Spectral Analysis of Growing Graphs
A Quantum Probability Point of View

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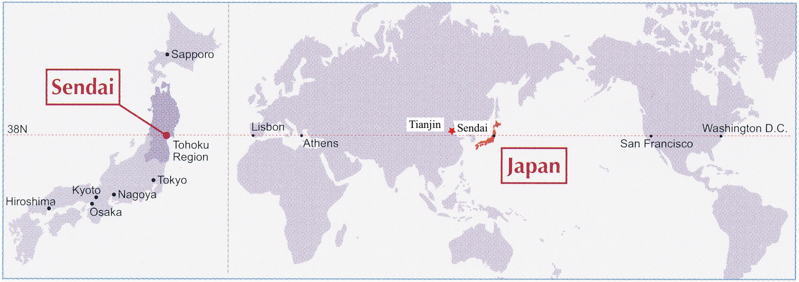
Introducing myself...



1. Tohoku University
   — The 3rd oldest national University of Japan, founded in 1907.
2. Graduate School of Information Sciences (GSIS)
   — One of the 17 Graduate Schools, founded in 1993.
3. Nobuaki Obata — Serving as Professor since 2001.
   Before then I was a member of Department of Mathematics in Nagoya University.
4. Research interests — Quantum probability, Quantum white noise analysis, Spectral analysis of graphs, Random graphs, and any topics related to network science.

My Motivations and Backgrounds

(1) Statistics in large scale discrete systems

1. A. M. Vershik's asymptotic combinatorics (1970s–)

   [1] Asymptotic combinatorics and algebraic analysis (ICM 1994)

   > *... the study of asymptotic problems in combinatorics is stimulated enormously by taking into account the various approaches from different branches of mathematics. ... The main question in this context is: What kind of limit behavior can have a combinatorial object when it "grows" ?*

   [2] *Between "very large" and "infinite"* (Bedlewo 2012)

   [3] Takagi lecture of Mathematical Society of Japan (Tohoku University 2015)

   [4] see also A. Hora: The limit shape problem for emsembles of Young diagrams, Springer 2017.

2. Complex networks — modelling real world large networks

   [1] A.-L. Barabási and R. Albert (1999) — scale free networks

   [2] D. J. Watts and S. H. Strogatz (1998) — small world networks

   [3] F. Chung and L. Lu (2006), R. Durrett (2007), L. Lovasz (2012).

My Motivations and Backgrounds

(2) Quantum probability = Noncommutative Probability = Algebraic Probability

1. J. von Neumann: Mathematische Grundlagen der Quantenmechanik (1932)

   Mathematical theory for the probabilistic interpretation in quantum mechanics in terms of operators on Hilbert spaces.

2. The term *quantum probability* was introduced by L. Accardi (Roma) around 1978.

3. R. Hudson and K. R. Parthasarathy (1984) initiated *quantum Ito calculus.*

4. P.-A. Meyer: *Quantum Probability for Probabilists*, LNM 1538 (1993).

5. N. Obata: Quantum probability + graph theory and network science since 1998.

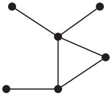
$\Downarrow \qquad \Downarrow$

A paradigm of non-commutative analysis

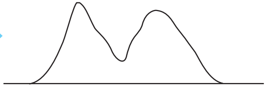
Main Theme: Asymptotic Spectral Analysis of Growing Graphs

- Spectral analysis of graphs

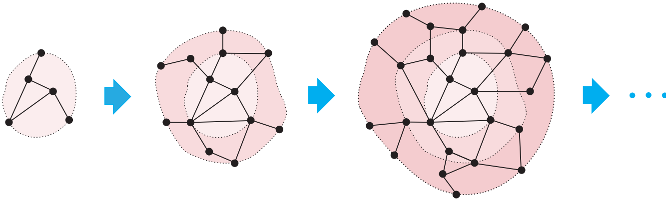
$G = (V, E)$ $\qquad$ $\mu(dx) = f(x)\,dx$



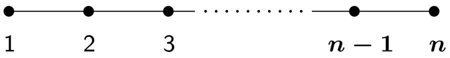
$$A = [A_{xy}]$$



- Growing graphs



A Motivating Example: P_n as $n \to \infty$



Adjacency matrix:

$$
A = \begin{bmatrix}
0 & 1 & & & & \\
1 & 0 & 1 & & & \\
 & 1 & 0 & 1 & & \\
 & & \ddots & \ddots & \ddots & \\
 & & & 1 & 0 & 1 \\
 & & & & 1 & 0
\end{bmatrix}
$$

Spectrum:

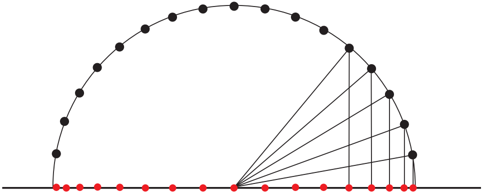
$$
\mathrm{Spec}\,(P_n) = \left\{ 2\cos\frac{k\pi}{n+1}\,;\, 1 \le k \le n \right\}
$$

- We are interested in $n \to \infty$.

A Motivating Example: P_n as $n \to \infty$

$$\mathbf{Spec}\,(P_n) = \left\{ 2\cos\frac{k\pi}{n+1}\,;\ 1 \le k \le n \right\}$$

just eigenvalues



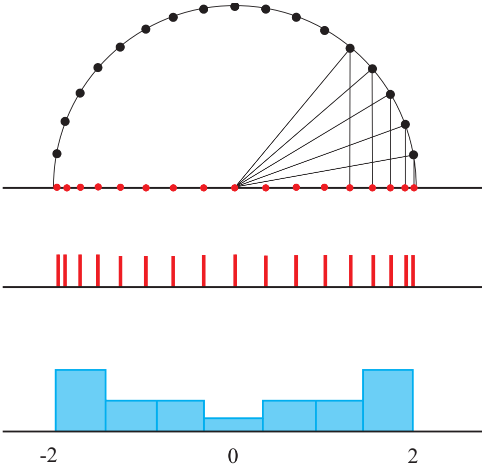
A Motivating Example: P_n as $n \to \infty$

$$\mathbf{Spec}\,(P_n) = \left\{ 2\cos\frac{k\pi}{n+1}\,;\ 1 \le k \le n \right\}$$

just eigenvalues

with multiplicities

histogram
showing “distribution”



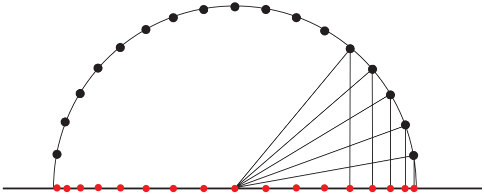
A Motivating Example: P_n as $n \to \infty$

Spectral distribution (= eigenvalue distribution):

$$\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{2\cos\frac{k\pi}{n+1}} \quad \Leftarrow \quad \mathrm{Spec}\,(P_n) = \left\{ 2\cos\frac{k\pi}{n+1}\,;\, 1 \le k \le n \right\}$$

where δ_a stands for the point mass at a:

$$\int_{-\infty}^{+\infty} f(x)\delta_a(dx) = f(a), \qquad f \in C_b(\mathbb{R}).$$



A Motivating Example: P_n as $n \to \infty$

Spectral distribution of P_n:

$$\mu_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{2\cos\frac{k\pi}{n+1}}$$

For $f \in C_b(\mathbb{R})$ we have

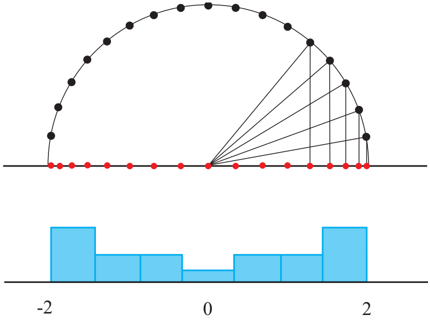
$$\int_{-\infty}^{+\infty} f(x)\mu_n(dx)$$

$$= \frac{1}{n} \sum_{k=1}^{n} f\left(2\cos\frac{k\pi}{n+1}\right)$$

$$\to \int_0^1 f(2\cos \pi t)dt$$

$$= \int_{-2}^{+2} f(x)\, \frac{dx}{\pi\sqrt{4-x^2}}.$$

This is the limit distribution (arcsine law).



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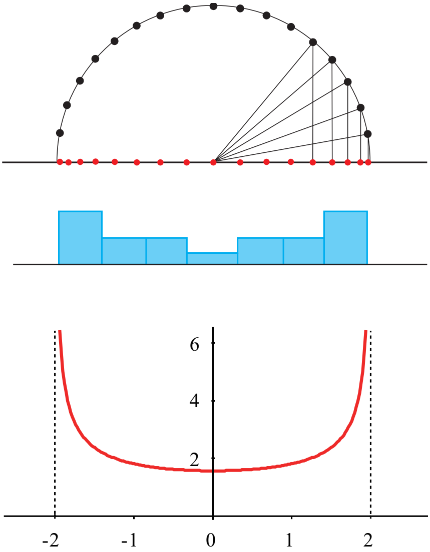

Plan

1. Basic Concepts of Quantum Probability
2. Interacting Fock Space and Quantum Decomposition
3. Spectral Distributions of Graphs
4. Quantum Walks on Spidernets
5. Asymptotic Spectral Distributions of Regular Graphs
6. Graph Products and Concepts of Independence
7. Counting Walks
8. Bivariate Extension: An Example

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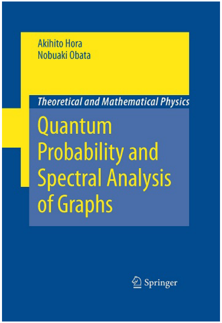
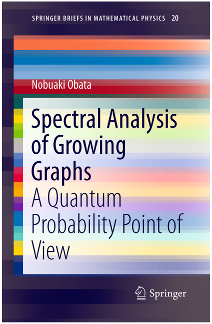
- Main References

[1] A. Hora and N. Obata: *Quantum Probability and Spectral Analysis of Graphs*, Springer, 2007.

[2] N. Obata: *Spectral Analysis of Growing Graphs. A Quantum Probability Point of View*, Springer, 2017.





1. Basic Concepts of Quantum Probability

1.1. Algebraic Probability Spaces

Definition

A pair $(\mathcal{A}, \varphi)$ is called an *algebraic probability space* if $\mathcal{A}$ is a unital $*$-algebra over $\mathbb{C}$ and φ a *state* on it, i.e.,

(i) $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear function;

(ii) positive, i.e., $\varphi(a^*a) \geq 0$;

(iii) normalized, i.e., $\varphi(1_{\mathcal{A}}) = 1$.

Definition

Each $a \in \mathcal{A}$ is called an *(algebraic) random variable*. It is called *real* if $a = a^*$.

