Spectral Analysis of Growing Graphs
A Quantum Probability Point of View

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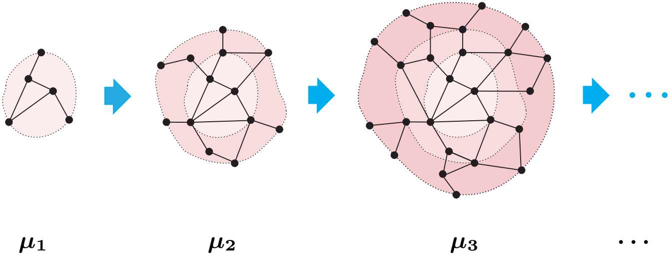
5. Asymptotic Spectral Analysis of Growing Regular Graphs

5.1. Main Theme

▶ Growing graphs and spectral distributions

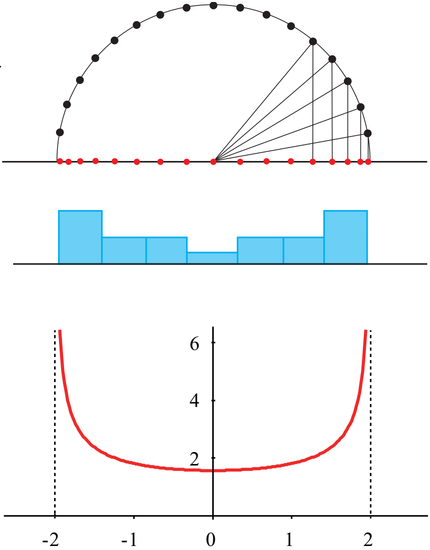


Our Main Theme

The asymptotic behavior of μ_n as $n \to \infty$. In fact, we will investigate the limit:

$$\lim_{n\to\infty} \mu_n$$

5.2. Simple Example (I) P_n as $n \to \infty$



P_n as $n \to \infty$

$$\mathrm{Spec}\,(P_n) = \left\{ 2\cos\frac{k\pi}{n+1}\,;\, 1 \le k \le n \right\}$$

$$\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{2\cos\frac{k\pi}{n+1}}$$

For $f \in C_b(\mathbb{R})$ we have

$$\int_{-\infty}^{+\infty} f(x)\mu_n(dx)$$

$$= \frac{1}{n}\sum_{k=1}^{n} f\left(2\cos\frac{k\pi}{n+1}\right)$$

$$\to \int_0^1 f(2\cos\pi t)dt$$

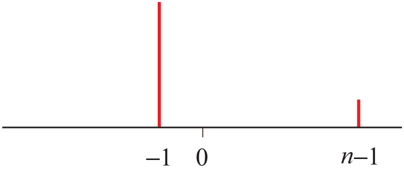
$$= \int_{-2}^{+2} f(x)\,\frac{dx}{\pi\sqrt{4-x^2}}.$$

5.2. Simple Example (II) K_n as $n \to \infty$

K_n as $n \to \infty$

$$\mathrm{Spec}\,(K_n) = \{-1(n-1), n-1(1)\}$$

$$\mu_n = \frac{1}{n}\,\delta_{n-1} + \frac{n-1}{n}\,\delta_{-1}$$



▶ Let us see what happens in the limit μ_n as $n \to \infty$

For $f \in C_b(\mathbb{R})$ we have

$$\int_{-\infty}^{+\infty} f(x)\mu_n(dx) = \frac{1}{n} f(n-1) + \frac{n-1}{n}\, f(-1)$$

$$\to f(-1) = \int_{-\infty}^{+\infty} f(x)\delta_{-1}(dx) \quad \text{as } n \to \infty$$

This means that $\mu_n \to \delta_{-1}$

Can we accept it? What about the mean values?

5.2. Simple Example (II) K_n as $n \to \infty$

▶ Normalization is a basic idea in probability theory to grasp the limit distribution.

E.g., central limit theorem (CLT) and its variants.

Definition (normalization)

For a probability distribution μ its *normalization* is a probability distribution $\tilde{\mu}$ defined by

$$\int f(x)\,\tilde{\mu}(dx) = \int f\Big(\frac{x-m}{\sigma}\Big)\,\mu(dx),$$

where

$$m = \mathrm{mean}(\mu), \quad \sigma^2 = \mathrm{var}(\mu).$$

Then we have

$$\mathrm{mean}(\tilde{\mu}) = 0, \qquad \mathrm{var}(\tilde{\mu}) = 1.$$

5.2. Simple Example (II) K_n as $n \to \infty$

K_n as $n \to \infty$

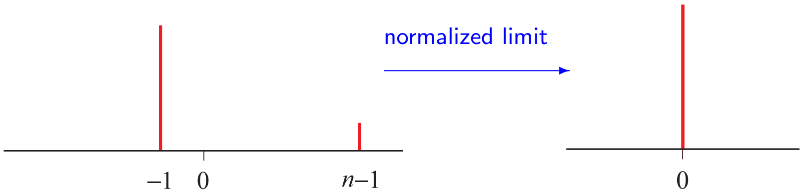
Spectral distribution (eigenvalue distribution): $\mu_n = \frac{1}{n}\,\delta_{n-1} + \frac{n-1}{n}\,\delta_{-1}$

Since $\mathrm{mean}(\mu_n) = 0$ and $\mathrm{var}(\mu_n) = n-1$, after normalization we have

$$\int_{-\infty}^{+\infty} f(x)\tilde{\mu}_n(dx) = \frac{1}{n} f\Big(\frac{n-1}{\sqrt{n-1}}\Big) + \frac{n-1}{n}\, f\Big(\frac{-1}{\sqrt{n-1}}\Big)$$

$$\to f(0) = \int_{-\infty}^{+\infty} f(x)\delta_0(dx) \quad \text{as } n \to \infty.$$

This means that $\tilde{\mu}_n \to \delta_0$.



5.3. Formulation of Question in General

A difference between K_n and P_n as $n \to \infty$

$$\mu_{P_n} = \frac{1}{n} \sum_{k=1}^{n} \delta_{2\cos\frac{k\pi}{n+1}}, \qquad \mu_{K_n} = \frac{1}{n}\,\delta_{n-1} + \frac{n-1}{n}\,\delta_{-1}$$

mean value

$$\mathrm{mean}(\mu_{P_n}) = \mathrm{mean}(\mu_{K_n}) = 0$$

variance

$$\mathrm{var}(\mu_{P_n}) = \frac{2(n-1)}{n} \to 2, \qquad \mathrm{var}(\mu_{K_n}) = n-1 \to \infty$$

- In general, it is not reasonable to consider $\lim \mu_n$ when $\mathrm{var}(\mu_n) \to \infty$.

  We should take *normalized limit* $\lim \tilde{\mu}_n$.

5.3. Formulation of Question in General

$G_\nu = (V_\nu, E_\nu)$: growing graphs

$(\mathcal{A}(G_\nu), \langle\cdot\rangle_\nu)$: adjacency algebra with a state (algebraic probability space)

μ_ν: spectral distribution of the adjacency matrix A_ν of G_ν, i.e.,

$$\langle A_\nu^m \rangle = \int_{-\infty}^{+\infty} x^m \mu_\nu(dx), \qquad m = 0, 1, 2, \dots.$$

Note: $\mathrm{mean}(A_\nu) = \langle A_\nu \rangle$ and $\mathrm{var}(A_\nu) = \langle (A_\nu - \mathrm{mean}(A_\nu))^2 \rangle$.

Main question in genaral

For the normalization $\tilde{\mu}_\nu$ of μ_ν find the limit spectral distribution:

$$\mu = \lim_\nu \tilde{\mu}_\nu \,.$$

In other words,

$$\lim_\nu \left\langle \left(\frac{A_\nu - \mathrm{mean}(A_\nu)}{\sqrt{\mathrm{var}(A_\nu)}} \right)^m \right\rangle_\nu = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 0, 1, 2, \dots.$$

5.4. Growing Distance-Regular Graphs (DRGs)

Definition

A graph $G = (V, E)$ is called *distance regular* if the intersection numbers:

$$p^k_{i,j} = |\{z \in V\,;\, d(x,z) = i,\ d(y,z) = j\}|,$$

is constant for all pairs x, y such that $d(x,y) = k$.

- Examples: Hamming graphs, Johnson graphs, odd graphs, homogeneous trees, ...

- We are interested in growing distance-regular graphs, e.g.,

$$\begin{aligned}
&H(d,N) \quad \text{as } d \to \infty \text{ and } N \to \infty \\
&J(v,d) \quad \text{as } v \to \infty \text{ and } d \to \infty \\
&O_k \quad \text{as } k \to \infty \\
&T_k \quad \text{as } k \to \infty \\
&\qquad \dots
\end{aligned}$$

5.4. Growing Distance-Regular Graphs (DRGs)

Some general facts on a distance-regular graph G (exercise)

1. Let $A = A^+ + A^- + A^\circ$ be the quantum decomposition (with respect to a fixed origin $o \in V$). Then

$$A^+\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1}, \quad A^-\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1}, \quad A^\circ\Phi_n = \alpha_{n+1}\Phi_n,$$

   where

$$\omega_n = p^n_{1,n-1}p^{n-1}_{1,n}\,, \qquad \alpha_n = p^{n-1}_{1,n-1}\,.$$

2. In particular, $(\Gamma(G), \{\Phi_n\}, A^+, A^\circ, A^-)$ is an IFS associated to $(\{\omega_n\}, \{\alpha_n\})$.
3. mean value and variance:

$$\mathrm{mean}(A) = \langle A\rangle = 0, \qquad \mathrm{var}(A) = \langle A^2\rangle = \deg(o) = p^0_{11}$$

4. If G is a finite distance-regular graph, the tracial and vacuum states coincide:

$$\langle A^m\rangle_{\mathrm{tr}} = \langle A^m\rangle_o = \langle e_o, A^m e_o\rangle, \quad m = 1, 2, \dots .$$

5.5. Growing DRGs: An Example $H(d, N)$

$H(d, N) = K_N \times \cdots \times K_N$ (d times): Hamming graph

$$p^0_{1,1} = \deg H(d, N) = d(N-1),$$

$$p^n_{1,n-1} = n, \quad p^{n-1}_{1,n} = (d-n)(N-1), \quad p^{n-1}_{1,n-1} = (n-1)(N-2).$$

Theorem

Let $\mu_{d,N}$ be the vacuum spectral distribution of $H(d, N)$ (in coincidence with the eigenvalue distribution). Then the Jacobi parameters of $\mu_{d,N}$ are given by

$$\omega_n = p^n_{1,n-1} p^{n-1}_{1,n} = n(d-n+1)(N-1), \quad 1 \le n \le d,$$
$$\alpha_n = p^{n-1}_{1,n-1} = (n-1)(N-2), \quad 1 \le n \le d+1.$$

In fact, the vacuum spectral distribution of A is the binomial distribution.

