \langle Q\delta_o, a\delta_o \rangle$, where $Q = [q^{\partial(x,y)}]$.

2.4. Spectral distributions

Definition (Spectral distribution)

Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. For a real random variable $a = a^* \in \mathcal{A}$ there exists a probability measure μ on $\mathbb{R} = (-\infty, +\infty)$ such that

$$\varphi(a^m) = \int_{-\infty}^{+\infty} x^m \mu(dx) \equiv M_m(\mu), \quad m = 1, 2, \dots.$$

This μ is called the *spectral distribution* of a (with respect to the state φ).

- Existence of μ by Hamburger's theorem using Hanckel determinants.
- In general, μ is not uniquely determined (indeterminate moment problem).

(I) Normalized trace $\leftrightarrow$ the eigenvalue distribution of A:

$$\langle A^m \rangle_{\mathrm{tr}} = \int_{-\infty}^{+\infty} x^m \mu(dx) \quad \Longleftrightarrow \quad \mu = \frac{1}{|V|} \sum m_i \delta_{\lambda_i}$$

(II) Vacuum state $\leftrightarrow$ counting walks

$$\langle A^m \rangle_o = \langle \delta_o \,, A^m \delta_o \rangle = |\{m\text{-step walks from } o \text{ to } o\}| = \int_{-\infty}^{+\infty} x^m \mu(dx)$$

2.4. Main Questions

1. Given a graph $G = (V, E)$ and a state $\langle\cdot\rangle$ on $\mathcal{A}(G)$, find the spectral distribution of A in the state $\langle\cdot\rangle$, i.e., a probability distribution $\mu = \mu(G)$ on $\mathbb{R}$ satisfying

$$\langle A^m \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots.$$

2. Given a growing graph $G^{(\nu)} = (V^{(\nu)}, E^{(\nu)})$ and a state $\langle\cdot\rangle_\nu$ on $\mathcal{A}(G^{(\nu)})$, find a probability measure μ on $\mathbb{R}$ satisfying

$$\langle (A^{(\nu)})^m \rangle_\nu \approx \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots.$$

   μ is called the *asymptotic spectral distribution* of $G^{(\nu)}$ in the state $\langle\cdot\rangle_\nu$.

3. Assume that $G = G_1 \# G_2$ is a "product" of two graphs G_1 and G_2. Find the mechanism of getting the spectral distribution $\mu(G)$ in terms of $\mu(G_1)$ and $\mu(G_2)$. We expect a new convolution product $\mu(G) = \mu(G_1) \# \mu(G_2)$.

3. Cartesian Product and Subgraphs

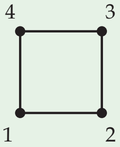
3.1. Cartesian Product

Definition

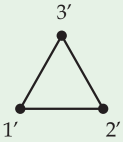
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *Cartesian product* or *direct product* of G_1 and G_2, denoted by $G_1 \times G_2$, is a graph on $V = V_1 \times V_2$ with adjacency relation:

$$(x, y) \sim (x', y') \iff \begin{cases} x = x' \\ y \sim y' \end{cases} \quad \text{or} \quad \begin{cases} x \sim x' \\ y = y'. \end{cases}$$

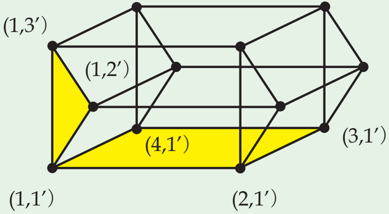
Example ($C_4 \times C_3$)



C_4



C_3



$C_4 \times C_3$

Theorem

Let G_1, G_2 be two graphs. The adjacency matrix A of $G = G_1 \times G_2$ is regarded as an operator acting on $\ell^2(V) = \ell^2(V_1 \times V_2) \cong \ell^2(V_1) \otimes \ell^2(V_2)$. Then,

$$A = A_1 \otimes I_2 + I_1 \otimes A_2$$

Example (Hypercubes and Hamming graphs)

The n-dim hypercube is the n-fold Cartesian power of K_2 ($\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet$). More generally, the Hamming graph $H(n, N)$ is the n-fold Cartesian power of the complete graph K_N:

$$H(n, N) \cong K_N \times \cdots \times K_N \quad (n\text{-times}).$$

The adjacency matrix is given by

$$A = A_{n,N} = \sum_{i=1}^{n} \overbrace{I \otimes \cdots \otimes I}^{i-1} \otimes B \otimes \overbrace{I \otimes \cdots \otimes I}^{n-i}$$

where B is the adjacency matrix of K_N.