- $(\Omega, \mathcal{F}, P)$: classical (Kolmogorovian) probability space

$$\mathcal{A} = L^{\infty-}(\Omega, \mathcal{F}, P) = \bigcap_{1 \leq p < \infty} L^p(\Omega, \mathcal{F}, P)$$

$$= \{X : \Omega \to \mathbb{C}\,;\, \mathbf{E}[|X|^m] < \infty \text{ for all } m \geq 1\}$$

$$\varphi(X) = \mathbf{E}[X], \qquad X \in \mathcal{A}.$$

1.2. Statistics of Algebraic Random Variables

Definition

(1) For a random variable a in $(\mathcal{A},\varphi)$ the *mixed moments* are defined by

$$\varphi(a^{\epsilon_m}\cdots a^{\epsilon_2}a^{\epsilon_1}), \qquad \epsilon_1,\epsilon_2\dots,\epsilon_m\in\{1,*\}.$$

(2) For a real random variable $a=a^*\in\mathcal{A}$ the mixed moments are reduced to the *moment sequence*:

$$\varphi(a^m), \quad m=1,2,\dots.$$

Definition

(1) Two algebraic random variables a in $(\mathcal{A},\varphi)$ and b in $(\mathcal{B},\psi)$ are called *stochastically equivalent* $a\stackrel{\mathrm{m}}{=}b$ if their all mixed moments coincide:

$$\varphi(a^{\epsilon_m}\cdots a^{\epsilon_2}a^{\epsilon_1})=\psi(b^{\epsilon_m}\cdots b^{\epsilon_2}b^{\epsilon_1}).$$

(2) For two real random variables $a=a^*$ in $(\mathcal{A},\varphi)$ and $b=b^*$ in $(\mathcal{B},\psi)$,

$$a\stackrel{\mathrm{m}}{=}b \quad\Longleftrightarrow\quad \varphi(a^m)=\psi(b^m) \text{ for all } m=1,2,\dots.$$

1.3. Spectral Distributions

Theorem (spectral distribution)

For a real random variable $a = a^* \in \mathcal{A}$ *there exists a probability measure* μ *on* $\mathbb{R} = (-\infty, +\infty)$ *such that*

$$\varphi(a^m) = \int_{-\infty}^{+\infty} x^m \mu(dx) \equiv M_m(\mu), \quad m = 1, 2, \dots.$$

This μ *is called the spectral distribution of* a *in the state* φ*.*

1. Existence proof is by Hamburger's theorem using Hanckel determinants.
2. μ is not uniquely determined in general (determinate moment problem).
3. μ is unique, for example, if

$$\sum_{m=1}^{\infty} M_{2m}^{-\frac{1}{2m}} = +\infty \quad \text{(Carleman's moment test)}$$

1.4. Classical Probability vs Quantum Probability

	Classical Probability	Quantum Probability
probability space	$(\Omega, \mathcal{F}, P)$	$(\mathcal{A}, \varphi)$
random variable	$X : \Omega \to \mathbb{R}$	$a = a^* \in \mathcal{A}$
expectation	$\mathbf{E}[X] = \int_\Omega X(\omega)\, P(d\omega)$	$\varphi(a)$
moments	$\mathbf{E}[X^m]$	$\varphi(a^m)$
distribution	$\mu_X((-\infty, x]) = P(X \le x)$	NA
	$\mathbf{E}[X^m] = \int_{-\infty}^{+\infty} x^m \mu_X(dx)$	$\varphi(a^m) = \int_{-\infty}^{+\infty} x^m \mu_a(dx)$
independence	$\mathbf{E}[X^m Y^n] = \mathbf{E}[X^m]\mathbf{E}[Y^n]$	many variants
LLN	$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} X_k$	many variants
CLT	$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k$	many variants

1.5. Matrix Algebras and States

Equipped with the usual matrix operations,

$$\mathcal{A} = M(n, \mathbb{C}) = \{a = [a_{ij}]\,;\, a_{ij} \in \mathbb{C}\}$$

becomes a unital $*$-algebra over $\mathbb{C}$.

(i) *normalized trace*:

$$\varphi(a) = \frac{1}{n}\mathrm{Tr}\,(a) = \frac{1}{n}\sum_{i=1}^{n} a_{ii}\,, \qquad a = [a_{ij}].$$

(ii) *vector state*:

$$\varphi(a) = \langle \xi, a\xi \rangle, \qquad \xi \in \mathbb{C}^n, \quad \|\xi\| = 1.$$

Lemma (exercise)

A general form of a state on $M(n, \mathbb{C})$ is given by

$$\varphi(a) = \mathrm{Tr}\,(\rho a),$$

where ρ is a density matrix, i.e., $\rho = \rho^* \geq 0$ and $\mathrm{Tr}\,(\rho) = 1$.

Exercises

Exercise 1 Let $(\mathcal{A}, \varphi)$ be an algebraic probability space and $a, b, \dots \in \mathcal{A}$.

(1) Show that $\varphi(a^*) = \overline{\varphi(a)}$.

(2) Show that $|\varphi(a^*b)|^2 \le \varphi(a^*a)\varphi(b^*b)$.

(3) Show that $a \overset{\mathrm{m}}{=} 0$ if $\varphi(a) = \varphi(a^*) = 0$ and $\varphi(a^*a) = \varphi(aa^*) = 0$.

(4) Show that $a \overset{\mathrm{m}}{=} \varphi(a)1_{\mathcal{A}}$ if $\varphi(a^*a) = \varphi(aa^*) = |\varphi(a)|^2$.

Exercise 2 A matrix $\rho \in M(n, \mathbb{C})$ is called a *density matrix* if $\rho = \rho^* \ge 0$ and $\mathrm{Tr}\,\rho = 1$. Show that for any state φ on $M(n, \mathbb{C})$ there exists a unique density matrix ρ such that $\varphi(a) = \mathrm{Tr}\,(\rho a)$.

Exercise 3 Consider a sequence of cycles C_n.

(1) Write the adjacency matrix A_n of C_n.

(2) Find $\mathbf{Spec}\,(C_n)$.

(3) Write the eigenvalue distribution μ_n of C_n.

(4) Find the limit of μ_n (after normalization if necessary).

2. Interacting Fock Space and Quantum Decomposition

2.1. Jacobi Coefficients

Definition

A pair of sequences $(\{\omega_n\}, \{\alpha_n\})$ is called *Jacobi coeffcients* if

1. (infinite type) $\{\omega_n\}$ and $\{\alpha_n\}$ are infinite sequences such that $\omega_n > 0$ and $\alpha_n \in \mathbb{R}$ for all $n = 1, 2, \dots$;

or

2. (finite type) there exists $d \geq 1$ such that $\{\omega_n\} = \{\omega_1, \dots, \omega_{d-1}\}$ is a positive sequence of $d-1$ terms and $\{\alpha_n\} = \{\alpha_1, \dots, \alpha_d\}$ is a real sequence of d terms. (For $d = 1$ we tacitly understand $\{\omega_n\}$ is an empty sequence.) This d is called the *length* of $(\{\omega_n\}, \{\alpha_n\})$.

Set

$$\mathfrak{J} = \mathfrak{J}_\infty \cup \mathfrak{J}_{\text{fin}}\,, \qquad \mathfrak{J}_{\text{fin}} = \bigcup_{1 \leq d < \infty} \mathfrak{J}_d\,,$$

where $\mathfrak{J}_\infty$ is the set of Jacobi coefficients of infinite type and $\mathfrak{J}_d$ is the set of Jacobi coefficients of finite length d.

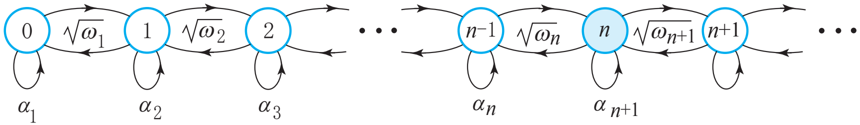
2.1. Interacting Fock Space (IFS)

$(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$: Jacobi coefficients of length d ($1 \le d \le \infty$),

Γ: d-dimensional Hilbert space with CONS $\{\Phi_n\} = \{\Phi_0, \Phi_1, \Phi_2, \dots\}$

Define three linear operators A^+, A^-, A° by

$$A^+\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1}, \quad A^-\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1}, \quad A^\circ\Phi_n = \alpha_{n+1}\Phi_n.$$



- More precisely, A^+, A^-, A° are linear operators defined on the domain $\Gamma_0 =$ linear span of $\{\Phi_n\} \subset \Gamma$.

Definition

The quintuple $(\Gamma, \{\Phi_n\}, A^+, A^-, A^\circ)$ is called an *interacting Fock space (IFS)* associated with Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$. We call A^+, A^- and A° the *creation, annihilation and conservation operators*, respectively.

2.2. Vacuum Spectral Distributions

Given IFS $(\Gamma, \{\Phi_n\}, A^+, A^-, A^\circ)$, we consider the algebraic probability space

$$\mathcal{A} = *\text{-algebra generated by } A^+, A^-, A^\circ$$

with the vacuum state

$$\langle a \rangle = \langle \Phi_0, a\Phi_0 \rangle, \qquad a \in \mathcal{A}.$$

In particular, we are interested in the real random variable

$$A^+ + A^- + A^\circ$$

called the *canonical random variable* of the IFS $(\Gamma, \{\Phi_n\}, A^+, A^-, A^\circ)$.

Definition (Vacuum spectral distribution)

A probability distribution μ characterized by

$$\langle \Phi_0, (A^+ + A^- + A^\circ)^m \Phi_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots,$$

is called the *vacuum spectral distribution* of IFS $(\Gamma, \{\Phi_n\}, A^+, A^-, A^\circ)$.

2.3. Boson, Fermion and Free Fock Spaces

1. Boson Fock space $(\{\omega_n = n\}, \{\alpha_n \equiv 0\})$

$$A^- A^+ - A^+ A^- = I \qquad \text{(canonical commutation relation)}$$

The vacuume spectral distribution: $\mu(dx) = \dfrac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ (normal distribution)

2. Free Fock space $(\{\omega_n \equiv 1\}, \{\alpha \equiv 0\})$

$$A^- A^+ = I$$

The vacuume spectral distribution: $\mu(dx) = \dfrac{1}{2\pi} \sqrt{4 - x^2}\, dx$ (semi-circle law)

3. Fermioin Fock space $(\{\omega_1 = 1, \omega_2 = \omega_2 = \cdots = 0\}, \{\alpha_n \equiv 0\})$

$$A^- A^+ + A^+ A^- = I \qquad \text{(canonical anti-commutation relation)}$$

The vacuume spectral distribution: $\mu = \dfrac{1}{2} \delta_{-1} + \dfrac{1}{2} \delta_{+1}$ (Bernoulli distribution)

4. q-Fock space $(\{\omega_n = [n]_q\}, \{\alpha_n \equiv 0\})$

$$A^- A^+ - q A^+ A^- = I \qquad (q\text{-commutation relation})$$

The vacuume spectral distribution: μ_q in terms of Jacobi theta function.