The IFS structure:

$$A^+ \Phi_n = \sqrt{\omega_{n+1}}\, \Phi_{n+1} = \sqrt{(n+1)(d-n)(N-1)}\, \Phi_{n+1},$$
$$A^- \Phi_n = \sqrt{\omega_n}\, \Phi_{n-1} = \sqrt{n(d-n+1)(N-1)}\, \Phi_{n-1},$$
$$A^\circ \Phi_n = \alpha_{n+1}\, \Phi_n = n(N-2)\Phi_n,$$

5.5. Growing DRGs: An Example $H(d, N)$

$$A^{+}\Phi_n = \sqrt{\omega_{n+1}}\,\Phi_{n+1} = \sqrt{(n+1)(d-n)(N-1)}\,\Phi_{n+1},$$
$$A^{-}\Phi_n = \sqrt{\omega_n}\,\Phi_{n-1} = \sqrt{n(d-n+1)(N-1)}\,\Phi_{n-1},$$
$$A^{\circ}\Phi_n = \alpha_{n+1}\,\Phi_n = n(N-2)\Phi_n,$$

- What happens when $N \to \infty$ and $d \to \infty$?
- Normalization: $\mathrm{mean}(A) = \langle A \rangle = 0$ and $\mathrm{var}(A) = \langle A^2 \rangle = d(N-1)$.

$$\frac{A^{+}}{\sqrt{d(N-1)}}\,\Phi_n = \sqrt{(n+1)\Big(1-\frac{n}{d}\Big)}\,\Phi_{n+1},$$
$$\frac{A^{-}}{\sqrt{d(N-1)}}\,\Phi_n = \sqrt{n\Big(1-\frac{n-1}{d}\Big)}\,\Phi_{n-1},$$
$$\frac{A^{\circ}}{\sqrt{d(N-1)}}\,\Phi_n = n\sqrt{\frac{N-2}{d}}\sqrt{\frac{N-2}{N-1}}\,\Phi_n,$$

- We thus find the proper scaling:

$$N \to \infty, \quad d \to \infty, \quad \frac{N}{d} \to \tau \geq 0.$$

5.5. Growing DRGs: An Example $H(d,N)$

- Taking the limit as $N \to \infty$, $d \to \infty$ and $\dfrac{N}{d} \to \tau \geq 0$, we have

$$\frac{A^+}{\sqrt{d(N-1)}}\,\Phi_n = \sqrt{(n+1)\Big(1-\frac{n}{d}\Big)}\,\Phi_{n+1} \to \sqrt{n+1}\,\text{“}\Phi_{n+1}\text{”}\,,$$

$$\frac{A^-}{\sqrt{d(N-1)}}\,\Phi_n = \sqrt{n\Big(1-\frac{n-1}{d}\Big)}\,\Phi_{n-1} \to \sqrt{n}\,\text{“}\Phi_{n-1}\text{”}\,,$$

$$\frac{A^\circ}{\sqrt{d(N-1)}}\,\Phi_n = n\sqrt{\frac{N-2}{d}}\sqrt{\frac{N-2}{N-1}}\,\Phi_n \to n\sqrt{\tau}\,\text{“}\Phi_n\text{”}.$$

- Recall the Boson Fock space $(\Gamma, \{\Psi_n\}, B^+, B^-)$ is defined by

$$B^+\Psi_n = \sqrt{n+1}\,\Psi_{n+1}, \quad B^-\Psi_n = \sqrt{n}\,\Psi_{n-1}.$$

- Note also that

$$B^+B^-\Psi_n = n\Psi_n\,.$$

5.5. Growing DRGs: An Example $H(d, N)$

Theorem (Quantum central limit theorem (QCLT) for $H(d, N)$)

Let $A = A^+ + A^- + A^\circ$ be the quantum decomposition of the adjacency matrix of $H(d, N)$. Let $(\Gamma, \{\Psi_n\}, B^+, B^-)$ be the Boson Fock space. Then we have

$$\left(\frac{A^+}{\sqrt{d(N-1)}}, \frac{A^-}{\sqrt{d(N-1)}}, \frac{A^\circ}{\sqrt{d(N-1)}}\right) \xrightarrow{\mathrm{m}} (B^+, B^-, \sqrt{\tau}\, B^+ B^-),$$

as $N \to \infty$, $d \to \infty$ and $\dfrac{N}{d} \to \tau \geq 0$.

where $\xrightarrow{\mathrm{m}}$ means the convergence of all mixed moments.

Deteiled proof is omitted (exercise).

5.5. Growing DRGs: An Example $H(d,N)$

Finding the asymptotic spectral distribution for $\boldsymbol{H(d,N)}$

$$\left(\frac{A^+}{\sqrt{d(N-1)}},\frac{A^-}{\sqrt{d(N-1)}},\frac{A^\circ}{\sqrt{d(N-1)}}\right)\xrightarrow{\mathrm{m}}(B^+,B^-,\sqrt{\tau}\,B^+B^-)$$

implies that

$$\left\langle e_o\left(\frac{A}{\sqrt{d(N-1)}}\right)^m e_o\right\rangle\to\langle\Psi_0,(B^++B^-+\sqrt{\tau}\,B^+B^-)^m\Psi_0\rangle.$$

On the other hand, by observing moments or generating functions, we see that

$$\langle\Psi_0,(B^++B^-+\sqrt{\tau}\,B^+B^-)^m\Psi_0\rangle=\int_{-\infty}^{+\infty}x^m\mu(dx),$$

where

$$\mu=\begin{cases}N(0,1), & \tau=0,\\ \text{affine transformed }\mathrm{Po}(\tau^{-1}), & \tau>0.\end{cases}$$

This μ is the asymptotic spectral (= eigenvalue) distribution of $H(d,N)$.

5.6. Growing DRGs: General Results

$\{G_\nu\}$: growing DRGs with adjacency matrices A_ν

▶ Using $\mathbf{mean}(A_\nu) = \langle A_\nu \rangle = 0$ and $\mathbf{var}(A_\nu) = \langle A_\nu^2 \rangle = \mathbf{deg}(G_\nu) = p^0_{11}(\nu)$, the normalization of A_ν is given by

$$\frac{A_\nu - \mathbf{mean}(A_\nu)}{\sqrt{\mathbf{var}(A_\nu)}} = \frac{A_\nu^+}{\sqrt{\mathbf{deg}(G_\nu)}} + \frac{A_\nu^\circ}{\sqrt{\mathbf{deg}(G_\nu)}} + \frac{A_\nu^-}{\sqrt{\mathbf{deg}(G_\nu)}}\,.$$

Theorem (Quantum CLT for growing DRGs)

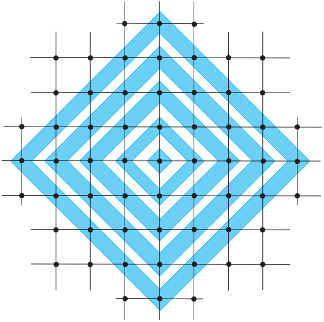
Assume that for all $n = 1, 2, \dots$ the limits

$$\omega_n = \lim_\nu \frac{p^n_{1,n-1}(\nu)\, p^{n-1}_{1,n}(\nu)}{p^0_{1,1}(\nu)}, \qquad \alpha_n = \lim_\nu \frac{p^{n-1}_{1,n-1}(\nu)}{\sqrt{p^0_{1,1}(\nu)}},$$

exist. Let $(\Gamma, \{\Phi_n\}, B^+, B^-, B^\circ)$ be the interacting Fock space associated with $(\{\omega_n\}, \{\alpha_n\})$. Then we have

$$\left(\frac{A_\nu^+}{\sqrt{\mathbf{deg}(G_\nu)}}, \frac{A_\nu^-}{\sqrt{\mathbf{deg}(G_\nu)}}, \frac{A_\nu^\circ}{\sqrt{\mathbf{deg}(G_\nu)}} \right) \xrightarrow{\mathrm{m}} (B^+, B^-, B^\circ).$$

5.7. Growing Regular Graphs — Going Slightly Beyond DRGs



$\mathbf{Z}^N$ as $N \to \infty$

1. $\Gamma(\mathbf{Z}^N)$ is asymptotically invariant under A^ϵ:

$$A^+\Phi_n = \sqrt{2N}\,\sqrt{n+1}\;\Phi_{n+1} + O(1),$$

$$A^-\Phi_n = \sqrt{2N}\,\sqrt{n}\;\Phi_{n-1} + O(N^{-1/2}).$$

2. Normalized adjacency matrices:

$$\frac{A_N^\epsilon}{\sqrt{\deg(A_N)}} = \frac{A_N^\epsilon}{\sqrt{2N}} \to B^\epsilon$$

3. The interacting Fock space in the limit:

$$B^+\Psi_n = \sqrt{n+1}\;\Psi_{n+1},$$

$$B^-\Phi_n = \sqrt{n}\;\Psi_{n-1}, \quad B^\circ = 0.$$

This is Boson Fock space!

4. The asymptotic spectral distribution is the standard Gaussian distribution:

$$\lim_{N\to\infty}\left\langle e_o, \left(\frac{A_N}{\sqrt{2N}}\right)^m e_o\right\rangle = \langle \Psi_0, (B^+ + B^-)^m \Psi_0\rangle$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^m e^{-x^2/2}\,dx.$$

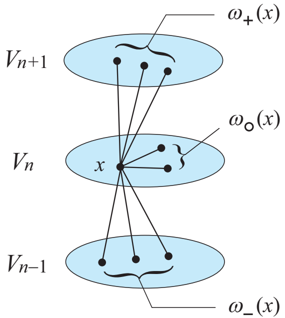
5.7. Growing Regular Graphs — Going Slightly Beyond DRGs

Statistics of $\omega_\epsilon(x)$

$$M(\omega_\epsilon|V_n) = \frac{1}{|V_n|}\sum_{x\in V_n}|\omega_\epsilon(x)|$$

$$\Sigma^2(\omega_\epsilon|V_n) = \frac{1}{|V_n|}\sum_{x\in V_n}\{|\omega_\epsilon(x)| - M(\omega_\epsilon|V_n)\}^2$$

$$L(\omega_\epsilon|V_n) = \max\{|\omega_\epsilon(x)|\,;\,x\in V_n\}.$$



Conditions for growing regular graphs $G_\nu = (V^{(\nu)}, E^{(\nu)})$

(A1) $\lim_\nu \kappa(\nu) = \infty$, where $\kappa(\nu) = \deg(G_\nu)$.