3.3. Limit Distributions — Commutative CLT

For growing graphs $G^{(n)} = G \times \cdots \times G$ (n-times) we have

$$A^{(n)} = \sum_{i=1}^{n} \overbrace{I \otimes \cdots \otimes I}^{i-1} \otimes B \otimes \overbrace{I \otimes \cdots \otimes I}^{n-i} \equiv \sum_{i=1}^{n} B_i,$$

where B is the adjacency matrix of G.

▶ Can check that $B_1, \dots, B_n$ are identically distributed, commutative independent random variables in the product vacuum state (as well as in the normalized trace).

Theorem (Commutative CLT)

Let $(\mathcal{A}, \varphi)$ be an algebraic probability space. Let $a_n = a_n^* \in \mathcal{A}$ be a sequence of real random variables, normalized such as $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$, and commutative independent. Then, we have

$$\lim_{n\to\infty} \varphi\left[\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k\right)^m\right] = \int_{-\infty}^{+\infty} x^m \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx, \quad m = 1, 2, \dots.$$

Applying the above general theorem, we obtain

$$\frac{1}{\sqrt{n}} \frac{A^{(n)}}{\sqrt{\deg_G(o)}} \xrightarrow{m} g, \quad \deg_G(o) = \langle B_i^2 \rangle = \langle B^2 \rangle, \quad g \sim N(0, 1).$$

3.4. Comb Product

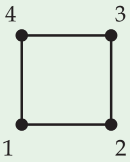
Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We fix a vertex $o_2 \in V_2$. For $(x, y), (x', y') \in V_1 \times V_2$ we write $(x, y) \sim (x', y')$ if one of the following conditions is satisfied:

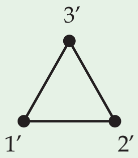
(i) $x = x'$ *and* $y \sim y'$*;* *(ii)* $x \sim x'$ *and* $y = y' = o_2$*.*

Then $V_1 \times V_2$ becomes a graph, denoted by $G_1 \rhd_{o_2} G_2$, and is called the *comb product* or the *hierarchical product*.

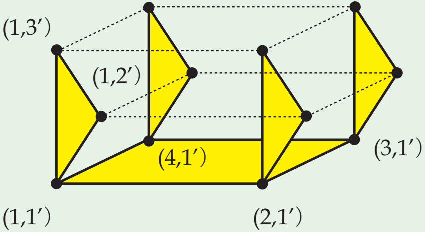
Example ($C_4 \rhd_{o_2} C_3$ with $o_2 = 1'$)



C_4



C_3



$C_4 \rhd C_3$

Theorem

Let G_1, G_2 *be two graphs with* $o_2 \in V_2$. *The adjacency matrix* A *of* $G_1 \rhd_{o_2} G_2$ *is regarded as an operator acting on* $\ell^2(V) = \ell^2(V_1 \times V_2) \cong \ell^2(V_1) \otimes \ell^2(V_2)$. *Then,*

$$A = A_1 \otimes P_2 + I_1 \otimes A_2$$

where P_2 *is the rank-one projection onto the space spanned by* δ_{o_2}. *Moreover, the right-hand side is the sum of monotone independent random variables.*

Theorem (Accardi–Ben Ghobal–O. (2004))

For $G^{(n)} = G \rhd_o G \rhd_o \cdots \rhd_o G$ *(n-times) the adjacency matrix is given by*

$$A^{(n)} = \sum_{i=1}^{n} \overbrace{I \otimes \cdots \otimes I}^{i-1} \otimes B \otimes \overbrace{P \otimes \cdots \otimes P}^{n-i},$$

where B *is the adjacency matrix of* G. *Moreover,*

$$\lim_{n\to\infty} \left\langle \left(\frac{A^{(n)}}{\sqrt{n}\sqrt{\deg(o)}} \right)^m \right\rangle = \int_{-\sqrt{2}}^{+\sqrt{2}} x^m \frac{dx}{\pi\sqrt{2-x^2}}, \quad m = 1, 2, \ldots.$$

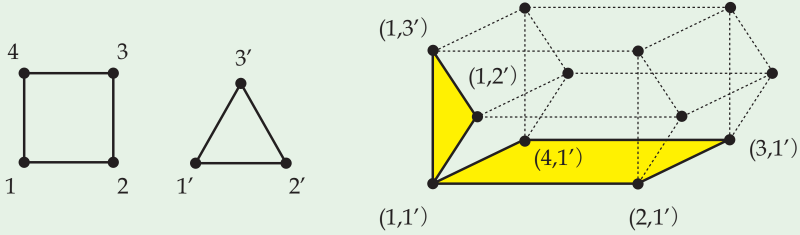
Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with distinguished vertices $o_1 \in V_1$ and $o_2 \in V_2$. Define a subset of $V_1 \times V_2$ by

$$V_1 \star V_2 = \{(x, o_2)\,;\, x \in V_1\} \cup \{(o_1, y)\,;\, y \in V_2\}$$

The induced subgraph of $G_1 \times G_2$ spanned by $V_1 \star V_2$ is called the *star product* of G_1 and G_2 (with contact vertices o_1 and o_2), and is denoted by $G_1 \star G_2 = G_1 \,{}_{o_1}\!\star_{o_2}\, G_2$.