2.4. Orthogonal Polynomials

$\mu(dx) \in \mathfrak{P}_{\mathrm{fm}}(\mathbb{R})$: a probability distribution with finite moments of all orders
Define an inner product by

$$\langle f, g\rangle = \int_{-\infty}^{+\infty} f(x)g(x)\mu(dx), \qquad f, g \in L^2(\mathbb{R}, \mu; \mathbb{R}).$$

Definition (Orthogonal polynomials)

Applying the Gram-Schmidt orthogonalization to $1, x, x^2, \dots, x^n, \dots$ we obtain a sequence of polynomials:

$$P_0(x) = 1, \quad P_1(x) = x - \frac{\langle x, P_0\rangle}{\langle P_0, P_0\rangle} P_0(x), \quad P_n(x) = x^n - \sum_{k=0}^{n-1} \frac{\langle x^n, P_k\rangle}{\langle P_k, P_k\rangle} P_k(x).$$

We call $\{P_n(x)\}$ the *orthogonal polynomials* associated to μ.

Note: The orthogonalization process stops at $n = d$ if $\langle P_d, P_d\rangle = 0$ happens. In that case we consider $\{P_0(x), P_1(x), \dots, P_{d-1}(x)\}$ as the orthogonal polynomials. That happens if and only if $|\mathrm{supp}\,\mu| = d$ (exercise).

2.5. Three-Term Recurrence Relation

Theorem (Three-term recurrence relation)

Assume that $|\mathrm{supp}\,\mu| = \infty$. Let $\{P_n(x)\}$ be the orthogonal polynomials associated to μ. Then there exist Jacobi coeficients $(\{\omega_n\}, \{\alpha_n\})$ of infinite type such that

$$P_0 = 1, \quad P_1 = x - \alpha_1, \quad xP_n = P_{n+1} + \alpha_{n+1} P_n + \omega_n P_{n-1}\,.$$

Note: If $|\mathrm{supp}\,\mu| = d < \infty$, we get Jacobi coeficients $(\{\omega_n\}, \{\alpha_n\})$ of length d and the same recurrence relation holds.

Proof (exercise)

- We note that

$$\alpha_1 = \int_{-\infty}^{+\infty} x\mu(dx) = \text{mean}(\mu),$$

$$\omega_1 = \int_{-\infty}^{+\infty} (x-\alpha_1)^2 \mu(dx) = \text{variance}(\mu),$$

$$\omega_n \omega_{n-1} \cdots \omega_1 = \int_{-\infty}^{+\infty} P_n(x)^2 \mu(dx),$$

2.5. Three-Term Recurrence Relation

▶ Three-Term Recurrence Relation $\Longrightarrow$ IFS Structure

1. The three-term recurrence relation:

$$P_0 = 1, \quad P_1 = x - \alpha_1, \quad xP_n = P_{n+1} + \alpha_{n+1} P_n + \omega_n P_{n-1}\,.$$

2. Define

$$\Phi_n(x) = \frac{1}{\|P_n\|} P_n(x) = \frac{1}{\sqrt{\omega_n \omega_{n-1} \cdots \omega_1}} P_n(x).$$

   Then $\{\Phi_n(x)\}$ becomes an orthonormal set in $L^2(\mathbb{R}, \mu)$.

3. Let Γ be the Hilbert space spanned by $\{\Phi_n(x)\}$ (not necessarily $\Gamma = L^2(\mathbb{R}, \mu)$).

4. Define

$$A^+ P_n = P_{n+1}\,, \qquad A^\circ P_n = \alpha_{n+1} P_n\,, \qquad A^- P_n = \omega_n P_{n-1}\,.$$

   Then

$$A^+ \Phi_n = \sqrt{\omega_{n+1}}\, \Phi_{n+1}, \quad A^\circ \Phi_n = \alpha_{n+1} \Phi_n, \quad A^- \Phi_n = \sqrt{\omega_n}\, \Phi_{n-1}\,.$$

   Namely, $(\Gamma, \{\Phi_n\}, A^+, A^\circ, A^-)$ is an IFS.

2.5. Three-Term Recurrence Relation

- Computing the vacuum spectral distribution of $(\Gamma, \{\Phi_n\}, A^+, A^\circ, A^-)$

1. Set $A = A^+ + A^\circ + A^-$. Then

$$\begin{aligned} AP_n(x) &= A^+ P_n(x) + A^\circ P_n(x) + A^- P_n(x) \\ &= P_{n+1}(x) + \alpha_{n+1} P_n(x) + \omega_n P_{n-1}(x) \\ &= x P_n(x) \end{aligned}$$

2. Hence for $\Phi_0(x) = P_0(x) = 1$ we have

$$A^m \Phi_0(x) = x^m \Phi_0(x) = x^m.$$

3. Then,

$$\langle \Phi_0, (A^+ + A^\circ + A^-)^m \Phi_0 \rangle = \langle 1, x^m \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx),$$

which means that the vacuum spectral distribution of $(\Gamma, \{\Phi_n\}, A^+, A^\circ, A^-)$ is the initial μ.

2.6. IFS Structure in Orthogonal Polynomials

Summing up,

Theorem

Let $(\Gamma, A^+, A^-, A^\circ)$ *be an interacting Fock space given by*

$$A^+\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1}, \quad A^-\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1}, \quad A^\circ\Phi_n = \alpha_{n+1}\Phi_n.$$

Then the vacuum spectral distribution of $A = A^+ + A^\circ + A^-$ *is a probability distribution* μ *of which the orthogonal polynomials* $\{P_n(x)\}$ *are given by*

$$P_0 = 1, \quad P_1 = x - \alpha_1, \quad xP_n = P_{n+1} + \alpha_{n+1}P_n + \omega_n P_{n-1}\,.$$

Namely, we have

$$\langle \Phi_0, A^m\Phi_0\rangle = \langle \Phi_0, (A^+ + A^\circ + A^-)^m\Phi_0\rangle = \int_{-\infty}^{+\infty} x^m \mu(dx).$$

2.6. IFS Structure in Orthogonal Polynomials

$\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbb{R})$ $\overset{\star}{=\!=\!=\!=}$ vaccum spectral distribution

$\downarrow$

$\{P_n(x)\}$ orthogonal polynomials

$\downarrow$

$(\{\omega_n\}, \{\alpha_n\})$ Jacobi coefficients

$\downarrow$

$(\Gamma, \{\Phi_n\}, A^+, A^\circ, A^-)$ IFS

$\star$ means: $\langle \Phi_0, (A^+ + A^\circ + A^-)^m \Phi_0 \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx)$

Note: μ is not uniquely determined by $(\{\omega_n\}, \{\alpha_n\})$ when μ is not a solution to the determinate moment problem.

2.7. Quantum Decomposition

Theorem (quantum decomposition)

Let $(\mathcal{A}, \varphi)$ *be an algebraic probability space and* $a = a^* \in \mathcal{A}$ *a real random variable. Then there exists an interacting Fock space* $(\Gamma, \{\Phi_n\}, A^+, A^-, A^\circ)$ *such that*

$$a \overset{\mathrm{m}}{=} A^+ + A^- + A^\circ.$$

In particular, if a classical random variable X *has finite moments of all orders, there exists an IFS* $(\Gamma, \{\Phi_n\}, A^+, A^-, A^\circ)$ *such that*

$$X \overset{\mathrm{m}}{=} A^+ + A^- + A^\circ.$$

Proof. Let μ be the spectral distribution of a. Consider the Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\})$ and the associated IFS $(\Gamma, \{\Phi_n\}, A^+, A^-, A^\circ)$. Then we have

$$\varphi(a^m) = \int_{-\infty}^{+\infty} x^m \mu(dx) = \langle \Phi_0, (A^+ + A^\circ + A^-)^m \Phi_0 \rangle.$$

▶ We apply the above idea to the adjacency matrix of a graph.

2.8. How to Explicitly Calculate μ from $(\{\omega_n\}, \{\alpha_n\})$

Determinate moment problem

In general, $\mu \in \mathfrak{P}_{\mathrm{fm}}(\mathbb{R})$ is not uniquely determined by the moments. Namely, it may happen that $\mu \neq \nu$ but

$$\int_{-\infty}^{+\infty} x^m \mu(dx) = \int_{-\infty}^{+\infty} x^m \nu(dx) = M_m, \qquad m = 0, 1, 2, \dots .$$

We say that *μ is the unique solution to a determinate moment problem* if μ is uniquely determined by its moments.

Some sufficient conditions for uniqueness of the determinate moment problem:

(i) $\operatorname{supp} \mu$ is finite.

(ii) μ is supported by a compact set.

(iii) (Carleman's moment test) $\displaystyle\sum_{m=1}^{\infty} M_{2m}^{-\frac{1}{2m}} = \infty$.

(iv) (Carleman) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{\omega_n}} = \infty$.

- In fact, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

2.8. How to Explicitly Calculate μ from $(\{\omega_n\}, \{\alpha_n\})$

Continued fraction For saving space we write

$$\cfrac{1}{z-\alpha_1-\cfrac{\omega_1}{z-\alpha_2-\cfrac{\omega_2}{z-\alpha_3-\cfrac{\omega_3}{z-\alpha_4-\cdots}}}}$$

$$= \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \frac{\omega_3}{z-\alpha_4} - \cdots$$

$$= \lim_{n\to\infty} \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \cdots - \frac{\omega_{n-1}}{z-\alpha_n}$$

2.8. How to Explicitly Calculate μ from $(\{\omega_n\}, \{\alpha_n\})$

Theorem (Cauchy–Stieltjes transform and inversion formula)

If μ is a unique solution to the determinate moment problem, we have

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \frac{\omega_3}{z-\alpha_4} - \cdots$$

where the right-hand side is convergent in $\{\mathrm{Im}\, z \neq 0\}$. Moreover, the absolutely continuous part of μ is given by

$$\rho(x) = -\frac{1}{\pi} \lim_{y \to +0} \mathrm{Im}\, G_\mu(x+iy)$$

Useful properties of $G(z)$

1. $G(z)$ is holomorphic in $\{\mathrm{Im}\, z \neq 0\}$.
2. $G(\bar{z}) = \overline{G(z)}$.
3. $\mathrm{Im}\, G(z) < 0$ for $\mathrm{Im}\, z > 0$.

2.8. How to Explicitly Calculate μ from $(\{\omega_n\}, \{\alpha_n\})$

Exercise: Consider the Jacobi coefficients $(\{\omega_n \equiv 1\}, \{\alpha_n \equiv 0\})$.