(A2) for each $n = 1, 2, \dots,$

$$\exists \lim_\nu M(\omega_-|V_n^{(\nu)}) = \omega_n < \infty, \quad \lim_\nu \Sigma^2(\omega_-|V_n^{(\nu)}) = 0, \quad \sup_\nu L(\omega_-|V_n^{(\nu)}) < \infty.$$

(A3) for each $n = 0, 1, 2, \dots,$

$$\exists \lim_\nu \frac{M(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} = \alpha_{n+1} < \infty, \quad \lim_\nu \frac{\Sigma^2(\omega_\circ|V_n^{(\nu)})}{\kappa(\nu)} = 0, \quad \sup_\nu \frac{L(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty.$$

5.7. Growing Regular Graphs — Going Slightly Beyond DRGs

Theorem (QCLT for growing regular graphs)

Let $\{G_\nu = (V^{(\nu)}, E^{(\nu)})\}$ *be a growing regular graph satisfying*

(A1) $\lim_\nu \kappa(\nu) = \infty$, *where* $\kappa(\nu) = \deg(G_\nu)$.

(A2) *for each* $n = 1, 2, \dots$,

$$\exists \lim_\nu M(\omega_-|V_n^{(\nu)}) = \omega_n < \infty, \quad \lim_\nu \Sigma^2(\omega_-|V_n^{(\nu)}) = 0, \quad \sup_\nu L(\omega_-|V_n^{(\nu)}) < \infty.$$

(A3) *for each* $n = 0, 1, 2, \dots$,

$$\exists \lim_\nu \frac{M(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} = \alpha_{n+1} < \infty, \quad \lim_\nu \frac{\Sigma^2(\omega_\circ|V_n^{(\nu)})}{\kappa(\nu)} = 0, \quad \sup_\nu \frac{L(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty.$$

Let $(\Gamma, \{\Psi_n\}, B^+, B^-, B^\circ)$ *be the interacting Fock space associated with the Jacobi parameters* $(\{\omega_n\}, \{\alpha_n\})$. *Then*

$$\left(\frac{A_\nu^+}{\sqrt{\kappa(\nu)}}, \frac{A_\nu^-}{\sqrt{\kappa(\nu)}}, \frac{A_\nu^\circ}{\sqrt{\kappa(\nu)}} \right) \xrightarrow{\text{m}} (B^+, B^-, B^\circ)$$

In particular, the asymptotic spectral distribution of the normalized A_ν *in the vacuum state is a probability distribution determined by* $(\{\omega_n\}, \{\alpha_n\})$.

Outline of Proof

(1) $\dfrac{A^{\epsilon}}{\sqrt{\kappa}}\Phi_n = \gamma^{\epsilon}_{n+\epsilon}\Phi_{n+\epsilon} + S^{\epsilon}_{n+\epsilon}, \qquad \epsilon \in \{+,-,\circ\}, \quad n = 0,1,2,\dots.$

$$\gamma^{+}_{n} = M(\omega_{-}|V_n)\left(\frac{|V_n|}{\kappa|V_{n-1}|}\right)^{1/2}, \quad \gamma^{-}_{n} = M(\omega_{+}|V_n)\left(\frac{|V_n|}{\kappa|V_{n+1}|}\right)^{1/2}, \quad \gamma^{\circ}_{n} = \frac{M(\omega_{\circ}|V_n)}{\sqrt{\kappa}}.$$

(2) $|V_n| = \left\{\prod_{k=1}^{n} M(\omega_{-}|V_k)\right\}^{-1}\kappa^n + O(\kappa^{n-1}).$

(3) $\lim_{\nu}\gamma^{+}_{n} = \sqrt{\omega_n}, \qquad \lim_{\nu}\gamma^{-}_{n} = \sqrt{\omega_{n+1}}, \qquad \lim_{\nu}\gamma^{\circ}_{n} = \alpha_{n+1}.$

(4)

$$\frac{A^{\epsilon_m}}{\sqrt{\kappa}}\cdots\frac{A^{\epsilon_1}}{\sqrt{\kappa}}\Phi_n = \gamma^{\epsilon_1}_{n+\epsilon_1}\gamma^{\epsilon_2}_{n+\epsilon_1+\epsilon_2}\cdots\gamma^{\epsilon_m}_{n+\epsilon_1+\cdots+\epsilon_m}\Phi_{n+\epsilon_1+\cdots+\epsilon_m}$$
$$+\sum_{k=1}^{m}\underbrace{\gamma^{\epsilon_1}_{n+\epsilon_1}\cdots\gamma^{\epsilon_{k-1}}_{n+\epsilon_1+\cdots+\epsilon_{k-1}}}_{(k-1)\text{ times}}\underbrace{\frac{A^{\epsilon_m}}{\sqrt{\kappa}}\cdots\frac{A^{\epsilon_{k+1}}}{\sqrt{\kappa}}}_{(m-k)\text{ times}}S^{\epsilon_k}_{n+\epsilon_1+\cdots+\epsilon_k}.$$

(5) Estimate the error terms and show that

$$\lim_{\nu}\left\langle \Phi^{(\nu)}_j, \frac{A^{\epsilon_m}}{\sqrt{\kappa(\nu)}}\cdots\frac{A^{\epsilon_{k+1}}}{\sqrt{\kappa(\nu)}}S^{\epsilon_k}_{n+\epsilon_1+\cdots+\epsilon_k}\right\rangle = 0.$$

5.8. Deformed Vacuum States on $\mathcal{A}(G)$

Definition (Q-matrix and deformed vacuum functional)

The *Q-matrix* of a graph $G = (V, E)$ is defined by

$$Q = Q_q = [q^{d(x,y)}]_{x,y \in V}\,, \qquad d(x,y) = \text{graph distance},$$

where q is a parameter (in fact, we are interested only in the case of $-1 \le q \le 1$). The *deformed vacuum functional* is defined by

$$\langle a \rangle_q = \langle Q_q e_o, a e_o \rangle, \qquad a \in \mathcal{A}(G).$$

1. For $q = 0$ we have $Q_0 = I$ so that $\langle \cdot \rangle_q$ coincides with the vacuum state.
2. Qe_o does not necessarily belong to $\ell^2(V)$ but $\langle a \rangle_q$ is well-defined for $a \in \mathcal{A}(G)$.
3. $\mathcal{A}(G) \ni a \mapsto \langle a \rangle_q$ is a merely a *normalized linear function*.
4. Positivity of $\langle \cdot \rangle_q$ is an interesting question from several aspects.

5.8. Deformed Vacuum States on $\mathcal{A}(G)$

▶ Let G be a κ-regular graph and consider the deformed vacuum functional on $\mathcal{A}(G)$:

$$\langle a \rangle_q = \langle Q_q e_o, a e_o \rangle, \qquad a \in \mathcal{A}(G).$$

We have

$$\langle A \rangle_q = \kappa q,$$
$$\Sigma_q^2(A) = \langle (A - \langle A \rangle_q)^2 \rangle_q = \kappa(1-q)\{1 + q + qM(\omega_\circ | V_1)\}$$

so that the quantum decomposition of the normalized adjacency matrix is given by

$$\frac{A - \langle A \rangle_q}{\Sigma_q(A)} = \frac{A^+}{\Sigma_q(A)} + \frac{A^-}{\Sigma_q(A)} + \frac{A^\circ - \langle A \rangle_q}{\Sigma_q(A)}$$

▶ Let $\{G_\nu\}$ be growing regular graphs. We need to find a proper scaling balance between $\kappa(\nu)$ and $q(\nu)$.

The balance condition found from the actions of A^ϵ and explicit form of $Q_q e_0$:

$$\lim_\nu \kappa(\nu) = \infty, \qquad \lim_\nu q(\nu) = 0, \qquad \lim_\nu q(\nu)\sqrt{\kappa(\nu)} = \gamma \in \mathbb{R}.$$

5.8. Deformed Vacuum States on $\mathcal{A}(G)$

(A1) $\lim_\nu \kappa(\nu) = \infty$, where $\kappa(\nu) = \deg(G_\nu)$.
(A2) for each $n = 1, 2, \dots,$

$$\exists \lim_\nu M(\omega_-|V_n^{(\nu)}) = \omega_n < \infty, \quad \lim_\nu \Sigma^2(\omega_-|V_n^{(\nu)}) = 0, \quad \sup_\nu L(\omega_-|V_n^{(\nu)}) < \infty.$$

(A3) for each $n = 0, 1, 2, \dots,$

$$\exists \lim_\nu \frac{M(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} = \alpha_{n+1} < \infty, \quad \lim_\nu \frac{\Sigma^2(\omega_\circ|V_n^{(\nu)})}{\kappa(\nu)} = 0, \quad \sup_\nu \frac{L(\omega_\circ|V_n^{(\nu)})}{\sqrt{\kappa(\nu)}} < \infty.$$

(A4) (scaling balance) $\lim_\nu q(\nu) = 0$ and $\lim_\nu q(\nu)\sqrt{\kappa(\nu)} = \gamma \in \mathbb{R}$ (constant).

Lemma

Under (A1)–(A4) we have

$$Qe_o = \sum_{n=0}^{\infty} q^n \sqrt{|V_n|}\, \Phi_n \longrightarrow \sum_{n=0}^{\infty} \frac{\gamma^n}{\sqrt{\omega_n \cdots \omega_1}}\, \Psi_n = \Omega_\gamma$$

The above Ω_γ is reasonably called a *coherent vector* of the interacting Fock space since

$$B^- \Omega_\gamma = \gamma \Omega_\gamma \,.$$

See e.g., P. K. Das: IJTP **41** (2002), 1099–1106.