Example ($C_4 \star C_3$)



C_4 C_3 $C_4 \star C_3$

Theorem

Let G_1, G_2 *be two graphs with* $o_1 \in V_1$ *and* $o_2 \in V_2$*. The adjacency matrix of* $G_1 \star G_2$ *is regarded as an operator acting on* $\ell^2(V_1 \times V_2) \cong \ell^2(V_1) \otimes \ell^2(V_2)$*. Then,*

$$A = A_1 \otimes P_2 + P_1 \otimes A_2,$$

where P_i *is the rank-one projection onto the space spanned by* δ_{o_i}*. Moreover, the right-hand side is the sum of Boolean independent random variables.*

Theorem (O. (2004))

For $G^{(n)} = G \star G \star \cdots \star G$ *(*n*-times) the adjacency matrix is given by*

$$A^{(n)} = \sum_{i=1}^{n} \overbrace{P \otimes \cdots \otimes P}^{i-1} \otimes B \otimes \overbrace{P \otimes \cdots \otimes P}^{n-i}$$

where B *is the adjacency matrix of* G*. Moreover,*

$$\lim_{n\to\infty} \left\langle \left(\frac{A^{(n)}}{\sqrt{n}\sqrt{\deg(o)}} \right)^m \right\rangle = \int_{-\infty}^{+\infty} x^m \frac{1}{2}(\delta_{-1} + \delta_{+1})(dx), \quad m = 1, 2, \dots.$$

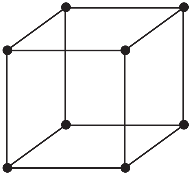
4. Distance-k Graphs

Definition (Distance-k graph)

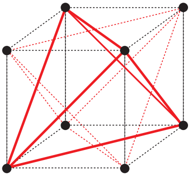
Let $G=(V,E)$ be a graph. For $k\ge1$ the *distance-k graph* of G is a graph

$$G^{[k]}=(V,E^{[k]}),\quad E^{[k]}=\{\{x,y\}\,;\,x,y\in V,\ \partial_G(x,y)=k\},$$

where $\partial_G(x,y)$ is the graph distance.



$K_2\times K_2\times K_2$



$(K_2\times K_2\times K_2)^{[2]}$

- The adjacency matrix of $G^{[k]}$ coincides with the k-th distance matrix of G defined by

$$D_k=[(D_k)_{xy}]_{x,y\in V}\qquad (D_k)_{xy}=\begin{cases}1, & \partial_G(x,y)=k,\\ 0, & \text{otherwise.}\end{cases}$$

4.2. Asymptotic Spectral Distribution of $G^{[N,k]}$ for $k = 1$

$G = (V, E)$: a finite graph with $|V| \geq 2$, D_l: the l-th distance matrix of G
$G^N = G \times \cdots \times G$: N-fold direct power ($N \geq 1$)
$G^{[N,k]}$: the distance-k graph of G^N ($1 \leq k \leq N$)
$A^{[N,k]}$: the adjacency matrix of $G^{[N,k]}$
regarded as a real random variable of $(\mathcal{A}(G^{[N,k]}), \varphi_{\mathrm{tr}})$

For $k = 1$ we have $G^{[N,1]} = G^N$ (Cartesian product) and

$$A^{[N,1]} = \sum_{i=1}^{N} 1 \otimes \cdots \otimes D_1 \otimes \cdots \otimes 1 \qquad (D_1 \text{ at } i\text{-th position}),$$

where D_1 is the adjacency matrix (1st distance matrix) of G

Theorem (Commutative CLT)

$$\frac{A^{[N,1]}}{N^{1/2}} \xrightarrow{m} \left(\frac{2|E|}{|V|}\right)^{1/2} g, \qquad g \sim N(0,1).$$

We only need to note that

$$\varphi_{\mathrm{tr}}(D_1) = 0, \qquad \varphi_{\mathrm{tr}}(D_1^2) = \frac{2|E|}{|V|} = (\text{mean degree of } G)$$