1. Check Carleman's condition.
2. Calculate the continued fraction:
$$G(z) = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \frac{\omega_3}{z-\alpha_4} - \cdots$$
3. Apply the inversion formula to get the absolutely continuous part:
$$\rho(x) = -\frac{1}{\pi} \lim_{y \to +0} \operatorname{Im} G(x+iy)$$
4. Check
$$\int_{-\infty}^{+\infty} \rho(x)dx = 1.$$

Theorem (free Fock space)

The vacuum spectral distribution of free Fock space is given by the semi-circle law:
$$\mu(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\, 1_{[-2,2]}(x)dx$$

2.9. Chebyshev Polynomials (exercise)

1st kind $\{T_n(x)\}$ defined by $T_n(\cos\theta) = \cos n\theta$, $x = \cos\theta$

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0.$$

Modifiying $T_n(x)$ as

$$\tilde{T}_0(x) = 1, \quad \tilde{T}_n(x) = \Big(\frac{1}{\sqrt{2}}\Big)^{n-2} T_n\Big(\frac{x}{\sqrt{2}}\Big), \quad n \ge 1.$$

Three-term recurrence relation:

$$x\tilde{T}_1(x) = \tilde{T}_2(x) + \tilde{T}_0(x), \quad x\tilde{T}_n(x) = \tilde{T}_{n+1}(x) + \frac{1}{2}\,\tilde{T}_{n-1}(x), \quad n \ge 2.$$

Orthogonal relation wrt *normalized arcsine law*:

$$\int_{-\sqrt{2}}^{\sqrt{2}} \tilde{T}_m(x)\tilde{T}_n(x)\,\frac{dx}{\pi\sqrt{2-x^2}} = 0, \qquad m \neq n.$$

Jacobi coefficients: $(\{\omega_n\} = \{1, 1/2, 1/2, \dots\}, \{\alpha_n \equiv 0\})$

2.9. Chebyshev Polynomials (exercise)

2nd kind $\{U_n(x)\}$ defined by $U_n(\cos\theta) = \dfrac{\sin(n+1)\theta}{\sin\theta}$, $x = \cos\theta$

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) - 2xU_n(x) + U_{n-1}(x) = 0,$$

Modifying $U_n(x)$ as

$$\tilde{U}_n(x) = U_n\Big(\frac{x}{2}\Big), \qquad n \geq 0.$$

Three-term recurrence relation:

$$\tilde{U}_0(x) = 1, \quad \tilde{U}_1(x) = x, \quad x\tilde{U}_n(x) = \tilde{U}_{n+1}(x) + \tilde{U}_{n-1}(x), \quad n \geq 1.$$

Orthogonal relation wrt *Wigner's semi-circle law*:

$$\int_{-2}^{2} \tilde{U}_m(x)\tilde{U}_n(x)\, \frac{1}{2\pi}\sqrt{4-x^2}\, dx = 0, \quad m \neq n.$$

Jacobi coefficients: $(\{\omega_n \equiv 1\}, \{\alpha_n \equiv 0\})$

2.10. Some Topics Relevant to Quantum Decomposition

[1] Quantum walks [Konno–Obata–Segawa, CMP (2013)]

[2] Random walks [Y. Kang, Physica (2016)]

[3] Another growing graphs [Kurihara–Hibino, IDAQP (2011), Gaxiola (2017)]

[4] S. Jafarizadeh and R. Sufiani: Evaluation of effective resistances in pseudo-distance-regular resistor networks, J. Stat. Phys. 139 (2010).

[5] Hecke algebras for p-adic PGL_2 [Hasegawa et al. arXiv:1803.02217]

[6] see also R. Schott and G. S. Staple: "Operator Calculus on Graphs" (2012).

[7] Stochastic processes
Of course the root is the quantum stochastic calculus due to Hudson–Parthasarathy (1984), and many others.
Quantum white noise calculus [Ji and others, Obata also]
Quantum decomposition of Lévy processes [Y-J. Lee and H.-H. Shih, others]

[8] Quantum decomposition without moments [Accardi–Rebei–Riahi (2013)]

Exercises

Exercise 4 Let $\{P_n(x)\}$ be the orthogonal polynomials associated to a probability distribution μ with $|\mathrm{supp}\,\mu| = \infty$. Derive the three-term recurrence relation:

$$P_0 = 1, \quad P_1 = x - \alpha_1, \quad xP_n = P_{n+1} + \alpha_{n+1} P_n + \omega_n P_{n-1}\,,$$

where $(\{\omega_n\}, \{\alpha_n\})$ are Jacobi coeficients of infinite type. Moreover, show that

$$\alpha_1 = \int_{-\infty}^{+\infty} x\mu(dx) = \text{mean}(\mu),$$

$$\omega_1 = \int_{-\infty}^{+\infty} (x - \alpha_1)^2 \mu(dx) = \text{variance}(\mu),$$

$$\omega_n \omega_{n-1} \cdots \omega_1 = \int_{-\infty}^{+\infty} P_n(x)^2 \mu(dx),$$

Exercises

Exercise 5 Consider the IFS $(\Gamma, \{\Phi_n\}, A^+, A^-)$ associated to Jacobi coefficients:

$$\omega_1 = 2, \quad \omega_2 = 1, \quad \omega_3 = 2, \quad (\omega_n = 0, \ \ n \ge 4);$$
$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \quad (\alpha_n = 0, \ \ n \ge 5).$$

(1) Calculate the continued fraction:

$$G(z) = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \frac{\omega_3}{z-\alpha_4} - \cdots$$

(2) Find the probability distribution μ such that

$$G(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x}\,.$$

(3) Show that

$$\langle \Phi_0, (A^+ + A^-)^{2m-1}\Phi_0 \rangle = 0,$$
$$\langle \Phi_0, (A^+ + A^-)^{2m}\Phi_0 \rangle = \frac{1}{3}(4^m + 2).$$

Exercises

Exercise 6 (free Fock space) Consider the free Fock space $(\Gamma, \{\Phi_n\}, A^+, A^-)$, i.e., the IFS associated to Jacobi coefficients $(\{\omega_n \equiv 1\}, \{\alpha_n \equiv 0\})$.

(1) Check Carleman's condition.

(2) Calculate the continued fraction:

$$G(z) = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \frac{\omega_3}{z-\alpha_4} - \cdots$$

(3) Apply the inversion formula to get the absolutely continuous part:

$$\rho(x) = -\frac{1}{\pi} \lim_{y \to +0} \operatorname{Im} G(x+iy)$$

(4) Check

$$\int_{-\infty}^{+\infty} \rho(x)dx = 1.$$

(5) Show that

$$\langle \Phi_0, (A^+ + A^-)^{2m} \Phi_0 \rangle = \frac{1}{m+1} \binom{2m}{m} \quad \text{(Catalan number)}$$

3. Spectral Distributions of Graphs

3.1. Graphs and Matrices

Definition (graph)

A (finite or infinite) *graph* is a pair $G = (V, E)$, where V is the set of *vertices* and E the set of *edges*. We write $x \sim y$ (adjacent) if they are connected by an edge.

Definition (adjacency matrix)

The *adjacency matrix* $A = [A_{xy}]$ is defined by $A_{xy} = \begin{cases} 1, & x \sim y, \\ 0, & \text{otherwise.} \end{cases}$

Assumption 1 [connected] Any pair of distinct vertices are connected by a walk.
Assumption 2 [locally finite] $\deg_G(x) = (\text{degree of } x) < \infty$ for all $x \in V$.

Definition (adjacency algebra)

Let $G = (V, E)$ be a graph. The $*$-algebra generated by the adjacency matrix A is called the *adjacency algebra* of G and is denoted by $\mathcal{A}(G)$. In fact, $\mathcal{A}(G)$ is the set of polynomials in A.

- Equipped with a state φ, $(\mathcal{A}(G), \varphi)$ becomes an algebraic probability space.

3.2. Tracial States for Finite Graphs

$$\varphi_{\mathrm{tr}}(a) = \langle a \rangle_{\mathrm{tr}} = \frac{1}{|V|}\,\mathrm{Tr}\,(a) = \frac{1}{|V|}\sum_{x\in V}\langle e_x\,, a e_x\rangle, \qquad a \in \mathcal{A},$$

where $\{e_x\,;\, x\in V\}$ is the canonical basis of $C(V)$.

Lemma

The spectral distribution of A in φ_{tr} coincides with the eigenvalue distribution of G, namely, letting μ be the eigenvalue distribution of G, we have

$$\langle A^m \rangle_{\mathrm{tr}} = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 1, 2, \dots .$$

Proof. Let $\mathbf{Spec}\,(G) = \{\lambda_1(m_1), \dots, \lambda_s(m_s)\}$ be the spectrum of G, where λ_i is an eigenvalue of A with multiplicity m_i. The eigenvalue distribution is defined by

$$\mu = \frac{1}{|V|}\sum_{i=1}^{s} m_i \delta_{\lambda_i}\,.$$

Then we have

$$\langle A^m \rangle_{\mathrm{tr}} = \frac{1}{|V|}\,\mathrm{Tr}\,(A^m) = \frac{1}{|V|}\sum m_i \lambda_i^m = \int_{-\infty}^{+\infty} x^m \mu(dx).$$

3.3. Vacuum State (at a fixed origin $o \in V$)

Fix a vertex $o \in V$ as an origin (root).
The *vacuum state at* $o \in V$ is the vector state defined by

$$\varphi(a) = \langle a \rangle_o = \langle \delta_o \,, a\delta_o \rangle, \qquad a \in \mathcal{A}(G).$$

Lemma

Let μ be the spectral distribution of A. Then we have

$$\langle A^m \rangle_o = \langle \delta_o, A^m \delta_o \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx) = |\{m\text{-step walks from } o \text{ to } o\}|.$$

Proof. We need only to note that

$$\langle \delta_o, A^m \delta_o \rangle = (A^m)_{oo} = \sum A_{ox_1} A_{x_1x_2} \cdots A_{x_{m-1}o} \,,$$

where $A_{ox_1} A_{x_1x_2} \cdots A_{x_{m-1}o} = 1$ if $o \sim x_1 \sim x_2 \sim \cdots \sim x_{m-1} \sim o$ and $= 0$ otherwise.

3.4. Our Main Questions

1. Given a graph $G = (V, E)$ and a state $\langle \cdot \rangle$ on $\mathcal{A}(G)$, find the spectral distribution of A, i.e., a probability distribution μ on $\mathbb{R}$ satisfying

$$\langle A^m \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx)$$

2. Given growing graphs $G_\nu = (V_\nu, E_\nu)$ and states $\langle \cdot \rangle_\nu$ on $\mathcal{A}(G_\nu)$, find the asymptotic spectral distribution, i.e., a probability measure μ on $\mathbb{R}$ satisfying

$$\lim_{\nu} \left\langle \left(\frac{A_\nu - \langle A_\nu \rangle_\nu}{\Sigma(A_\nu)} \right)^m \right\rangle_\nu = \int_{-\infty}^{+\infty} x^m \mu(dx).$$

- Quantum Probabilistic Approaches (Use of Non-Commutativity)

1. Method of quantum decomposition:

$$A = A^+ + A^- + A^\circ$$

2. Sum of independent random variables and quantum central limit theorem (CLT):

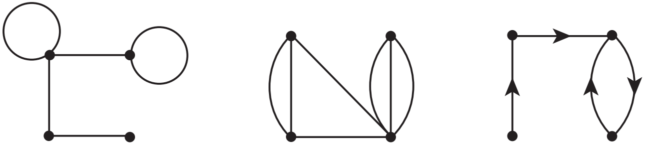
$$A = B_1 + B_2 + \cdots + B_n$$

Of course, we may focus on generalizations of graphs

Lemma

A matrix A with index set $V \times V$ is the adjacency matrix of a graph on V if and only if (i) $(A)_{xy} \in \{0, 1\}$; (ii) $(A)_{xy} = (A)_{yx}$; (iii) $(A)_{xx} = 0$.