5.8. Deformed Vacuum States on $\mathcal{A}(G)$

Theorem (Deformed QCLT for growing regular graphs)

Let $\{G_\nu = (V^{(\nu)}, E^{(\nu)})\}$ *be a growing regular graph satisying conditions (A1)–(A3) and* A_ν *its adjacency matrix. Let* $(\Gamma, \{\Psi_n\}, B^+, B^-, B^\circ)$ *be the IFS associated to* $(\{\omega_n\}, \{\alpha_n\})$. *Under (A4) we have*

$$\lim_{\substack{\kappa\to\infty, q\to 0 \\ q\sqrt{\kappa}\to\gamma}} \left\langle Qe_o, \frac{\tilde{A}^{\epsilon_m}}{\Sigma_q(A)} \cdots \frac{\tilde{A}^{\epsilon_1}}{\Sigma_q(A)} e_o \right\rangle = \langle \Omega_\gamma, \tilde{B}^{\epsilon_m} \cdots \tilde{B}^{\epsilon_1} \Psi_0 \rangle,$$

where

$$\tilde{A}^\pm = A^\pm_\nu, \quad \tilde{A}^\circ = A^\circ_\nu - \langle A_\nu \rangle_q, \quad \tilde{B}^\pm = \frac{B^\pm}{\sqrt{1+\gamma\alpha_2}}, \quad \tilde{B}^\circ = \frac{B^\circ - \gamma}{\sqrt{1+\gamma\alpha_2}}.$$

In particular,

$$\lim_{\substack{\kappa\to\infty, q\to 0 \\ q\sqrt{\kappa}\to\gamma}} \left\langle \left(\frac{A_\nu - \langle A\rangle_q}{\Sigma_q(A_\nu)} \right)^m \right\rangle_q = \left\langle \Omega_\gamma, \left(\frac{B^+ + B^- + B^\circ - \gamma}{\sqrt{1+\gamma\alpha_2}} \right)^m \Psi_0 \right\rangle.$$

▶ Challenging Exercise: Examine the above argument for T_κ as $\kappa \to \infty$ and find the limit distribution (free Poisson distribution = Marchenko–Pastur distribution).

Some concrete examples: Asymptotic spectral distributions

graphs	IFS	vacuum state	deformed vacuum state
Hamming graphs $H(d,N)$	$\omega_n = n$ (Boson)	Gaussian $(N/d \to 0)$ Poisson $(N/d \to \lambda^{-1} > 0)$	Gaussian or Poisson
Johnson graphs $J(v,d)$	$\omega_n = n^2$	exponential $(2d/v \to 1)$ geometric $(2d/v \to p \in (0,1))$	'Poissonization' of exponential distribution
odd graphs O_k	$\omega_{2n-1} = n$ $\omega_{2n} = n$	two-sided Rayleigh	?
homogeneous trees T_κ	$\omega_n = 1$ (free)	Wigner semicircle	free Poisson
integer lattices $\mathbb{Z}^N$	$\omega_n = n$ (Boson)	Gaussian	Gaussian
symmetric groups $\mathfrak{S}_n$ (Coxeter)	$\omega_n = n$ (Boson)	Gaussian	Gaussian
Coxeter groups (Fendler)	$\omega_n = 1$ (free)	Wigner semicircle	free Poisson
Spidernets $S(a,b,c)$	$\omega_1 = 1$ $\omega_2 = \cdots = q$	free Meixner law	(free Meixner law)

6. Concepts of Independence and Graph Products

6.1. (Classical) Independence and Central Limit Theorem

$X, Y, \dots$: random variables on a classical probability space $(\Omega, \mathcal{F}, P)$

Definition

Two random variables X and Y are called *independent* if

$$P(X \le a, Y \le b) = P(X \le a)P(Y \le b), \qquad a, b \in \mathbb{R}.$$

Theorem (multiplicativity of mean values)

If two random variables X, Y are independent, then

$$\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y].$$

Moreover,

$$\mathbf{E}[X^m Y^n] = \mathbf{E}[X^m]\mathbf{E}[Y^n]$$

whenever the mean values exist.

6.1. (Classical) Independence and Central Limit Theorem

$X_1, X_2, \dots$: sequence of random variables such that

(i) independent

(ii) identically distributed

(iii) normalized, i.e., $\mathbf{E}[X_n] = 0$, $\mathbf{V}[X_n] = \mathbf{E}[X_n^2] = 1$

- Law of Large Numbers (LLN) says that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X_n = 0 \quad \text{almost surely.}$$

- Central Limit Theorem (CLT) describes the fluctuation of

$$\lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n$$

6.1. (Classical) Independence and Central Limit Theorem

Theorem (Central limit theorem (CLT))

Let $X_1, X_2, \dots$ be a sequence of random variables such that (i) independent, (ii) identically distributed, and (iii) normalized. Then

$$\frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n$$

obeys the standard normal distribution $N(0,1)$ in the limit.

$$\lim_{N\to\infty} P\left(\frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \le a \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-x^2/2} dx,$$

or equivalently, for any $f \in C_b(\mathbb{R})$,

$$\lim_{N\to\infty} \mathbf{E}\left[f\left(\frac{1}{\sqrt{N}} \sum_{n=1}^{N} X_n \right) \right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-x^2/2} dx.$$

6.1. (Classical) Independence and Central Limit Theorem

Theorem (Algebraic Version of CLT)

Let $X_1, X_2, \dots$ be a sequence of random variables such that (i) independent, (ii) identically distributed, and (iii) normalized. If X_n has finite moments of all orders, we have

$$\lim_{N\to\infty} \mathbf{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} X_n\right)^m\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} x^m e^{-x^2/2}dx.$$

In other words,

$$\lim_{N\to\infty} \mathbf{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} X_n\right)^{2m}\right] = \frac{(2m)!}{2^m m!},$$

$$\lim_{N\to\infty} \mathbf{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} X_n\right)^{2m-1}\right] = 0.$$

6.1. (Classical) Independence and Central Limit Theorem

Combinatorial Proof

$$\mathbf{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} X_n\right)^m\right] = \frac{1}{N^{m/2}} \sum_{n_1,\dots,n_m=1}^{N} \mathbf{E}[X_{n_1} X_{n_2} \cdots X_{n_m}]$$

- We pick up the essential terms $\mathbf{E}[X_{n_1} X_{n_2} \cdots X_{n_m}]$ that contributes to the limit.

1.
$$\mathbf{E}[\underbrace{X_{n_1} X_{n_2} \cdots\cdots X_{n_m}}_{\exists X_i \text{ appears only once}}] = \mathbf{E}[X_i]\mathbf{E}[\cdots\cdots] = 0.$$

2. Hence we only need to count the terms
$$\mathbf{E}[\underbrace{X_{n_1} X_{n_2} \cdots\cdots X_{n_m}}_{\#\text{ of distinct } X_i\text{'s} \le [\frac{m}{2}]}]$$

6.1. (Classical) Independence and Central Limit Theorem

$$\mathbf{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} X_n\right)^m\right] = \frac{1}{N^{m/2}} \sum_{n_1,\dots,n_m=1}^{N} \mathbf{E}[X_{n_1}X_{n_2}\cdots X_{n_m}]$$

2. Hence we only need to count the terms

$$\mathbf{E}[\underbrace{X_{n_1}X_{n_2}\cdots\cdots X_{n_m}}_{\#\text{ of distinct } X_i\text{'s} \le [\frac{m}{2}]}]$$

3. Let s be the number of distinct X_i's. The number of such terms is

$$\binom{N}{s} \times \#\{\text{arrangements of } X_1,\dots,X_s\} \sim N^s C(s).$$

4. Thus the terms of $s < m/2$ have no contribution in the limit.
5. Namely, only the terms of $s = m/2$ have contribution in the limit.

6.1. (Classical) Independence and Central Limit Theorem

$$\mathbf{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} X_n\right)^m\right] = \frac{1}{N^{m/2}} \sum_{n_1,\dots,n_m=1}^{N} \mathbf{E}[X_{n_1} X_{n_2} \cdots X_{n_m}]$$

5. Namely, only the terms of $s = m/2$ have contribution in the limit.
6. If m is odd,

$$\lim_{N\to\infty} \mathbf{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} X_n\right)^m\right] = 0.$$

7. Suppose that $m = 2s$ is even.

$$\mathbf{E}[\underbrace{X_{n_1} X_{n_2} \cdots X_{n_m}}_{\substack{s \text{ distinct } X_i\text{'s} \\ \text{each appears twice}}}] = \mathbf{E}[X_{i_1}^2 X_{i_2}^2 \cdots X_{i_s}^2] = \mathbf{E}[X_{i_1}^2]\mathbf{E}[X_{i_2}^2] \cdots \mathbf{E}[X_{i_s}^2] = 1.$$

8. Consequently,

$$\lim_{N\to\infty} \mathbf{E}\left[\left(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} X_n\right)^{2s}\right] = \lim_{N\to\infty} \frac{1}{N^s} \binom{N}{s} \frac{(2s)!}{2^s} = \frac{(2s)!}{2^s s!}.$$

6.2. Independence in Quantum Probability and Quantum CLT

- Algebraic version of CLT is proved by
  1. using factorization rule of mixed moments $\mathbf{E}[X_{n_1} X_{n_2} \cdots X_{n_m}]$,
  2. picking up the essential terms that contribute to the limit.

<u>Factorization rule</u>

- For classical random variables X and Y, obviously we have

$$\mathbf{E}[YXX] = \mathbf{E}[XYX] = \mathbf{E}[XXY] = \mathbf{E}[X^2Y] = \mathbf{E}[X^2]\mathbf{E}[Y], \quad \ldots$$

- But for $a = a^*, b = b^*$ in $(\mathcal{A}, \varphi)$ we wonder

$$\varphi(baa) \overset{?}{=} \varphi(aba) \overset{?}{=} \varphi(aab) = ??? \quad \ldots$$

There are many possibilities arising from non-commutativity.

Our viewpoint

- Independence is formulated as a "good" factorization rule.
- There are four basic concepts of independence in quantum probability.

6.2. Independence in Quantum Probability and Quantum CLT

- Suppose we are given a concept of *independence* in $(\mathcal{A}, \varphi)$.
- Then we may consider a sequence $\{a_n\}$ of random variables in $(\mathcal{A}, \varphi)$ such that

(0) real, i.e., $a_n = a_n^*$,

(i) independent,

(ii) identically distributed,

(iii) normalized, i.e., $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$.

- Then we ask for the probability distribution μ such that

$$\lim_{N\to\infty} \varphi\left[\left(\frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n\right)^m\right] = \int_{-\infty}^{+\infty} x^m \mu(dx), \quad m = 1, 2, \dots.$$

We call μ the *central limit distribution*.