4.3. Asymptotic Spectral Distribution of $G^{[N,k]}$ for general k

$G=(V,E)$: a finite graph with $|V|\ge 2$
$G^N=G\times\cdots\times G$: N-fold direct power $(N\ge 1)$
$G^{[N,k]}$: the distance k-graph of G^N $(1\le k\le N)$
$A^{[N,k]}$: the adjacency matrix of $G^{[N,k]}$
regarded as a real random variable of $(\mathcal{A}(G^{[N,k]}),\varphi_{\mathrm{tr}})$

Theorem (Hibino-Lee-O. (2013))

For any $k\ge 1$ we have

$$\frac{A^{[N,k]}}{N^{k/2}} \xrightarrow{m} \left(\frac{2|E|}{|V|}\right)^{k/2}\frac{1}{k!}\,\tilde{H}_k(g),$$

where g is a real algebraic random variable $\sim N(0,1)$, and $\tilde{H}_k(g)=g^k+\cdots$ is the Hermite polynomial.

1. The limit distribution does not depend on the detailed structure of G.
2. For $k\ge 3$ the uniqueness of the limit distribution is not known. Probably uniqueness does not hold, cf. [Berg (Ann. Prob. 1988)].

4.4. Outline of the Proof

1. Write $A^{[N,k]} = B^{[N,k]} + C(N,k)$, where
$$B^{[N,k]} = \sum 1 \otimes \cdots \otimes D_1 \otimes \cdots \otimes D_1 \otimes \cdots \otimes 1 \quad (D_1 \text{ appears } k \text{ times})$$
2. By the commutative CLT,
$$\left(\frac{2|E|}{|V|}\right)^{-1/2} \frac{B^{[N,1]}}{N^{1/2}} \xrightarrow{m} g = \tilde{H}_1(g)$$
3. By induction
$$(k+1)! \left(\frac{2|E|}{|V|}\right)^{-(k+1)/2} \frac{B^{[N,k+1]}}{N^{(k+1)/2}} \xrightarrow{m} g\tilde{H}_k(g) - k\tilde{H}_{k-1}(g) = \tilde{H}_{k+1}(g)$$
and hence
$$\frac{B^{[N,k]}}{N^{k/2}} \xrightarrow{m} \left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!}\, \tilde{H}_k(g),$$
4. Show that
$$\frac{C(N,k)}{N^{k/2}} \xrightarrow{m} 0$$
5. Show that
$$A^{[N,k]} = B^{[N,k]} + C(N,k) \xrightarrow{m} \left(\frac{2|E|}{|V|}\right)^{k/2} \frac{1}{k!}\, \tilde{H}_k(g),$$

Definition (Convergence in moments)

For $a_n = a_n^*$ in $(\mathcal{A}_n, \varphi_n)$ and $a = a^*$ in $(\mathcal{A}, \varphi)$ we say that

$$a_n \xrightarrow{m} a \quad \iff \quad \lim_{n\to\infty} \varphi_n(a_n^m) = \varphi(a^m), \qquad m = 1, 2, \dots .$$

- For any polynomial $p(x)$ we have

$$a_n \xrightarrow{m} a \quad \Longrightarrow \quad p(a_n) \xrightarrow{m} p(a).$$

- However, it does not hold in general that

$$a_n \xrightarrow{m} a, \quad b_n \xrightarrow{m} b \quad \Longrightarrow \quad p(a_n, b_n) \xrightarrow{m} p(a, b)$$

for a non-commutative polynomial $p(x, y)$.

4.5. Algebraic Convergence Lemma — A Technical Tool (cont)

Lemma (Hibino-Lee-O. (2013))

Let $a_n = a_n^*, z_{1n} = z_{1n}^*, \dots, z_{kn} = z_{kn}^*$ be real random variables in $(\mathcal{A}_n, \varphi_n)$, $n = 1, 2, \dots$. Assume the following conditions are satisfied:

(i) *There exist a real random variable $a = a^* \in \mathcal{A}$ and $\zeta_1, \dots, \zeta_k \in \mathbb{R}$ such that*

$$a_n \xrightarrow{m} a, \qquad z_{in} \xrightarrow{m} \zeta_i 1, \quad i = 1, 2, \dots, k;$$