1. Graph with loops. Dropping (iii) allows a loop connecting a vertex with itself.
2. Multigraph. Relaxing (i) as $(A)_{xy} \in \{0, 1, 2, \dots\}$ allows a *multi-edge*.
3. Digraph (directed graph). Dropping (ii) gives rise to orientation of edges, namely, $(A)_{xy} = 1 \Leftrightarrow x \to y$.
4. Network. In a broad sense, an arbitrary matrix A with index set $V \times V$ gives rise to a *network*, where each directed edge $x \to y$ is associated with the value $(A)_{xy}$ whenever $(A)_{xy} \neq 0$. A transition diagram of a Markov chain is an example.



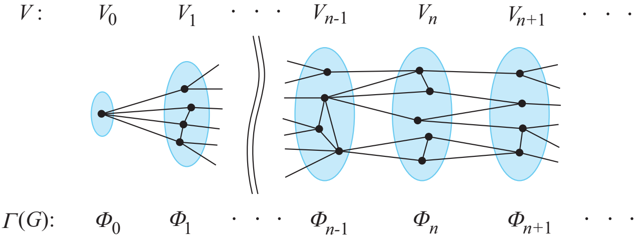
and more matrices associated to graphs ...

- Matrices with index set $V \times V$:
  1. Adjacency matrix: $A = [A_{xy}]$
  2. Combinatorial Laplacian: $L = D - A$, where $D = [\delta_{xy} \deg x]$ (degree matrix).
  3. Signless Laplacian: $D + A$
  4. Transition matrix: $T = [T_{xy}]$, where $T_{xy} = \deg(x)^{-1} A_{xy}$.
  5. Normalized transition matrix: $\hat{T} = D^{1/2} T D^{-1/2}$.
  6. Random walk Laplacian: $I - T = D^{-1} L$
  7. Normalized Laplacian: $\hat{L} = D^{-1/2} L D^{-1/2} = I - \hat{T}$
  8. Distance matrix: $D = [d_G(x, y)]$
  9. Q-matrix: $Q = [q^{d(x,y)}]$
- Other matrices with index set $V \times E$:
  incidence matrix, oriented incidence matrix (coboundary matrix), ...

3.5. Fock Spaces Associated to Graphs — Stratification

1. Fix an origin $\boldsymbol{o} \in \boldsymbol{V}$ of $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E})$.
2. Stratification (Distance Partition)

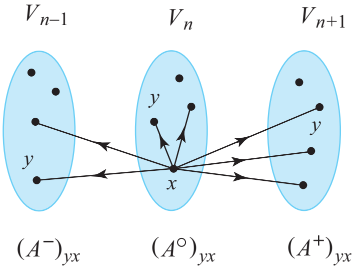
$$\boldsymbol{V} = \bigcup_{n=0}^{\infty} \boldsymbol{V_n}\,, \qquad \boldsymbol{V_n} = \{x \in \boldsymbol{V}\,;\, d(o,x) = n\}$$



3. Associated Hilbert space $\boldsymbol{\Gamma(G) \subset \ell^2(V)}$

$$\boldsymbol{\Phi_n} = |\boldsymbol{V_n}|^{-1/2} \sum_{x \in \boldsymbol{V_n}} \boldsymbol{e_x}\,, \qquad \boldsymbol{\Gamma(G)} = \sum_{n=0}^{\infty} \oplus \mathbf{C}\boldsymbol{\Phi_n}\,.$$

3.5. Fock Spaces Associated to Graphs — Quantum Decomposition

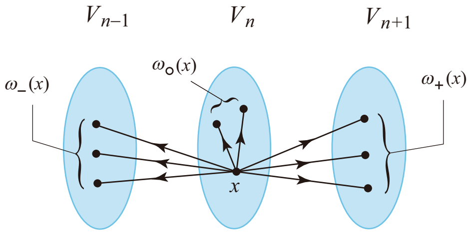


4. Quantum decomposition

$$A = A^+ + A^- + A^\circ, \qquad (A^+)^* = A^-, \quad (A^\circ)^* = A^\circ.$$

5. Is $(\Gamma(G), \{\Phi_n\}, A^+, A^\circ, A^-)$ an IFS?
   - Yes, if $\Gamma(G)$ is invariant under the actions of A^+, A^-, A°.
   - Yes in the limit, if $\Gamma(G)$ is asymptotically invariant under A^+, A^-, A°.

3.6. IFS Structure Associated to Graphs



- For $x \in V_n$ and $\epsilon = +, -, \circ$ we set $\omega_\epsilon(x) = \{y \in V_{n+\epsilon}\,;\, y \sim x\}$.

Lemma (exercise)

$\Gamma(G)$ is invariant under A^ϵ if and only if $\omega_\epsilon(x)$ is constant on each V_n. In that case there exist Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$ such that

$$A^+\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1}, \quad A^-\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1}, \quad A^\circ\Phi_n = \alpha_{n+1}\Phi_n.$$

In other words, $(\Gamma(G), \{\Phi_n\}, A^+, A^-, A^\circ)$ is an interacting Fock space (IFS).

3.7. IFS Structure Associated to Homogeneous Trees

Let T_κ denote the homogeneous tree of degree $\kappa \geq 2$.

For T_4

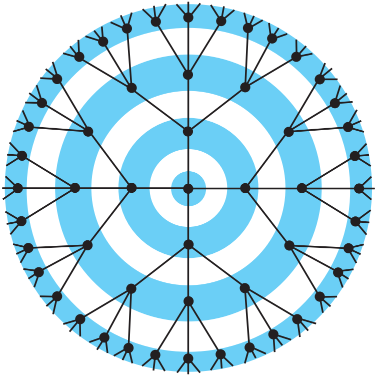
$$\Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} e_x$$

$$A = A^+ + A^- + A^\circ$$

$$A^+\Phi_0 = \sqrt{4}\,\Phi_1$$
$$A^+\Phi_n = \sqrt{3}\,\Phi_{n+1} \quad (n \geq 1)$$

$$A^-\Phi_0 = 0$$
$$A^-\Phi_1 = \sqrt{4}\,\Phi_0$$
$$A^-\Phi_n = \sqrt{3}\,\Phi_{n-1} \quad (n \geq 2)$$

$$A^\circ = 0$$



3.7. IFS Structure Associated to Homogeneous Trees

Let T_κ denote the homogeneous tree of degree $\kappa \geq 2$.

For a general T_κ

$$\Phi_n = |V_n|^{-1/2} \sum_{x \in V_n} e_x$$

$$A = A^+ + A^- + A^\circ$$

$$A^+\Phi_0 = \sqrt{\kappa}\, \Phi_1$$

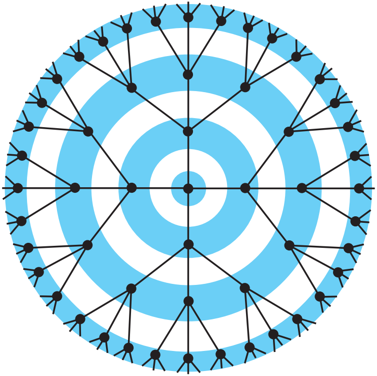
$$A^+\Phi_n = \sqrt{\kappa - 1}\, \Phi_{n+1} \qquad (n \geq 1)$$

$$A^-\Phi_0 = 0$$

$$A^-\Phi_1 = \sqrt{\kappa}\, \Phi_0$$

$$A^-\Phi_n = \sqrt{\kappa - 1}\, \Phi_{n-1} \qquad (n \geq 2)$$

$$A^\circ = 0$$



3.7. IFS Structure Associated to Homogeneous Trees

1. Quantum decomposition: $A = A^+ + A^-$

$$A^+\Phi_0 = \sqrt{\kappa}\,\Phi_1, \quad A^+\Phi_n = \sqrt{\kappa-1}\,\Phi_{n+1} \quad (n \ge 1)$$
$$A^-\Phi_0 = 0, \quad A^-\Phi_1 = \sqrt{\kappa}\,\Phi_0, \quad A^-\Phi_n = \sqrt{\kappa-1}\,\Phi_{n-1} \quad (n \ge 2)$$

2. Jacobi coefficients: $(\{\omega_1 = \kappa,\ \omega_2 = \omega_3 = \cdots = \kappa - 1\}, \{\alpha_n \equiv 0\})$

3. Cauchy–Stieltjes transform:

$$\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = G_\mu(z) = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \frac{\omega_3}{z-\alpha_4} - \cdots$$
$$= \frac{(\kappa-2)z - \kappa\sqrt{z^2 - 4(\kappa-1)}}{2(\kappa^2 - z^2)}$$

4. Vacuum spectral distribution: $\mu(dx) = \rho_\kappa(x)dx$

$$\rho_\kappa(x) = \frac{\kappa}{2\pi} \frac{\sqrt{4(\kappa-1) - x^2}}{\kappa^2 - x^2}, \qquad |x| \le 2\sqrt{\kappa-1}\,.$$

This is called the *Kesten distribution* (1959).

3.7. IFS Structure Associated to Homogeneous Trees

Theorem (see also Kesten (1959))

Let $A = A_\kappa$ be the adjacency matrix of T_κ. Then we have

$$\langle A^m \rangle = \langle e_o, A^m e_o \rangle = \int_{-2\sqrt{\kappa-1}}^{2\sqrt{\kappa-1}} x^m \rho_\kappa(x)dx, \quad m = 0, 1, 2, \dots.$$

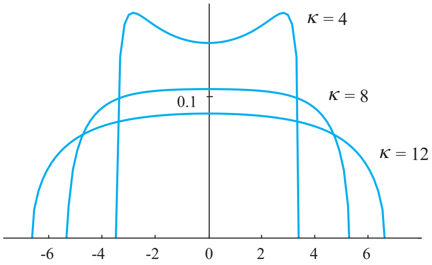
- Kesten measure

$$\rho_\kappa(x) = \frac{\kappa}{2\pi} \frac{\sqrt{4(\kappa-1)-x^2}}{\kappa^2 - x^2}$$

$$|x| \le 2\sqrt{\kappa - 1}$$

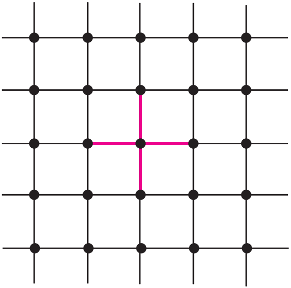
- Semi-circle law

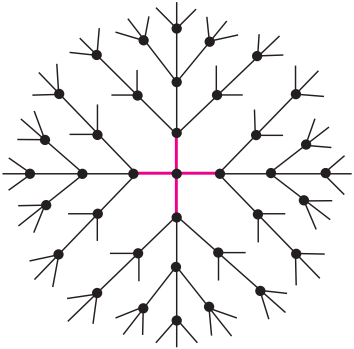
$$\lim_{\kappa\to\infty} \sqrt{\kappa}\rho_\kappa(\sqrt{\kappa}\,x) = \frac{1}{2\pi}\sqrt{4 - x^2}.$$