6.2. Independence in Quantum Probability and Quantum CLT

Four Concepts of Independence and Quantum CLTs

- Factorization rules are shown only for three mixed moments of low orders.

	commutative	free	Boolean	monotone
$\varphi(aba)$	$\varphi(a^2)\varphi(b)$	$\varphi(a^2)\varphi(b)$	$\varphi(a)^2\varphi(b)$	$\varphi(a^2)\varphi(b)$
$\varphi(bab)$	$\varphi(a)\varphi(b^2)$	$\varphi(a)\varphi(b^2)$	$\varphi(a)\varphi(b)^2$	$\varphi(a)\varphi(b)^2$
$\varphi(abab)$	$\varphi(a^2)\varphi(b^2)$	$\varphi(a)^2\varphi(b^2)$ $+\varphi(a^2)\varphi(b)^2$ $-\varphi(a)^2\varphi(b)^2$	$\varphi(a)^2\varphi(b)^2$	$\varphi(a^2)\varphi(b)^2$
CLM	Gaussian	Wigner	Bernoulli	arcsine

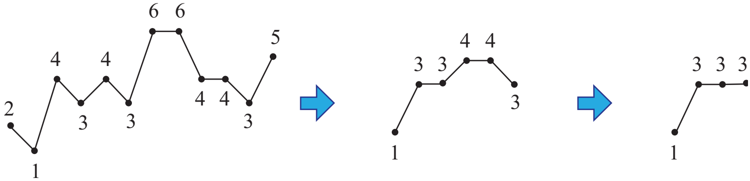
6.2. Independence in Quantum Probability and Quantum CLT

- One more: $\varphi(a_2a_1a_4a_3a_4a_3a_6a_6a_4a_4a_3a_5) = \varphi(\mathbf{214343664435})$

1. [commutative independence]

$$\varphi(\mathbf{214343664435}) = \varphi(\mathbf{1})\varphi(\mathbf{2})\varphi(\mathbf{3^3})\varphi(\mathbf{4^4})\varphi(\mathbf{5})\varphi(\mathbf{6^2})$$

2. [monotone independence]



$$\begin{aligned}\varphi(\mathbf{214343664435}) &= \varphi(\mathbf{2})\varphi(\mathbf{4})\varphi(\mathbf{4})\varphi(\mathbf{66})\varphi(\mathbf{133443})\varphi(\mathbf{5}) \\ &= \varphi(\mathbf{2})\varphi(\mathbf{4})\varphi(\mathbf{4})\varphi(\mathbf{66})\varphi(\mathbf{44})\varphi(\mathbf{1333})\varphi(\mathbf{5}) \\ &= \varphi(\mathbf{2})\varphi(\mathbf{4})\varphi(\mathbf{4})\varphi(\mathbf{66})\varphi(\mathbf{44})\varphi(\mathbf{333})\varphi(\mathbf{1})\end{aligned}$$

3. [Boolean independence]

$$\varphi(\mathbf{214343664435}) = \varphi(\mathbf{2})\varphi(\mathbf{1})\varphi(\mathbf{4})\varphi(\mathbf{3})\varphi(\mathbf{4})\varphi(\mathbf{3})\varphi(\mathbf{66})\varphi(\mathbf{44})\varphi(\mathbf{3})\varphi(\mathbf{5})$$

6.2. Independence in Quantum Probability and Quantum CLT

Central limit distributions

$$\varphi\left[\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} a_k\right)^m\right] \to \int_{-\infty}^{+\infty} x^m \mu(dx).$$

Theorem (QCLT)

1. [commutative CLT] *If* $a_1, a_2, \dots$ *are commutative independent, we have*

$$\mu(dx) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \qquad \textit{(normal distribution)}$$

2. [monotone CLT] *If* $a_1, a_2, \dots$ *are monotone independent, we have*

$$\mu(dx) = \frac{dx}{\pi\sqrt{2-x^2}} \qquad \textit{(normalized arcsine law)}$$

3. [Boolean CLT] *If* $a_1, a_2, \dots$ *are Boolean independent, we have*

$$\mu = \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1} \qquad \textit{(normalized Bernoulli distribution)}$$

6.2. Independence in Quantum Probability and Quantum CLT

Outline of proof

$$\varphi\left[\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} a_k\right)^m\right] = \frac{1}{n^{m/2}} \sum_{k_1,\dots,k_m=1}^{n} \varphi[a_{k_1} a_{k_2} \cdots a_{k_m}]$$

- We pick up the essential terms $\varphi[a_{k_1} a_{k_2} \cdots a_{k_m}]$ that contributes to the limit.

1. $\varphi(a_{k_1} a_{k_2} \cdots a_{k_m}) = 0$ if there is a singleton.
2. $\varphi(a_{k_1} a_{k_2} \cdots a_{k_m})$ contributes to the limit only if the number s of distinct a_i's is $s = [m/2]$.
3. According to the independence evaluate $\varphi(a_{k_1} a_{k_2} \cdots a_{k_m})$, where distinct a_i's appear exact twice.

6.2. Independence in Quantum Probability and Quantum CLT

Outline of proof

4. Finally we get

$$\lim_{n\to\infty} \varphi\left[\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} a_k\right)^{2m-1}\right] = 0$$

for three cases and

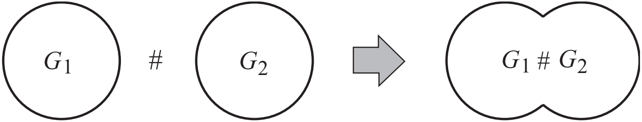
$$\lim_{n\to\infty} \varphi\left[\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} a_k\right)^{2m}\right] = \begin{cases} \dfrac{(2m)!}{2^m m!}, & \text{commutative independence,} \\ \dfrac{(2m)!}{2^m m! m!}, & \text{monotone independence,} \\ 1, & \text{Boolean independence.} \end{cases}$$

Cf. free CLT

$$\lim_{n\to\infty} \varphi\left[\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} a_k\right)^{2m}\right] = \frac{1}{m+1}\binom{2m}{m} = \int_{-2}^{2} x^m \frac{1}{2\pi}\sqrt{4-x^2}\,dx.$$

6.3. Graph Products

A binary operation of graphs



$$(G_1, G_2) \mapsto \Phi(G_1, G_2) = G_1 \# G_2$$

$$(A_1, A_2) \mapsto \Phi(A_1, A_2) = A[G_1 \# G_2]$$

$$(\mu_1, \mu_2) \mapsto \Phi(\mu_1, \mu_2) = \mu_1 \# \mu_2 \quad \text{(convolution)}$$

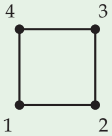
6.3. Graph Products — Cartesian Product

Definition

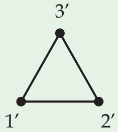
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *Cartesian product* or *direct product* of G_1 and G_2, denoted by $G_1 \times G_2$, is a graph on $V = V_1 \times V_2$ with adjacency relation:

$$(x, y) \sim (x', y') \iff \begin{cases} x = x' \\ y \sim y' \end{cases} \quad \text{or} \quad \begin{cases} x \sim x' \\ y = y'. \end{cases}$$

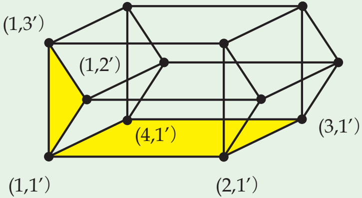
Example ($C_4 \times C_3$)



C_4



C_3



$C_4 \times C_3$

6.3. Graph Products — Comb Product

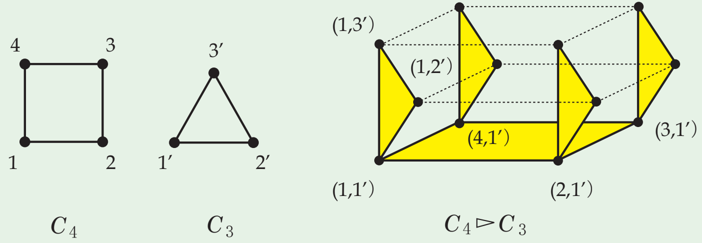
Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. We fix a vertex $o_2 \in V_2$. For $(x, y), (x', y') \in V_1 \times V_2$ we write $(x, y) \sim (x', y')$ if one of the following conditions is satisfied:

(i) $x = x'$ *and* $y \sim y'$; (ii) $x \sim x'$ *and* $y = y' = o_2$.

Then $V_1 \times V_2$ becomes a graph, denoted by $G_1 \rhd_{o_2} G_2$, and is called the *comb product* or the *hierarchical product*.

Example ($C_4 \rhd_{o_2} C_3$ with $o_2 = 1'$)



6.3. Graph Products — Star Product

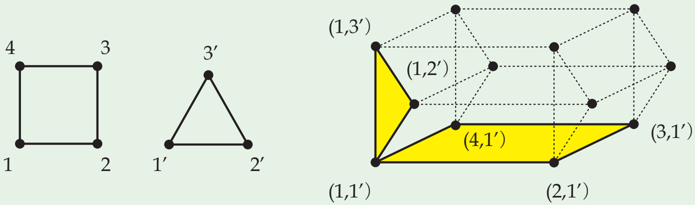
Definition

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with distinguished vertices $o_1 \in V_1$ and $o_2 \in V_2$. Define a subset of $V_1 \times V_2$ by

$$V_1 \star V_2 = \{(x, o_2)\,;\, x \in V_1\} \cup \{(o_1, y)\,;\, y \in V_2\}$$

The induced subgraph of $G_1 \times G_2$ spanned by $V_1 \star V_2$ is called the *star product* of G_1 and G_2 (with contact vertices o_1 and o_2), and is denoted by $G_1 \star G_2 = G_1 \,{}_{o_1}\!\star_{o_2}\, G_2$.

Example ($C_4 \star C_3$)



C_4 C_3 $C_4 \star C_3$

6.3. Graph Products — Adjacency Matrices

$G_1 = (V_1, E_1), G_2 = (V_2, E_2)$: two graphs

$G = G_1 \# G_2$: a graph product and assume that $V[G] = V_1 \times V_2$

$A_i = A[G_i]$: adjacency matrix of G_i acting on $\ell^2(V_i)$, $(i = 1, 2)$

$\Longrightarrow A = A[G_1 \# G_2]$ acts on

$$\ell^2(V) = \ell^2(V_1 \times V_2) \cong \ell^2(V_1) \otimes \ell^2(V_2).$$

Theorem

1. [Cartesian product]
$$A[G_1 \times G_2] = A_1 \otimes I_2 + I_1 \otimes A_2 \,.$$
2. [comb product]
$$A[G_1 \triangleright G_2] = A_1 \otimes P_2 + I_1 \otimes A_2 \,.$$
3. [star product]
$$A[G_1 \star G_2] = A_1 \otimes P_2 + P_1 \otimes A_2 \,.$$

Here, P_i is the rank one projection corresponding to o_i.

6.4. Quantum CLT for Graph Products

- Let φ_i be the vacuum state at o_i and consider the *product state* $\varphi = \varphi_1 \otimes \varphi_2$.