(ii) $\{a_n, z_{1n}, \dots, z_{kn}\}$ *have uniformly bounded mixed moments in the sense that*

$$C_m = \sup_n \max \left\{ |\varphi(a_n^{\alpha_1} z_{1n}^{\beta_1} \cdots z_{kn}^{\gamma_1} \cdots a_n^{\alpha_i} z_{1n}^{\beta_i} \cdots z_{kn}^{\gamma_i} \cdots)| \, ; \begin{array}{l} \alpha_i, \beta_i, \gamma_i \geq 0 \text{ are integers} \\ \sum_i (\alpha_i + \beta_i + \cdots + \gamma_i) = m \end{array} \right\} < \infty;$$

(iii) φ_n *is a tracial state for $n = 1, 2, \dots$.*

Then, for any non-commutative polynomial $p(x, y_1, \dots, y_k)$ we have

$$p(a_n, z_{1n}, \dots, z_{kn}) \xrightarrow{m} p(a, \zeta_1 1, \dots, \zeta_k 1).$$

4.6. Relevant Results

- q-Deformation [Lee-Obata (2013)]
  1. Meyer's bébé Fock space $\cong$ weighted $G^{(N)} = K_2 \times \cdots \times K_2$ (N-dim hypercube)
  2. CLT for bébé Fock space [Biane (1997)]
  $$\frac{A^{[N,1]}}{N^{1/2}} \xrightarrow{m} g_q \quad (q\text{-Gaussian})$$
  3. Our claim:
  $$\frac{A^{[N,k]}}{N^{k/2}} \xrightarrow{m} \frac{1}{k!}\, \tilde{H}_k^q(g_q) \quad \text{for almost surely in } \epsilon.$$

- Distance k-graphs of another product graphs
  1. [Arizmendi-Gaxiola (2015)] Distance-k graphs of $G \star G \star \cdots \star G$ are again $\star$-product. Hence
  $$\frac{A^{[N,k]}}{\sqrt{N}\sqrt{\deg(o)}} \xrightarrow{m} \frac{1}{2}(\delta_{-1} + \delta_{+1})$$
  2. other cases in progress.

5. Mellin Product

H.-H. Lee and N. Obata:
Mellin product of graphs and counting walks, preprint, 2015

5.1. Definition and Elementary Properties

Definition (Mellin product)

Let $G_i = (V_i, E_i)$ be a graph with adjacency matrix $A^{(i)}$, $i = 1, 2$. The *Mellin product* $G_1 \times_M G_2$ is a graph on $V = V_1 \times V_2$ with the adjacency relation:

$$(x, y) \sim_M (x', y') \iff x \sim x', y \sim y'.$$

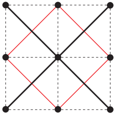
In other words, the adjacency matrix A of $G_1 \times_M G_2$ is given by

$$A = A^{(1)} \otimes A^{(2)}.$$

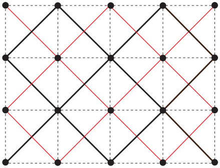
1. $G_1 \times_M G_2 \cong G_2 \times_M G_1$
2. $(G_1 \times_M G_2) \times_M G_3 \cong G_1 \times_M (G_2 \times_M G_3)$
3. If $|V_1| \geq 2$ and $|V_2| \geq 2$, then $G_1 \times_M G_2$ has at most two connected components.
4. Let $P_1 = K_1$ be the graph on a single vertex. Then for any graph $G = (V, E)$ the Mellin product $P_1 \times_M G$ is a graph on V with no edges, i.e., an empty graph on V.
5. $G_1 \times_M G_2$ is a subgraph (not necessarily induced subgraph) of the distance-2 graph of $G_1 \times G_2$.

5.2. Examples

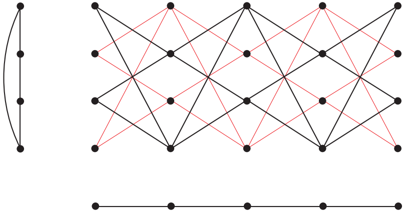
$P_3 \times_M P_3$



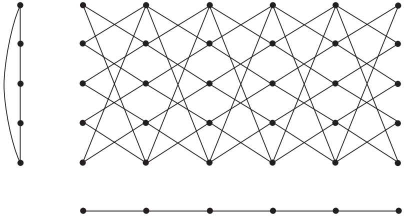
$P_4 \times_M P_5$



$C_4 \times_M P_5$



$C_5 \times_M P_5$



5.3. Spectral Distribution of Mellin Product Graphs

Theorem

For $i = 1, 2$ let $G_i = (V_i, E_i)$ be a graph with a distinguished vertex o_i. Let μ_i be the spectral distribution of the adjacency matrix A_i of G_i in the vector state at o_i. Assume that μ_i is symmetric, or equivalently that $M_{2m+1}(G_i, o_i) = 0$ for all $m = 0, 1, 2, \dots$ and $i = 1, 2$. Then we have

$$M_m(G_1 \times_M G_2, (o_1, o_2)) = M_m(\mu_1 *_M \mu_2), \qquad m = 0, 1, 2, \dots.$$

In other words, the spectral distribution of the Mellin product $G_1 \times_M G_2$ in the vector state at (o_1, o_2) is the Mellin convolution $\mu_1 *_M \mu_2$.