3.8. IFS Structure Associated to Spidernets

Lattices vs Trees

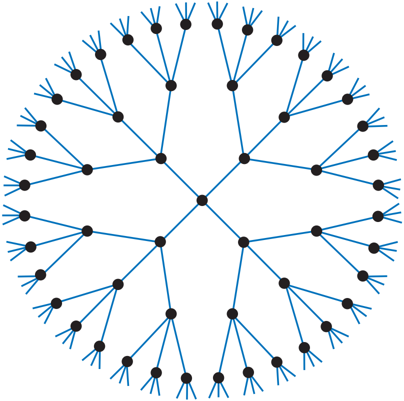




additive group $\mathbf{Z}^n$	free group F_n
many cycles	no cycles
binomial coefficients	Catalan numbers
commutative independence	free independence
Normal distribution	Wigner semi-circle law

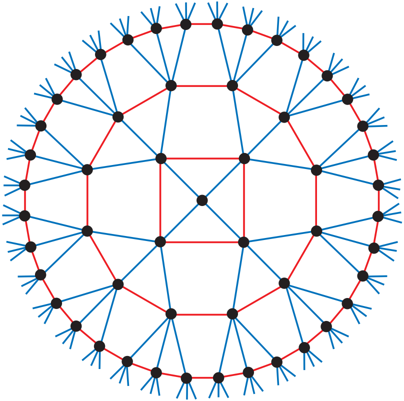
3.8. IFS Structure Associated to Spidernets

Spidernet = Homogeneous tree + large cycles



3.8. IFS Structure Associated to Spidernets

Spidernet = Homogeneous tree + large cycles



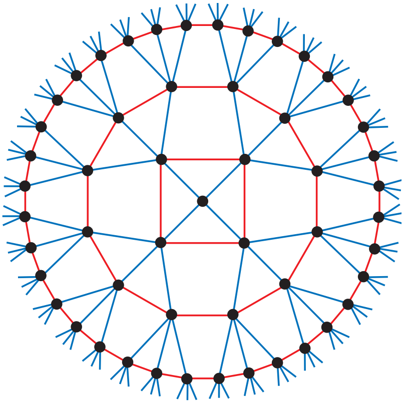
3.8. IFS Structure Associated to Spidernets

- Parametrization: $S(a,b,c)$

$$\deg(x) = \begin{cases} a & x = o \text{ (origin)} \\ b & x \neq o \end{cases}$$

For $x \neq o$ we have

$$\begin{cases} \omega_-(x) = 1 \\ \omega_+(x) = c \\ \omega_\circ(x) = b - 1 - c \end{cases}$$



$S(4,6,3)$

- Note: (a,b,c) does not necessarily determine a spidernet uniquely.

3.9. Vacuum Spectral Distribution of $S(a,b,c)$

$$\begin{cases}\omega_-(o)=0,\\ \omega_+(o)=a,\\ \omega_\circ(o)=0,\end{cases}\qquad \begin{cases}\omega_-(x)=1,\\ \omega_+(x)=c,\\ \omega_\circ(x)=b-1-c,\end{cases}\qquad x\neq o.$$

1. Quantum decomposition: $A=A^+ + A^- + A^\circ$

$$A^+\Phi_0=\sqrt{a}\,\Phi_1,\quad A^+\Phi_n=\sqrt{c}\,\Phi_{n+1}\quad (n\ge 1)$$
$$A^-\Phi_0=0,\quad A^-\Phi_1=\sqrt{a}\,\Phi_0,\quad A^-\Phi_n=\sqrt{c}\,\Phi_{n-1}\quad (n\ge 2)$$
$$A^\circ\Phi_0=0,\quad A^\circ\Phi_n=(b-1-c)\Phi_n\quad (n\ge 1)$$

2. Jacobi coefficients $(\{\omega_n\},\{\alpha_n\})$:

$$\omega_1=a,\quad \omega_2=\omega_3=\cdots=c,$$
$$\alpha_1=0,\quad \alpha_2=\alpha_3=b-1-c.$$

3. Cauchy–Stieltjes transform:

$$\int_{-\infty}^{+\infty}\frac{\mu(dx)}{z-x}=G_\mu(z)=\frac{1}{z-\alpha_1}\,\frac{}{-}\,\frac{\omega_1}{z-\alpha_2}\,\frac{}{-}\,\frac{\omega_2}{z-\alpha_3}\,\frac{}{-}\,\frac{\omega_3}{z-\alpha_4}\,\frac{}{-}\cdots$$

3.9. Vacuum Spectral Distribution of $S(a, b, c)$

Definition (Free Meixner distribution)

For $p > 0$, $q \geq 0$ and $a \in \mathbb{R}$ a probability distribution μ uniquely determined by

$$G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = \frac{1}{z} - \frac{p}{z-a} - \frac{q}{z-a} - \frac{q}{z-a} - \frac{q}{z-a} - \cdots$$

is called the *Free Meixner distribution* with parameters p, q, a.

1. Calculating the continued fraction:

$$G(z) = \frac{(2q-p)z + pa - p\sqrt{(z-a)^2 - 4q}}{2(q-p)z^2 + 2paz + 2p^2}.$$

2. The absolutely continuous part of μ is obtained by means of the inversion formula:

$$\rho_{p,q,a}(x) = \frac{p}{2\pi} \frac{\sqrt{4q-(x-a)^2}}{(q-p)x^2 + pax + p^2}, \qquad |x-a| \leq 2\sqrt{q}.$$

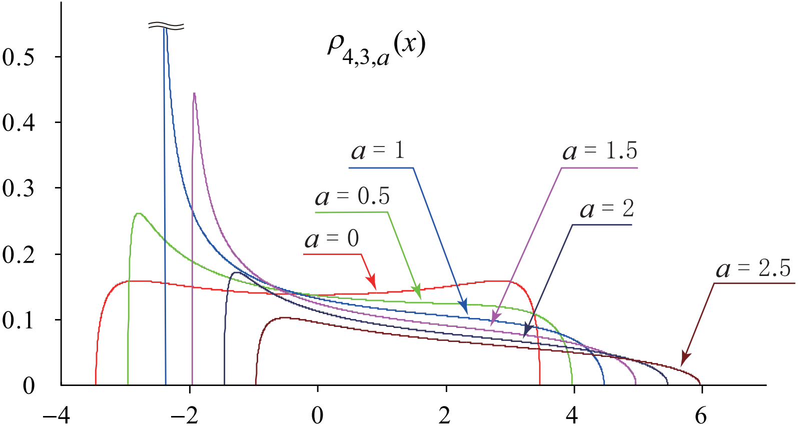
3. We obtain an explicit form of μ:

$$\mu(dx) = \rho_{p,q,a}(x)dx + w_+\delta_{\lambda_+} + w_-\delta_{\lambda_-} \quad \text{(at most two atoms)}$$

For further details see Hora–Obata (2007).

3.9. Vacuum Spectral Distribution of $S(a,b,c)$

Free Meixner distribution $\boldsymbol{\mu_{4,3,a}(dx) = \rho_{4,3,a}(x)dx + w_1\delta_{c_1} + w_2\delta_{c_2}}$



3.9. Vacuum Spectral Distribution of $S(a,b,c)$

Theorem

The vacuum spectral distribution of $S(a,b,c)$ is the free Meixner distribution with parameters $a, c, b-1-c$. namely,

$$\langle e_o, A^m e_o \rangle = \int_{-\infty}^{+\infty} x^m \mu_{a,c,b-1-c}(dx), \qquad m = 0, 1, 2, \dots .$$

Proof. Let μ be the vacuum spectral distribution of $S(a,b,c)$.
By graphical observation we have obtained the Jacobi coefficients:

$$\omega_1 = a, \quad \omega_2 = \omega_3 = \cdots = c,$$
$$\alpha_1 = 0, \quad \alpha_2 = \alpha_3 = b - 1 - c.$$

Then the Cauchy–Stieltjes transform of μ satisfies

$$\int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x} = G_\mu(z) = \frac{1}{z-\alpha_1} - \frac{\omega_1}{z-\alpha_2} - \frac{\omega_2}{z-\alpha_3} - \frac{\omega_3}{z-\alpha_4} - \cdots .$$

By definition the above μ is the free Meixner distribution $\mu_{a,c,b-1-c}$.

Exercises

Exercise 7 Let $G = (V, E)$ be a graph with fixed origin $o \in V$. Let $\Gamma(G)$ be the associated Fock space. Show that $\Gamma(G)$ is invariant under the actions A^+, A^-, A° if and only if $\omega_+(x)$, $\omega_-(x)$ and $\omega_\circ(x)$ are constant on each V_n. Then find the Jacobi coefficients $(\{\omega_n\}, \{\alpha_n\}) \in \mathfrak{J}$ such that

$$A^+\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1}, \quad A^-\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1}, \quad A^\circ\Phi_n = \alpha_{n+1}\Phi_n.$$

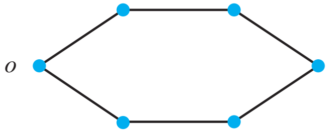
Exercise 8 Let $A = A^+ + A^-$ be the quantum decomposition of the adjacency matrix of the homogeneous tree T_κ $(\kappa \geq 2)$. Examine the actions of A^+ and A^-:

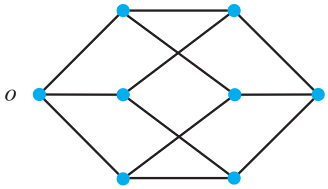
$$A^+\Phi_0 = \sqrt{\kappa}\,\Phi_1, \quad A^+\Phi_n = \sqrt{\kappa - 1}\,\Phi_{n+1} \quad (n \geq 1)$$

$$A^-\Phi_0 = 0, \quad A^-\Phi_1 = \sqrt{\kappa}\,\Phi_0, \quad A^-\Phi_n = \sqrt{\kappa - 1}\,\Phi_{n-1} \quad (n \geq 2)$$

Exercises

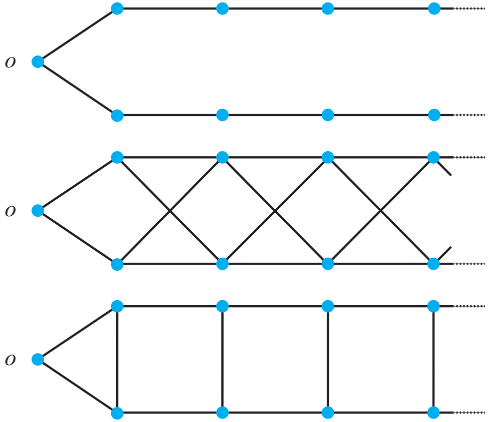
Exercise 9 Applying the method of quantum decomposition to the following graphs, derive the spectral distribution of at the vertex o.





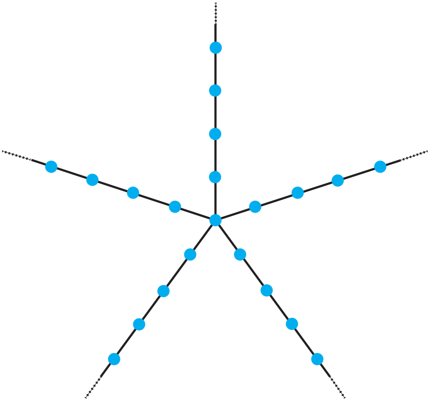
Exercises

Exercise 10 Applying the method of quantum decomposition to the following graphs, derive the spectral distribution of at the vertex o.