  $\Longrightarrow A = A[G_1 \# G_2]$ is a random variable in $(\mathcal{A}(G_1 \# G_2), \varphi)$.

Theorem

Let $A_i = A[G_i]$ be the adjacency matrix of G_i.

1. [Cartesian product]

$$A[G_1 \times G_2] = A_1 \otimes I_2 + I_1 \otimes A_2$$

is a sum of commutative independent random variables.

2. [comb product]

$$A[G_1 \triangleright G_2] = A_1 \otimes P_2 + I_1 \otimes A_2$$

is a sum of monotone independent random variables.

3. [star product]

$$A[G_1 \star G_2] = A_1 \otimes P_2 + P_1 \otimes A_2$$

is a sum of Boolean independent random variables.

6.4. Quantum CLT for Graph Products

Associativity of graph operations

1. [Cartesian product]

$$(G_1 \times G_2) \times G_3 \cong G_1 \times (G_2 \times G_3)$$

2. [Comb product]

$$(G_1 \rhd G_2) \rhd G_3 \cong G_1 \rhd (G_2 \rhd G_3)$$

   To be precise,

$$(G_1 \rhd_{o_2} G_2) \rhd_{o_3} G_3 \cong G_1 \rhd_{(o_2,o_3)} (G_2 \rhd_{o_3} G_3)$$

3. [Star product]

$$(G_1 \star G_2) \star G_3 \cong G_1 \star (G_2 \star G_3)$$

▶ Thus, we have naturally n-fold powers:

$$G^{\#n} = G\#G\# \cdots \#G \qquad (n \text{ times})$$

$$A[G^{\#n}] = B_1 + B_2 + \cdots + B_n$$

6.4. Quantum CLT for Graph Products

Theorem (CLT for Cartesian product graphs)

For the n-fold Cartesian power $G^{(n)} = G \times \cdots \times G$ (n-times),

$$\lim_{n\to\infty} \left\langle \left(\frac{A^{(n)}}{\sqrt{n}\,\sqrt{\deg(o)}} \right)^m \right\rangle = \int_{-\infty}^{+\infty} x^m \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} dx.$$

Theorem (CLT for comb product graphs)

For the n-fold monotone power $G^{(n)} = G \triangleright_o G \triangleright_o \cdots \triangleright_o G$ (n-times),

$$\lim_{n\to\infty} \left\langle \left(\frac{A^{(n)}}{\sqrt{n}\sqrt{\deg(o)}} \right)^m \right\rangle = \int_{-\sqrt{2}}^{+\sqrt{2}} x^m \frac{dx}{\pi\sqrt{2-x^2}}, \quad m = 1, 2, \dots .$$

Theorem (CLT for star product graphs)

For the n-fold star power $G^{(n)} = G \star G \star \cdots \star G$ (n-times) we have

$$\lim_{n\to\infty} \left\langle \left(\frac{A^{(n)}}{\sqrt{n}\sqrt{\deg(o)}} \right)^m \right\rangle = \int_{-\infty}^{+\infty} x^m \frac{1}{2}(\delta_{-1} + \delta_{+1})(dx), \quad m = 1, 2, \dots .$$

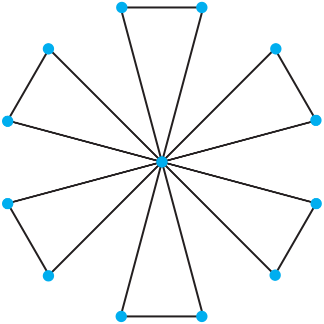
More Graph Products

products	$G_1 \# G_2$	$A[G_1 \# G_2]$	spectral distribution
Cartesian	$G_1 \times_C G_2$	$A_1 \otimes I_2 + I_1 \otimes A_2$	$\mu_1 * \mu_2$
monotone	$G_1 \rhd G_2$	$A_1 \otimes P_2 + I_2 \otimes A_2$	$\mu_1 \rhd \mu_2$
star	$G_1 \star G_2$	$A_1 \otimes P_2 + P_1 \otimes A_2$	$\mu_1 \uplus \mu_2$
lexicographic	$G_1 \rhd_L G_2$	$A_1 \otimes J_2 + P_1 \otimes A_2$	$D(\mu_1) \rhd \mu_2$
Kronecker	$G_1 \times_K G_2$	$A_1 \otimes A_2$	$\mu_1 *_M \mu_2$
strong	$G_1 \times_S G_2$	$A_1 \otimes I_2 + I_1 \otimes A_2$ $+A_1 \otimes A_2$	$S^{-1}(S\mu_1 *_M S\mu_2)$
free	$G_1 * G_2$	$A_1 * A_2$	$\mu_1 \boxplus \mu_2$

1. Every product except the free product is a graph on $V_1 \times V_2$.
2. There is a classification of graph products realized on $V_1 \times V_2$, see e.g., R. Hammack *et al.*: "Handbook of Product Graphs," CRC Press, 2011.

Exercises

Exercise 12 Let G_n be the graph obtained by joining n triangles ($K_3 \cong C_3$ at the origin o, also called the n-fold star product of K_3. (The following figure shows G_6.) Calculate explicitly the spectral distribution of G_n at o and study its asymptotic behevior as $n \to \infty$.



7. Counting Walks

N. Obata: "Spectral Analysis of Growing Graphs," Chapter 7, Springer, 2017.

H. H. Lee and N. Obata: *Kronecker product graphs and counting walks in restricted lattices*, arXiv:1607.06808.

7.1. Counting Walks and Spectral Distributions

$G = (V, E)$: a (finite or infinite) graph

$o \in V$: a fixed origin

$W_m(o; G) = |\{o \to o \ : \ m\text{-step walk}\}|$

Theorem

Let A be the adjacency matrix of G and μ the vacuum spectral distribution at $o \in V$. Then we have

$$W_m(o; G) = \langle e_o, A^m e_o \rangle = \int_{-\infty}^{+\infty} x^m \mu(dx), \qquad m = 0, 1, 2, \dots .$$

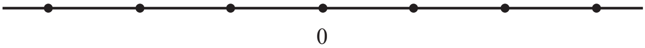
- we are interested in the correspondence

$$G \to \mu$$

from the point of view of counting walks.

7.1. Counting Walks and Spectral Distributions

Basic result (1) $\mathbb{Z}$



$$W_{2m}(0;\mathbb{Z}) = \binom{2m}{m} = \int_{-2}^{2} x^{2m} \alpha(dx), \qquad \alpha(x) = \frac{1}{\pi\sqrt{4-x^2}}\,.$$

Basic result (2) $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$

$$W_{2m}(0;\mathbb{Z}_+) = \frac{1}{m+1}\binom{2m}{m} = \int_{-2}^{2} x^{2m} w(dx), \quad w = \frac{1}{2\pi}\sqrt{4-x^2}\,.$$

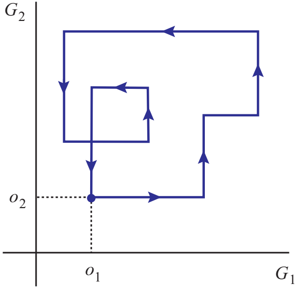
Catalan number

7.2. Cartesian Product: $W((o_1,o_2);G_1\times_C G_2)$

The adjacency matrix of $G_1\times_C G_2$ is

$$A=A_1\otimes I+I\otimes A_2,$$

where two matrices in RHD are commutative.



We then have

$$\begin{aligned}\langle e_{(o_1,o_2)},A^m e_{(o_1,o_2)}\rangle &=\langle e_{o_1}\otimes e_{o_2},(A_1\otimes I+I\otimes A_2)^m e_{o_1}\otimes e_{o_2}\rangle\\ &=\sum_{k=0}^m\binom{m}{k}\langle e_{o_1}\otimes e_{o_2},A_1^k\otimes A_2^{m-k}e_{o_1}\otimes e_{o_2}\rangle\\ &=\sum_{k=0}^m\binom{m}{k}\langle e_{o_1},A_1^k e_{o_1}\rangle\langle e_{o_2}\otimes A_2^{m-k}e_{o_2}\rangle\end{aligned}$$

Consequently,

$$W((o_1,o_2);G_1\times_C G_2)=\sum_{k=0}^m\binom{m}{k}W_k(o_1;G_1)W_{m-k}(o_2;G_2)$$

7.2. Cartesian Product: $W((o_1, o_2); G_1 \times_C G_2)$

μ_i: Spectral distribution of G_i at o_i

μ: Spectral distribution of $G = G_1 \times_C G_2$ at (o_1, o_2)

$$W_m(o_i; G_i) = \int_{-\infty}^{+\infty} x^m \mu_i(dx), \quad W_m((o_1, o_2); G_1 \times_C G_2) = \int_{-\infty}^{+\infty} x^m \mu(dx).$$

Then the identity

$$W((o_1, o_2); G_1 \times_C G_2) = \sum_{k=0}^{m} \binom{m}{k} W_k(o_1; G_1) W_{m-k}(o_2; G_2)$$

implies that

$$\int_{-\infty}^{+\infty} x^m \mu(dx) = \sum_{k=0}^{m} \binom{m}{k} \int_{-\infty}^{+\infty} x^k \mu_1(dx) \int_{-\infty}^{+\infty} x^{m-k} \mu_2(dx)$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x_1 + x_2)^m \mu_1(dx_1) \mu_2(dx_2).$$

Thus, $\mu = \mu_1 * \mu_2$ (classical) convolution.

7.3. Graph Products and Convolution of Distributions

products	$G_1 \# G_2$	$A[G_1 \# G_2]$	spectral distribution
Cartesian	$G_1 \times_C G_2$	$A_1 \otimes I_2 + I_1 \otimes A_2$	$\mu_1 * \mu_2$
comb	$G_1 \rhd G_2$	$A_1 \otimes P_2 + I_2 \otimes A_2$	$\mu_1 \rhd \mu_2$
star	$G_1 \star G_2$	$A_1 \otimes P_2 + P_1 \otimes A_2$	$\mu_1 \uplus \mu_2$
lexicographic	$G_1 \rhd_L G_2$	$A_1 \otimes J_2 + P_1 \otimes A_2$	$D(\mu_1) \rhd \mu_2$
Kronecker	$G_1 \times_K G_2$	$A_1 \otimes A_2$	$\mu_1 *_M \mu_2$
strong	$G_1 \times_S G_2$	$A_1 \otimes I_2 + I_1 \otimes A_2$ $+ A_1 \otimes A_2$	$S^{-1}(S\mu_1 *_M S\mu_2)$
free	$G_1 * G_2$	$A_1 * A_2$	$\mu_1 \boxplus \mu_2$

1. Every product *except the free product* is a graph on $V_1 \times V_2$.
2. There is a classification of graph products realized on $V_1 \times V_2$, see e.g., R. Hammack *et al.*: "Handbook of Product Graphs," CRC Press, 2011.