Mellin convolution

1. For symmetric probability distributions μ, ν on $\mathbb{R}$ we define
$$\int_{\mathbb{R}} h(x)\mu *_M \nu(dx) = \int_{\mathbb{R}} \int_{\mathbb{R}} h(xy)\mu(dx)\nu(dy), \qquad h \in C_{\mathrm{bdd}}(\mathbb{R}).$$
2. If $\mu(dx) = f(x)dx$ and $\nu(dx) = g(x)dx$ with symmetric density functions, then $\mu *_M \nu$ admits a symmetric density function $2f \star g(x)$, where
$$f \star g(x) = \int_0^\infty f(y) g\Big(\frac{x}{y}\Big)\frac{dy}{y} = \int_0^\infty f\Big(\frac{x}{y}\Big) g(y)\frac{dy}{y}, \quad x > 0.$$

$\mathbb{Z} \times_M \mathbb{Z}$: a graph on $\mathbb{Z}^2 = \{(u,v)\,;\, u,v \in \mathbb{Z}\}$ with adjacency relation:

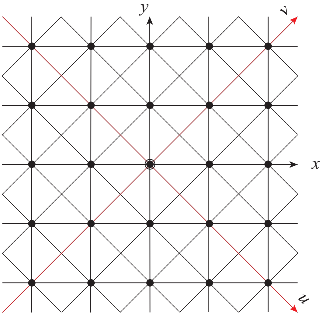
$$(u,v) \sim_M (u',v') \quad \Longleftrightarrow \quad u' = u \pm 1 \quad \text{and} \quad v' = v \pm 1.$$

$\mathbb{Z} \times_C \mathbb{Z}$ (2-d interger lattice): a graph on $\mathbb{Z}^2$ with adjacency relation:

$$(x,y) \sim (x',y') \quad \Longleftrightarrow \quad \begin{cases} x' = x \pm 1, \\ y' = y, \end{cases} \quad \text{or} \quad \begin{cases} x' = x, \\ y' = y \pm 1. \end{cases}$$

1. $\mathbb{Z} \times_M \mathbb{Z}$ has two connected components, each of which is isomorphic to $\mathbb{Z} \times_C \mathbb{Z}$.
2. Let $(\mathbb{Z} \times_M \mathbb{Z})^O$ denote the connected component of $\mathbb{Z} \times_M \mathbb{Z}$ containing $O = (0,0)$. Then

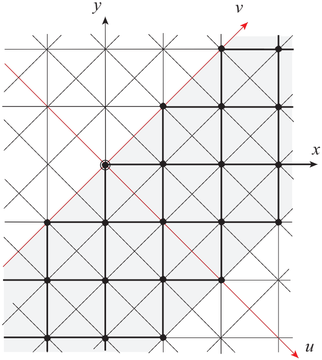
$$(\mathbb{Z} \times_M \mathbb{Z})^O \cong \mathbb{Z} \times_C \mathbb{Z}.$$



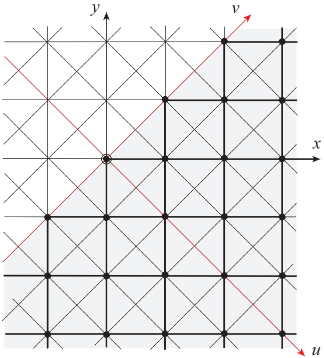
For a subset $D \subset \mathbb{Z}^2$ let $L[D]$ denote the lattice restricted to D, i.e., the induced subgraph of $\mathbb{Z} \times_C \mathbb{Z}$ spanned by the vertices in D.

1. $L\{(x,y) \in \mathbb{Z}^2\,;\, x \ge y \ge x-(n-1)\} \cong (P_n \times_M \mathbb{Z})^O$ for $n \ge 2$.
2. $L\{(x,y) \in \mathbb{Z}^2\,;\, x \ge y\} \cong (\mathbb{Z}_+ \times_M \mathbb{Z})^O$.