Exercises

Exercise 11 [Challenging Project] Let G_n be the graph obtained by joining n infinite paths at the endpoint o, also called the n-fold star product of $\mathbb{Z}_+$. (The following figure shows G_5.) Calculate explicitly the spectral distribution of G_n at o and study its asymptotic behavior as $n \to \infty$. Note: μ_n posesses two atoms.

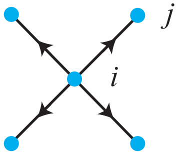


4. Quantum Walks on Spidernets

4.1. Random Walks on a Graph

$\{X_n\}$: isotropic random walk on $G = (V, E)$
determined by the transition probability:

$$P(X_{n+1} = j | X_n = i) = \begin{cases} \dfrac{1}{\deg(i)} & j \sim i, \\ 0, & \text{otherwise.} \end{cases}$$



Transition matrix $T = [P(X_{n+1} = j | X_n = i)]$ gives the n-step transition probability:

$$P(X_n = j | X_0 = i) = T^n(i, j) = \langle e_i, T^n e_j \rangle$$

▶ Asymptotic behavior of $P^n(i, j)$ as $n \to \infty$ is important from several points of view.

For example, $i \in V$ is *recurrent*, i.e.,

$$P(T_i < \infty | X_0 = i) = 1, \qquad T_i = \inf\{n \geq 1\,;\, X_n = i\},$$

if and only if

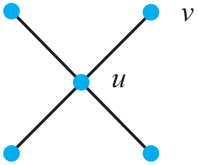
$$\sum_{n=1}^{\infty} T^n(i, i) = +\infty.$$

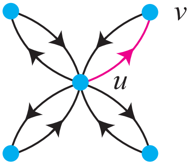
4.2. Grover Walks on Graphs

For a graph $G = (V, E)$ we consider the arcs (half-edges):

$$A(G) = \{(u,v) \in V \times V \,;\, u \sim v\}$$

and the associated Hilbert space: $\mathcal{H}(G) = \ell^2(A(G))$,
where $\{e_{(u,v)} \,;\, (u,v) \in A(G)\}$ becomes the canonical orthonormal basis.





- 1-step dynamics of random walk (RW) is given by the transition matrix:

$$Te_u = \sum_{v \sim u} T(u,v) e_v$$

- 1-step dynamics of quantum walk (QW) is given by a particular unitary matrix:

$$Ue_{(u,v)} = \sum_{w \sim u} \dots\dots e_{(w,u)}$$

4.2. Grover Walks on Graphs

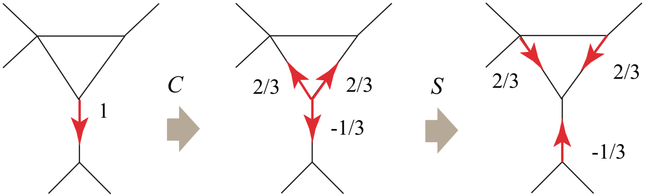
1. *Coin flip matrix* C is defined by

$$Ce_{(u,v)} = \sum_{w\sim u} \left(\frac{2}{\deg(u)} - \delta_{wv} \right) e_{(u,w)}$$

2. *Shift matrix* S is defined by

$$Se_{(u,v)} = e_{(v,u)}$$

3. *Time evolution matrix* U is the unitary matrix defined by $U = SC$.
4. Grover walk is given by $\{U^n \psi_0\}$ with an initial state $\psi_0 \in \mathcal{H}(G)$, $\|\psi_0\| = 1$,



4.2. Grover Walks on Graphs

Finding probability

$\{U^n\psi_0\}$: Grover walk with initial state $\psi_0 \in \mathcal{H}(G)$

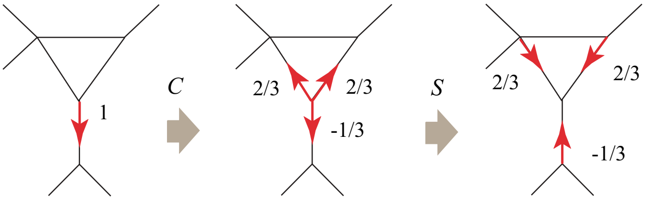
X_n^{GW}: position of the "Grover walker" at time n

$$P(X_n^{\mathrm{GW}} = u) = \sum_{v\sim u} |\langle e_{(u,v)}, U^n\psi_0\rangle|^2$$

- Note: Since $\{e_{(u,v)}\}$ is an orthonormal basis of $\mathcal{H}(G)$ and U is unitary,

$$\sum_{u\in V}\sum_{v\sim u} |\langle e_{(u,v)}, U^n\psi_0\rangle|^2 = \|U^n\psi_0\|^2 = \|\psi_0\|^2 = 1$$

- Check the finding probability (exercise)



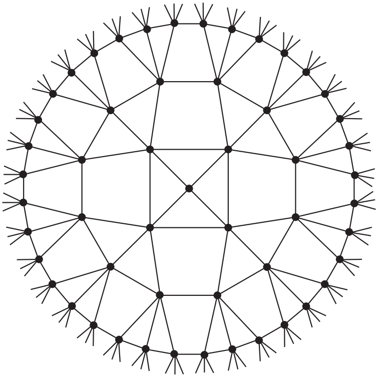
4.3. Main Question on Grover Walk on Spidernet $S(a, b, c)$

$\underline{\boldsymbol{G = S(a, b, c)}}$

$\boldsymbol{a = \deg(o)}$

$\boldsymbol{b = \deg(x)}$ for $\boldsymbol{x \neq o}$

$\boldsymbol{c = \omega_+(x)}$ for $\boldsymbol{x \neq o}$



$\boldsymbol{U = SC}$: time evolution matrix of Grover walk

$\boldsymbol{\{U^n \psi_0^+\}}$: Grover walk with initial state

$$\boldsymbol{\psi_0^+ = \frac{1}{\sqrt{a}} \sum_{u \sim o} e_{(o,u)}}$$

▶ By symmetry we have

$$\boldsymbol{P(X_n^{\mathrm{GW}} = 0) = |\langle \psi_0^+, U^n \psi_0^+ \rangle|^2,}$$

of which the asymptotic behavior is to be studied.

4.4. Isotropic Random Walk on $S(a,b,c)$

Theorem

For the adjacency matrix $\boldsymbol{A}$ *of* $\boldsymbol{S(a,b,c)}$ *we have*

$$\langle e_o, A^m e_o \rangle = \int_{-\infty}^{+\infty} x^m \mu_{a,c,b-1-c}(dx), \qquad m = 0, 1, 2, \dots,$$

where $\boldsymbol{\mu_{a,c,b-1-c}}$ *is the free Meixner distribution.*

4.4. Isotropic Random Walk on $S(a,b,c)$

In a similar manner we have

Theorem

For the transition matrix T of $S(a,b,c)$ we have

$$\begin{aligned} P(X_n^{\mathrm{RW}} = o \mid X_0^{\mathrm{RW}} = o) &= \langle e_o, T^n e_o \rangle \\ &= \int_{-\infty}^{+\infty} x^n \mu_{q,pq,r}(dx), \quad n = 0,1,2,\dots, \end{aligned}$$

Here $\mu_{q,pq,r}$ is the free Meixner distribution, where $p>0$, $q>0$ and $r \ge 0$ given by

$$p = \frac{c}{b}, \qquad q = \frac{1}{b}, \qquad r = \frac{b-c-1}{b}.$$

4.4. Isotropic Random Walk on $S(a,b,c)$

▶ Interesting case: $c \geq 2$ and $b > c+1 \geq 3$ (namely, avoiding trees)

Then

$$p > q > 0, \qquad r > 0, \qquad p+q+r=1. \tag{*}$$

In this case the free Meixner law is of the form:

$$\mu_{q,pq,r}(dx) = \rho_{q,pq,r}(x)dx + w\delta_c\,,$$

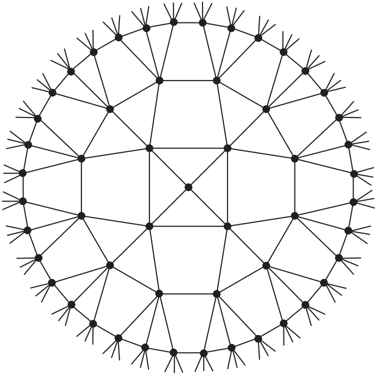
$$c = -\frac{q}{1-p}\,, \quad w = \max\left\{\frac{(1-p)^2 - pq}{(1-p)(1-p+q)}\,, 0\right\}$$

Since the support of $\mu_{q,pq,r}$ is properly contained in $(-1,1)$, we see that

$$\lim_{n\to\infty} P(X_n^{\mathrm{RW}} = o | X_0^{\mathrm{RW}} = o) = \lim_{n\to\infty} \int_{-\infty}^{+\infty} x^n \mu_{a,c,b-1-c}(dx) = 0.$$

4.4. Isotropic Random Walk on $S(a, b, c)$

- In fact, the return probability is obtained by means of 1-dimensional reduction.



$$a = \deg(o)$$

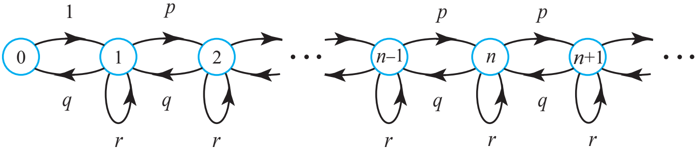
$$b = \deg(x) \text{ for } x \neq o$$

$$c = \omega_+(x) \text{ for } x \neq o$$

$$p = \frac{c}{b},$$

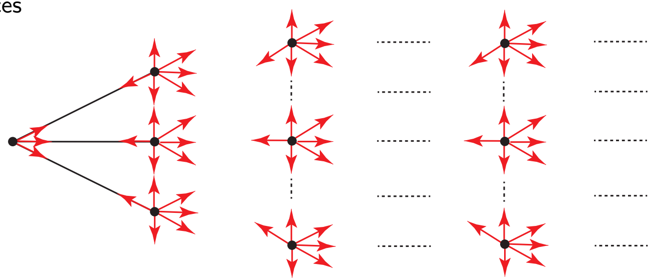
$$q = \frac{1}{b},$$

$$r = \frac{b-c-1}{b}$$

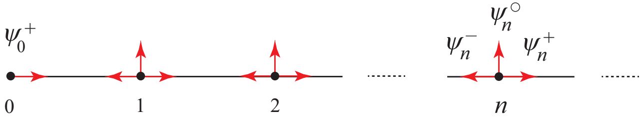


4.5. Reduction to (p, q)-Quantum Walk on $\mathbf{Z}_+$

- State spaces



$$\boldsymbol{\mathcal{H}(G) = \mathcal{H}_0^+ \oplus \sum_{n=1}^{\infty} \oplus (\mathcal{H}_n^- \oplus \mathcal{H}_n^\circ \oplus \mathcal{H}_n^+)}$$



$$\boldsymbol{\mathcal{H}(\mathrm{Z}_+) = \mathbb{C}\psi_0^+ \oplus \sum_{n=1}^{\infty} \oplus (\mathbb{C}\psi_n^+ \oplus \mathbb{C}\psi_n^\circ \oplus \mathbb{C}\psi_n^-)}$$