7.3. Graph Products and Convolution of Distributions

- Monotone convolution $\mu = \mu_1 \rhd \mu_2$ is characterized by

$$H_\mu(z) = H_{\mu_1}(H_{\mu_2}(z)),$$

where

$$H_\mu(z) = \frac{1}{G_\mu(z)}, \qquad G_\mu(z) = \int_{-\infty}^{+\infty} \frac{\mu(dx)}{z-x}.$$

- Boolean convolution $\mu = \mu_1 \uplus \mu_2$ is characterized by

$$\frac{1}{G_\mu(z)} = \frac{1}{G_{\mu_1}(z)} + \frac{1}{G_{\mu_2}(z)} - z$$

7.4. Kronecker Product

Definition (Kronecker product)

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. The *Kronecker product* $G_1 \times_K G_2$ is a graph on $V = V_1 \times V_2$ with the adjacency relation:

$$(x, y) \sim_K (x', y') \iff x \sim x', \; y \sim y'.$$

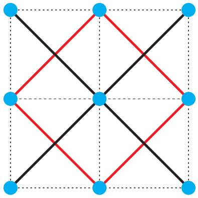
In other words, the adjacency matrix $A = A[G_1 \times_K G_2]$ is given by

$$A = A_1 \otimes A_2 \, .$$

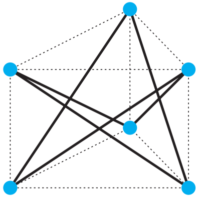
1. $G_1 \times_K G_2 \cong G_2 \times_K G_1$.
2. $(G_1 \times_K G_2) \times_K G_3 \cong G_1 \times_K (G_2 \times_K G_3)$.
3. (trivial case) For any graph $G = (V, E)$ the Kronecker product $K_1 \times_K G$ is a graph on V with no edges (i.e., an empty graph on V).

7.4. Kronecker Product

$P_3 \times_K P_3$



$K_2 \times_K K_3$



Lemma (exercise)

If $|V_1| \geq 2$ *and* $|V_2| \geq 2$, *then* $G_1 \times_K G_2$ *has at most two connected components.*

Lemma (exercise)

$G_1 \times_K G_2$ *is a subgraph of the distance-2 graph of* $G_1 \times_C G_2$. *(But not necessarily induced subgraph.)*

7.5. Counting Walks in Kronecker Product

$G_i = (V_i, E_i)$: a connected graph with fixed origin $o_i \in V_i$

$G = G_1 \times_K G_2$: Kronecker product with origin (o_1, o_2)

$G^o = (G_1 \times_K G_2)^o$: the connected component containing (o_1, o_2)

$$\begin{aligned} W_m((o_1, o_2); G) &= W_m((o_1, o_2); G^o) \\ &= \langle e_{(o_1,o_2)}, A^m e_{(o_1,o_2)} \rangle \\ &= \langle e_{o_1} \otimes e_{o_2}, (A_1 \otimes A_2)^m e_{o_1} \otimes e_{o_2} \rangle \\ &= \langle e_{o_1}, A_1^m e_{o_1} \rangle \langle e_{o_2}, A_2^m e_{o_2} \rangle \\ &= W_m(o_1; G_1) W_m(o_2; G_2) \end{aligned}$$

7.5. Counting Walks in Kronecker Product

$G_i = (V_i, E_i)$: a connected graph with fixed origin $o_i \in V_i$

$G = G_1 \times_K G_2$: Kronecker product with origin (o_1, o_2)

$G^o = (G_1 \times_K G_2)^o$: the connected component containing (o_1, o_2)

Thus,

$$W_m((o_1, o_2); G) = W_m(o_1; G_1) W_m(o_1; G_2).$$

μ_i: spectral distribution of the adjacency matrix A_i at o_i

μ: spectral distribution of the adjacency matrix $A = A[G]$ at (o_1, o_2)

$$\begin{aligned}\int_{-\infty}^{+\infty} x^m \mu(dx) &= \int_{-\infty}^{+\infty} x_1^m \mu_1(dx_1) \int_{-\infty}^{+\infty} x_2^m \mu_2(dx_2) \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x_1 x_2)^m \mu_1(dx_1) \mu_2(dx_2)\end{aligned}$$

This μ is called the *Mellin convolution* and denoted by $\mu = \mu_1 *_M \mu_2$.

7.5. Counting Walks in Kronecker Product

Theorem

For $i=1,2$ let $G_i=(V_i,E_i)$ be a graph with a distinguished vertex o_i. Let μ_i be the spectral distribution of the adjacency matrix $A_i=A[G_i]$ at o_i. Then the spectral distribution of $G=G_1\times_K G_2$ at (o_1,o_2) is given by the Mellin convolution:

$$\mu(G_1\times_K G_2)=\mu_1 *_M \mu_2.$$

1. $\delta_a *_M \delta_b=\delta_{ab}$ for $a,b\in\mathbb{R}$.

   [cf. $\delta_a * \delta_b=\delta_{a+b}$.]

2. If $\mu_i(dx)=f_i(x)dx$ and $f_i(-x)=f_i(x)$, then $\mu_1 *_M \mu_2$ admits a symmetric density function $2f_1\star f_2(x)$, where

$$f_1\star f_2(x)=\int_0^\infty f_1(y)f_2\Big(\frac{x}{y}\Big)\frac{dy}{y}=\int_0^\infty f_1\Big(\frac{x}{y}\Big)f_2(y)\frac{dy}{y}\,, \quad x>0.$$

In fact, this is the standard convolution of the multiplicative group $\mathbb{R}_{>0}$.

Exercises

Exercise 13 Observe that $(K_2 \times_K K_2)^o \cong K_2$ and examine the identity:

$$\left(\frac{1}{2}\,\delta_{-1} + \frac{1}{2}\,\delta_1\right) *_M \left(\frac{1}{2}\,\delta_{-1} + \frac{1}{2}\,\delta_1\right) = \frac{1}{2}\,\delta_{-1} + \frac{1}{2}\,\delta_1\,.$$

Exercise 14 Using $K_3 \times_K K_2 \cong C_6$, derive the spectral distribution of C_6 at a fixed origin (which in fact coincides with the eigenvalue distribution):

$$\frac{1}{6}\,\delta_{-2} + \frac{1}{3}\,\delta_{-1} + \frac{1}{3}\,\delta_1 + \frac{1}{6}\,\delta_2.$$

Exercise 15 Using $K_4 \times_K K_2 \cong K_2 \times_C K_2 \times_C K_2 = H(3,2)$, derive the spectral distribution of $H(3,2)$ at a fixed origin (which in fact coincides with the eigenvalue distribution):

$$\frac{1}{8}\,\delta_{-3} + \frac{3}{8}\,\delta_{-1} + \frac{3}{8}\,\delta_1 + \frac{1}{8}\,\delta_3.$$

Also examine the identity:

$$\left(\frac{3}{4}\,\delta_{-1} + \frac{1}{4}\,\delta_3\right) *_M \left(\frac{1}{2}\,\delta_{-1} + \frac{1}{2}\,\delta_1\right) = \left(\frac{1}{2}\,\delta_{-1} + \frac{1}{2}\,\delta_1\right)^{*3}.$$

7.6. Restricted Lattices

- $\mathbb{Z} \times_C \mathbb{Z}$ (2d interger lattice): a graph on $\mathbb{Z}^2$ with adjacency relation:

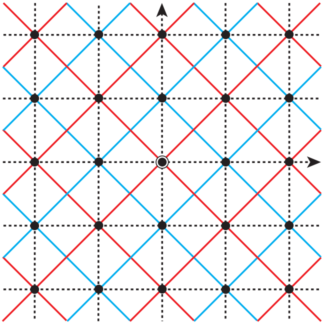
$$(x, y) \sim (x', y') \iff \begin{cases} x' = x \pm 1, \\ y' = y, \end{cases} \text{or} \begin{cases} x' = x, \\ y' = y \pm 1. \end{cases}$$

- $\mathbb{Z} \times_K \mathbb{Z}$: a graph on $\mathbb{Z}^2 = \{(u, v) ; u, v \in \mathbb{Z}\}$ with adjacency relation:

$$(u, v) \sim_K (u', v') \iff u' = u \pm 1 \text{ and } v' = v \pm 1.$$

1. $\mathbb{Z} \times_K \mathbb{Z}$ has two connected components, each of which is isomorphic to $\mathbb{Z} \times_C \mathbb{Z}$.
2. Let $(\mathbb{Z} \times_K \mathbb{Z})^O$ denote the connected component of $\mathbb{Z} \times_K \mathbb{Z}$ containing $O = (0, 0)$. Then

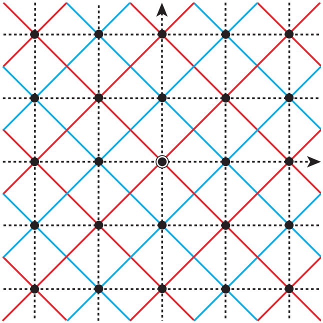
$$(\mathbb{Z} \times_K \mathbb{Z})^O \cong \mathbb{Z} \times_C \mathbb{Z}.$$



7.6. Restricted Lattices

1. $\mathbb{Z} \times_{\boldsymbol{K}} \mathbb{Z}$ has two connected components, each of which is isomorphic to $\mathbb{Z} \times_{\boldsymbol{C}} \mathbb{Z}$.
2. Let $(\mathbb{Z} \times_{\boldsymbol{K}} \mathbb{Z})^{\boldsymbol{O}}$ denote the connected component of $\mathbb{Z} \times_{\boldsymbol{K}} \mathbb{Z}$ containing $\boldsymbol{O} = (\mathbf{0}, \mathbf{0})$. Then

$$(\mathbb{Z} \times_{\boldsymbol{K}} \mathbb{Z})^{\boldsymbol{O}} \cong \mathbb{Z} \times_{\boldsymbol{C}} \mathbb{Z}.$$



Since the spectral distribution of $\mathbb{Z}$ at $\mathbf{0}$ is the arcsine law $\boldsymbol{\alpha}$, we have