$(P_5 \times_M \mathbb{Z})^O$

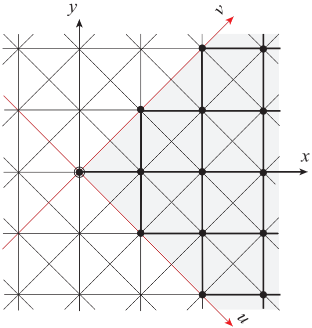


$(\mathbb{Z}_+ \times_M \mathbb{Z})^O$



1. $L\left\{(x,y)\in\mathbb{Z}^2\,;\ \begin{array}{l} 0\le x+y\le m-1, \\ 0\le x-y\le n-1 \end{array}\right\}$
$\cong (P_m \times_M P_n)^O$ for $m\ge 2$ and $n\ge 2$.

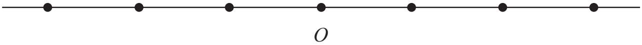
2. $L\{(x,y)\in\mathbb{Z}^2\,;\ x\ge y\ge -x\}\cong(\mathbb{Z}_+\times_M\mathbb{Z}_+)^O$,



5.6. Counting Walks

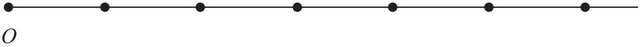
$M_k(G, o)$ = the number of k-step walks in G from o to itself

1. $\mathbb{Z}$.



$$M_{2m}(\mathbb{Z}, 0) = \binom{2m}{m}, \quad M_{2m+1}(\mathbb{Z}, 0) = 0.$$

2. $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$.



$$M_{2m}(\mathbb{Z}_+, 0) = C_m = \frac{1}{m+1}\binom{2m}{m}, \quad M_{2m+1}(\mathbb{Z}_+, 0) = 0,$$

where C_m is the renown *Catalan number*.

Theorem

For $L = L\{(x,y) \in \mathbb{Z}^2 \,;\, x \ge y\}$ *we have* $M_{2m+1}(L,(0,0)) = 0$ *and*

$$M_{2m}(L,(0,0)) = C_m \binom{2m}{m} = \frac{1}{m+1}\binom{2m}{m}^2, \quad m = 0,1,2,\dots.$$

Proof. We know that $L \cong (\mathbb{Z}_+ \times_M \mathbb{Z})^O$. Therefore,

$$M_m(L,(0,0)) = M_m(\mathbb{Z}_+ \times_M \mathbb{Z},(0,0)) = M_m(\mathbb{Z}_+,0)M_m(\mathbb{Z},0).$$

Theorem

For $L = L\{(x,y) \in \mathbb{Z}^2 \,;\, x \ge y \ge -x\}$ *we have* $M_{2m+1}(L,(0,0)) = 0$ *and*

$$M_{2m}(L,(0,0)) = C_m^2 = \frac{1}{(m+1)^2}\binom{2m}{m}^2, \quad m = 0,1,2,\dots.$$

Proof. Similarly we apply the Mellin convolution to $L \cong (\mathbb{Z}_+ \times_M \mathbb{Z}_+)^O$.

5.7. Spectral Distributions

Theorem

For $m = 0, 1, 2, \dots$ we have

$$M_m(\mathbb{Z}, 0) = M_m(\alpha), \qquad M_m(\mathbb{Z}_+, 0) = M_m(w).$$

1. Arcsine distribution.

$$\alpha(x) = \frac{1}{\pi\sqrt{4-x^2}}\, 1_{(-2,2)}(x), \qquad x \in \mathbb{R},$$

   The moments of even orders are given by

$$M_{2m}(\alpha) = \int_{-\infty}^{+\infty} x^{2m} \alpha(x)\, dx = \binom{2m}{m}, \qquad m = 0, 1, 2, \dots.$$

2. Semi-circle distribution.

$$w(x) = \frac{1}{2\pi}\sqrt{4-x^2}\, 1_{[-2,2]}(x), \qquad x \in \mathbb{R},$$

   The moments of even orders are given by

$$M_{2m}(w) = \int_{-\infty}^{+\infty} x^{2m} w(x)\, dx = C_m = \frac{1}{m+1}\binom{2m}{m}, \qquad m = 0, 1, 2, \dots,$$