4.5. Reduction to (p,q)-Quantum Walk on $\mathbf{Z}_+$

In fact,

$$\psi_n^+ = \frac{1}{\sqrt{ac^n}} \sum_{u\in V_n} \sum_{\substack{v\in V_{n+1} \\ v\sim u}} e_{(u,v)}\,, \qquad n\ge 0,$$

$$\psi_n^\circ = \frac{1}{\sqrt{a(b-c-1)c^{n-1}}} \sum_{u\in V_n} \sum_{\substack{v\in V_{n} \\ v\sim u}} e_{(u,v)}\,, \qquad n\ge 1,$$

$$\psi_n^- = \frac{1}{\sqrt{ac^{n-1}}} \sum_{u\in V_n} \sum_{\substack{v\in V_{n-1} \\ v\sim u}} e_{(u,v)}\,, \qquad n\ge 1,$$

$$\mathcal{H}(\mathbf{Z}_+) = \mathbb{C}\psi_0^+ \oplus \sum_{n=1}^{\infty} \oplus (\mathbb{C}\psi_n^+ \oplus \mathbb{C}\psi_n^\circ \oplus \mathbb{C}\psi_n^-)$$

$$C\psi_0^+ = \psi_0^+; \quad C\psi_n^+ = (2p-1)\psi_n^+ + 2\sqrt{pr}\,\psi_n^\circ + 2\sqrt{pq}\,\psi_n^-\,, \quad n\ge 1,$$

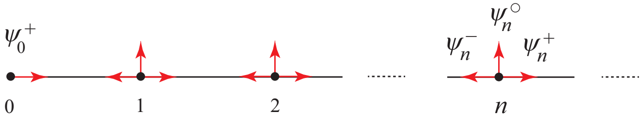
$$C\psi_n^\circ = 2\sqrt{pr}\,\psi_n^+ + (2r-1)\psi_n^\circ + 2\sqrt{qr}\,\psi_n^-\,, \qquad n\ge 1,$$

$$C\psi_n^- = 2\sqrt{pq}\,\psi_n^+ + 2\sqrt{qr}\,\psi_n^\circ + (2q-1)\psi_n^-\,, \qquad n\ge 1.$$

$$S\psi_n^+ = \psi_{n+1}^-\,, \quad n\ge 0; \quad S\psi_n^\circ = \psi_n^\circ\,, \quad n\ge 1; \quad S\psi_n^- = \psi_{n-1}^+\,, \quad n\ge 1.$$

4.5. Reduction to (p, q)-Quantum Walk on $\mathbf{Z}_+$

State space: $\boldsymbol{\mathcal{H}}(\mathbf{Z}_+) = \mathbb{C}\psi_0^+ \oplus \sum_{n=1}^{\infty} \oplus(\mathbb{C}\psi_n^+ \oplus \mathbb{C}\psi_n^\circ \oplus \mathbb{C}\psi_n^-)$



1. $\boldsymbol{\mathcal{H}}(\mathbf{Z}_+)$ is invariant under $U = SC$ (proof by computation).
2. We call $U = SC$ restricted to $\boldsymbol{\mathcal{H}}(\mathbf{Z}_+)$ a *(p, q)-quantum walk* on $\mathbf{Z}_+$.
3. Thus, the finding probability is obtained by

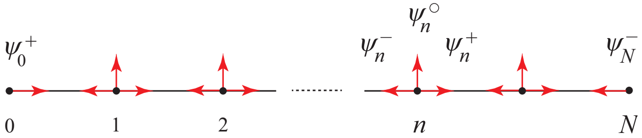
$$P(X_n^{\mathrm{GW}} = o) = |\langle \psi_0^+, U^n \psi_0^+ \rangle_{\mathcal{H}(\mathbb{Z}_+)}|^2$$

4.6. Calculating Probabbility Amplitude $\langle \psi_0^+, U^n \psi_0^+ \rangle$

- *Cut off* the (p,q)-quantum walk at N

State space: $\mathcal{H}(N) = \mathbb{C}\psi_0^+ \oplus \sum_{n=1}^{N-1} \oplus(\mathbb{C}\psi_n^+ \oplus \mathbb{C}\psi_n^\circ \oplus \mathbb{C}\psi_n^-) \oplus \mathbb{C}\psi_N^-$

Time evolution matrix: $U_N = S_N C_N \ (C_N \psi_N^- = \psi_N^-)$



Lemma

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \langle \psi_0^+, U_N^n \psi_0^+ \rangle_{\mathcal{H}(N)}, \qquad n < N.$$

4.6. Calculating Probability Amplitude $\langle \psi_0^+, U^n \psi_0^+ \rangle$

▶ Suppose $r > 0$ (the case of $r = 0$ is similar with small modification).

Eigenvalues of U_N

$$1(1), \quad e^{\pm i\theta_1}(1), \dots, \quad e^{\pm i\theta_N}(1), \quad -1(N-2)$$

where $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_N < \pi$ are determined in such a way that

$$\lambda_0 = 1 = \cos\theta_0, \quad \lambda_1 = \cos\theta_1\,, \quad \lambda_2 = \cos\theta_2\,, \quad \dots, \quad \lambda_N = \cos\theta_N$$

are the eigenvalues of

$$T_N = \begin{bmatrix} 0 & \sqrt{q} & & & & & \\ \sqrt{q} & r & \sqrt{pq} & & & & \\ & \sqrt{pq} & r & \sqrt{pq} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \sqrt{pq} & r & \sqrt{pq} & \\ & & & & \sqrt{pq} & r & \sqrt{p} \\ & & & & & \sqrt{p} & 0 \end{bmatrix}.$$

4.6. Calculating Probability Amplitude $\langle \psi_0^+, U^n \psi_0^+ \rangle$

Let Ω_j be the eigenvector of T_N with eigenvalue $\cos\theta_j$ (explicitly known but omitted).

1. By simple calculus we have
$$\langle \psi_0^+, U_N^n \psi_0^+ \rangle = \sum_{j=0}^{N} |\langle \psi_0^+, \Omega_j \rangle|^2 \cos n\theta_j \,.$$

2. Define a probability distribution
$$\mu_N = \sum_{j=0}^{N} |\langle \Omega_j, \psi_0^+ \rangle|^2 \delta_{\lambda_j} \,.$$
Then
$$\langle \psi_0^+, U_N^n \psi_0^+ \rangle = \int_{-1}^{1} \cos n\theta \, \mu_N(d\lambda), \qquad \lambda = \cos\theta.$$

3. The LHS may be replaced with $\langle \psi_0^+, U^n \psi_0^+ \rangle$ whenever $n < N$.

4. Letting $N \to \infty$, we have
$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \int_{-1}^{1} \cos n\theta \, \mu_{q,pq,r}(d\lambda), \qquad \lambda = \cos\theta,$$
where $\mu_{q,pq,r}$ is the free Meixner distribution with parameters q, pq, r.

4.6. Calculating Probability Amplitude $\langle \psi_0^+, U^n \psi_0^+ \rangle$

Outline of $\mu_N \to \mu_{q,pq,r}$

1. The limit of

$$T_N = \begin{bmatrix} 0 & \sqrt{q} & & & & & \\ \sqrt{q} & r & \sqrt{pq} & & & & \\ & \sqrt{pq} & r & \sqrt{pq} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \sqrt{pq} & r & \sqrt{pq} & \\ & & & & \sqrt{pq} & r & \sqrt{p} \\ & & & & & \sqrt{p} & 0 \end{bmatrix}$$

   determines an IFS of which vacuum spectral distribution is the free Meixner distribution with parameters q, pq, r.

2. $\mu_N \xrightarrow{\mathrm{m}} \mu_{q,pq,r}$.
3. $\mu_{q,pq,r}$ is determined uniquely by moments.
4. By general theory $\mu_N \xrightarrow{\mathrm{m}} \mu_{q,pq,r}$ implies weak convergence.

4.6. Calculating Probability Amplitude $\langle \psi_0^+, U^n \psi_0^+ \rangle$

Summing up,

Theorem (Integral representation)

Let U be the Grover walk on $S(a,b,c)$. We have

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \int_{-1}^{1} \cos n\theta \, \mu_{q,pq,r}(d\lambda), \qquad \lambda = \cos\theta.$$

Recall: For the isotropic random walk on $S(a,b,c)$ we have

$$P(X_n^{\mathrm{RW}} = o | X_0^{\mathrm{RW}} = o) = \langle e_o, T^n e_o \rangle = \int_{-1}^{1} x^n \mu_{q,pq,r}(dx).$$

4.7. Initial Value Localization

Using $\mu_{q,pq,r}(dx) = \rho_{q,pq,r}(x)dx + w\delta_c$ with

$$c = -\frac{q}{1-p}\,, \quad w = \max\left\{\frac{(1-p)^2 - pq}{(1-p)(1-p+q)}\,, 0\right\},$$

we have

$$\langle \psi_0^+, U^n \psi_0^+ \rangle = \int_{-1}^{1} \cos n\theta \, \rho_{q,pq,r}(\lambda) d\lambda + w \cos n\tilde{\theta},$$

$$\cos\tilde{\theta} = -\frac{q}{1-p}\,.$$

By Riemann–Lebesgue lemma we have

$$\langle \psi_0^+, U^n \psi_0^+ \rangle \sim w \cos n\tilde{\theta} \qquad \text{as } n \to \infty.$$

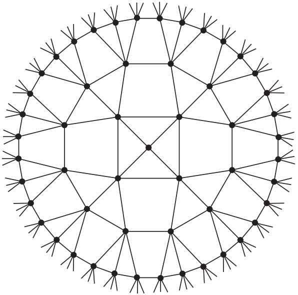
Theorem (Konno-Obata–Segawa)

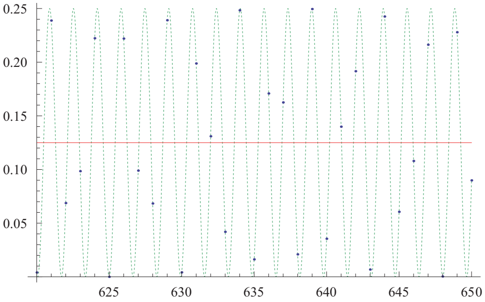
Grover walk on $S(a, b, c)$ exhibits localization if and only if $(1-p)^2 - pq > 0$.

4.7. Initial Value Localization

- For example, $S(\kappa, \kappa+2, \kappa-1)$ exhibits localization for $2 \le \kappa < 10$.
- large κ $\Longleftrightarrow$ density of large cycles is low $\Longleftrightarrow$ more likely tree

Example $S(4,6,3)$





$$P(X_n = o) \sim \frac{1}{4}\cos^2(n\tilde{\theta}), \qquad \tilde{\theta} = \arccos(-1/3)$$