Domain D	$M_{2m}(L[D], O)$	spectral distribution
$\mathbb{Z}$	$\binom{2m}{m}$	α
$\mathbb{Z}_+$	C_m	w
$\mathbb{Z}^2$	$\binom{2m}{m}^2$	$\alpha * \alpha = \alpha *_M \alpha$
$\{x \geq y\}$	$C_m \binom{2m}{m}$	$w *_M \alpha$
$\{x \geq y \geq -x\}$	C_m^2	$w *_M w$
$\{x \geq 0,\ y \geq 0\}$	(A)	$w * w$
$\{x \geq y \geq x - (n-1)\}$	(B)	$\pi_n *_M \alpha$
$\left\{ \begin{array}{l} 0 \leq x + y \leq k - 1, \\ 0 \leq x - y \leq l - 1 \end{array} \right\}$	(C)	$\pi_k *_M \pi_l$

$$(\mathrm{A}) = \sum_{k=0}^{m} \binom{2m}{2k} C_k C_{m-k},$$

$$(\mathrm{B}) = M_{2m}(P_n, 0) \binom{2m}{m}, \quad (\mathrm{C}) = M_{2m}(P_k, 0) M_{2m}(P_l, 0).$$

5.8. Calculating Density Functions

The density function of $w *_M \alpha$ is given by

$$2w \star \alpha(x) = 2\int_0^\infty w(y)\alpha\Big(\frac{x}{y}\Big)\frac{dy}{y} = \frac{1}{\pi^2}\int_{x/2}^2 \sqrt{\frac{4-y^2}{4y^2-x^2}}\,dy$$
$$= \frac{1}{\pi^2}\{K(\xi(x)) - E(\xi(x))\}, \qquad \xi(x) = \sqrt{1-\frac{x^2}{16}}\,.$$

For $k^2 < 1$, the elliptic integrals are defined by

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$
$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta = \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx,$$

Similarly, the density function of $\alpha *_M \alpha = \alpha * \alpha$ is given by

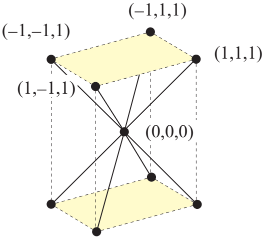
$$\frac{1}{2\pi^2}\,K(\xi(x))1_{[-4,4]}(x), \qquad x \in \mathbb{R},$$

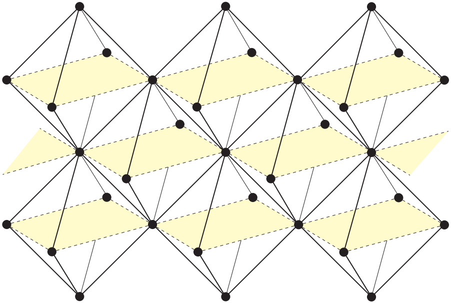
and the density function of $w *_M w$ by

$$\frac{2}{\pi^2}\left\{\left(1+\frac{x^2}{16}\right)K(\xi(x)) - 2E(\xi(x))\right\}1_{[-4,4]}(x), \qquad x \in \mathbb{R}.$$

5.9. An Example in 3-Dimension

$\mathbb{Z} \times_M \mathbb{Z} \times_M \mathbb{Z}$ has 4 connected components, which are mutually isomorphic. The connected component containing $O(0,0,0)$ looks like an octahedra honeycomb, built up by gluing octahedra or body-centered cubes.



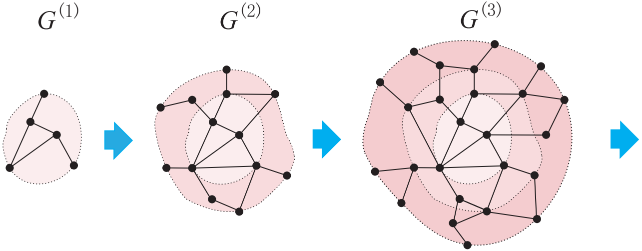


We have

$$M_{2m}(\mathbb{Z} \times_M \mathbb{Z} \times_M \mathbb{Z}, (0,0,0)) = \binom{2m}{m}^3, \qquad m = 0, 1, 2, \dots,$$

and the spectral distribution is given by $\mu = \alpha *_M \alpha *_M \alpha$.

Summary



1. For the adjacency matrices $\boldsymbol{A^{(n)}}$ of growing graphs $\boldsymbol{G^{(n)}}$ we expect
$$\boldsymbol{A^{(n+1)} \cong A^{(n)} + B^{(n)}}$$
where $\boldsymbol{B^{(n)}}$ has special relation to $\boldsymbol{A^{(n)}}$, i.e., a kind of independence.
2. The asymptotic spectrum $\boldsymbol{A^{(n)}}$ as $\boldsymbol{n \to \infty}$ is formulated within quantum probability theory.
3. The following cases are covered by our framework:
   Cartesian product, comb product, star product, distance-$\boldsymbol{k}$ graphs of Cartesian product, (free product, $\boldsymbol{q}$-deformation, distance-$\boldsymbol{k}$ graphs of star product, ...)
4. Asymptotic study of Mellin product is now in progress.