Theorem

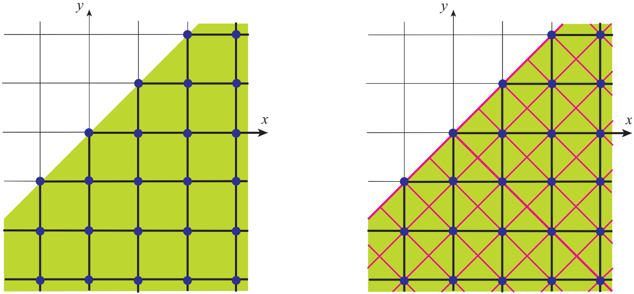
The spectral distribution of 2d lattice $\mathbb{Z}^2$ at $(\mathbf{0}, \mathbf{0})$ is given by

$$\boldsymbol{\alpha} *_M \boldsymbol{\alpha} = \boldsymbol{\alpha} * \boldsymbol{\alpha}$$

7.6. Restricted Lattices

- Let $L\{x \geq y\}$ denote the induced subgraph of $\mathbb{Z} \times_C \mathbb{Z}$ spanned by the vertices

$$\{(x, y) \in \mathbb{Z}^2 \,;\, x \geq y\}.$$



Theorem

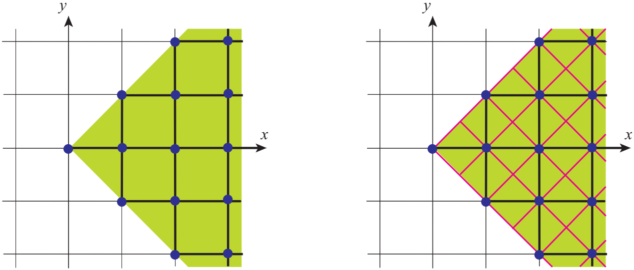
We have $L\{x \geq y\} \cong (\mathbb{Z}_+ \times_K \mathbb{Z})^O$ *and its spectral distribution at* $(0, 0)$ *is given by*

$$w *_M \alpha$$

7.6. Restricted Lattices

- Let $L\{x \geq y \geq -x\}$ denote the induced subgraph of $\mathbb{Z} \times_C \mathbb{Z}$ spanned by the vertices

$$\{(x, y) \in \mathbb{Z}^2 \,;\, x \geq y \geq -x\}.$$



Theorem

We have $L\{x \geq y \geq -x\} \cong (\mathbb{Z}_+ \times_K \mathbb{Z})^O$ *and its spectral distribution at* $(0, 0)$ *is given by*

$$w *_M w$$

7.6. Restricted Lattices

Domain D	$W_{2m}(L[D],O)$	spectral distribution
$\mathbb{Z}$	$\binom{2m}{m}$	α
$\mathbb{Z}_+$	$C_m = \frac{1}{m+1}\binom{2m}{m}$	w
$\mathbb{Z}^2$	$\binom{2m}{m}^2$	$\alpha * \alpha = \alpha *_M \alpha$
$\{x \ge y\}$	$C_m \binom{2m}{m}$	$w *_M \alpha$
$\{x \ge y \ge -x\}$	C_m^2	$w *_M w$
$\{x \ge 0,\ y \ge 0\}$	(A)	$w * w$
$\{x \ge y \ge x-(n-1)\}$	(B)	$\pi_n *_M \alpha$
$\left\{ \begin{array}{l} 0 \le x+y \le k-1, \\ 0 \le x-y \le l-1 \end{array} \right\}$	(C)	$\pi_k *_M \pi_l$

$$(\mathrm{A}) = \sum_{k=0}^{m} \binom{2m}{2k} C_k C_{m-k},$$

$$(\mathrm{B}) = W_{2m}(P_n,0)\binom{2m}{m}, \qquad (\mathrm{C}) = W_{2m}(P_k,0)W_{2m}(P_l,0).$$

7.6. Restricted Lattices — Density Functions

Elliptic integrals For $k^2 < 1$, the elliptic integrals are defined by

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2\theta}\, d\theta = \int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\, dx.$$

1. The density function of $w *_M \alpha$ is given by

$$\frac{1}{\pi^2}\{K(\xi(x)) - E(\xi(x))\}, \quad \xi(x) = \sqrt{1 - \frac{x^2}{16}}, \quad -4 \le x \le 4.$$

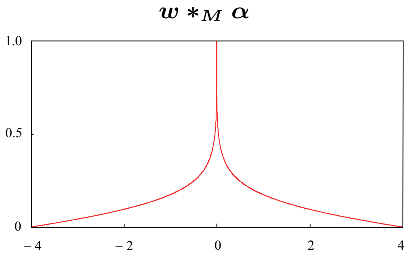
2. The density function of $\alpha *_M \alpha = \alpha * \alpha$ is given by

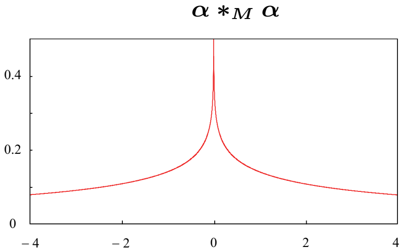
$$\frac{1}{2\pi^2} K(\xi(x)), \quad -4 \le x \le 4.$$

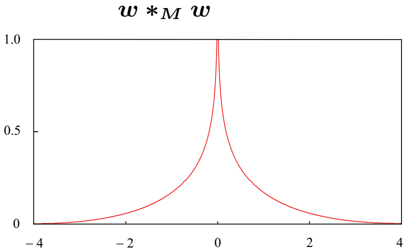
3. The density function of $w *_M w$ is given by

$$\frac{2}{\pi^2}\left\{\left(1 + \frac{x^2}{16}\right) K(\xi(x)) - 2E(\xi(x))\right\}, \quad -4 \le x \le 4.$$

7.6. Restricted Lattices — Density Functions

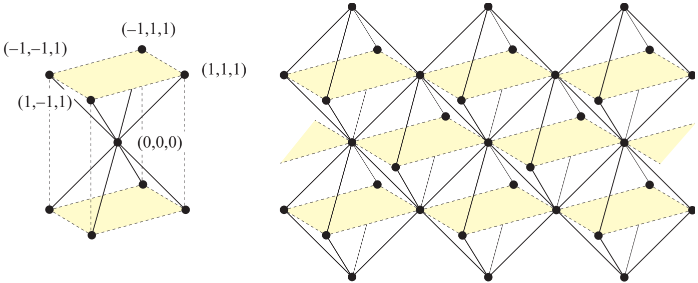






An Example in 3-Dimension: $\mathbb{Z} \times_K \mathbb{Z} \times_K \mathbb{Z}$

$\mathbb{Z} \times_K \mathbb{Z} \times_K \mathbb{Z}$ has 4 connected components, which are mutually isomorphic. The connected component containing $O(0,0,0)$ looks like an octahedra honeycomb, built up by gluing octahedra or body-centered cubes.



We have

$$W_{2m}(\mathbb{Z} \times_K \mathbb{Z} \times_K \mathbb{Z}, (0,0,0)) = \binom{2m}{m}^3, \qquad m = 0, 1, 2, \dots,$$

and the spectral distribution is given by $\mu = \alpha *_M \alpha *_M \alpha$.

8. Bivariate Extension: An Example

J. V. S. Morales, N. Obata and H. Tanaka: *Asymptotic joint spectra of Cartesian powers of strongly regular graphs and bivariate Charlier-Hermite polynomials*, arXiv:1809.03761, to appear in Colloq. Math.

Motivation

(I) Quantum CLT: $A_\nu \xrightarrow{\mathrm{m}} B$

$\Rightarrow$ The limit spectral distribution is a probability distribution on $\mathbb{R}^1$

$\Rightarrow$ Multi-variate extension: $(A_\nu^{(1)}, \dots, A_\nu^{(p)}) \xrightarrow{\mathrm{m}} (Z_1, \dots, Z_p)$?

See e.g., T. Espinasse and P. Rochet (2019), arXiv:1904.10720

— An extension of Boolean CLT

(II) Method of quantum decomposition $A = A^+ + A^\circ + A^-$

$\Rightarrow$ Orthogonal polynomials in one variable:

$$xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x)$$

$\Rightarrow$ Multi-variate extension?

potentially very interesting in connection to *multi-variate orthogonal polynomials*

8.1. Hamming Graphs $H(n, v)$

- $n \geq 1$, $v \geq 1$: natural numbers
- Alphabets $K = \{1, 2, \ldots, v\}$
- Words of length n:

$$V = \{x = (\xi_1, \xi_2, \ldots, \xi_n) \;\; \xi_i \in K\} = K^n$$

- Hamming distance between two words x and y:

$$\partial(x, y) = |\{1 \leq i \leq n \,|\, \xi_i \neq \eta_i\}|.$$

- A graph is defined with vertex set V and adjacency relation

$$x \sim y \quad \Leftrightarrow \quad \partial(x, y) = 1$$

$\Rightarrow$ This is the Hamming graph $H(n, v)$.

8.1. Hamming Graphs

- Product structure

$$H(n,v) = K_v \times \cdots \times K_v \quad (n\text{-fold Cartesian power})$$

where K_v is the complete graph on v vertices.

- The adjacency matrix of $H(n,v)$ is given by

$$A_{n,v} = \sum_{i=1}^{n} \overbrace{I \otimes \cdots \otimes I}^{i-1} \otimes A \otimes \overbrace{I \otimes \cdots \otimes I}^{n-i},$$

where $A = A[K_v]$ is the adjacency matrix of K_v.

- The eigenvalue distribution $\mu_{n,v}$ is specified by

$$\frac{1}{v^n}\,\mathrm{Tr}(A_{n,v}^m) = \int_{-\infty}^{+\infty} x^m \mu_{n,v}(dx), \qquad m = 0,1,2,\dots .$$

Question [CLT for Hamming graphs]

$$\mu_{n,v} \to ?? \quad \text{as } n \to \infty \text{ and } v \to \infty$$

8.1. Hamming Graphs

Review of Hora's argument (1998). This is *before* quantum decomposition

1. The adjacency matrix of K_v is given by $A = J - I$ (J: all-one matrix)
2. Then $C(K_v) = \mathbb{C}^v = U_{v-1} \oplus U_{-1}$ and

$$A \restriction U_{v-1} = v-1, \quad \dim U_{v-1} = 1; \quad A \restriction U_{-1} = -1, \quad \dim U_{-1} = v-1.$$

3. $A_{n,v} = \sum I \otimes \cdots \otimes A \otimes \cdots \otimes I$ acts on

$$(\mathbb{C}^v)^{\otimes n} = (U_{v-1} \oplus U_{-1}) \otimes \cdots \otimes (U_{v-1} \oplus U_{-1})$$

4. The eigenvalues of $A_{n,v}$ are

$$(v-1)(n-j) + (-1)j = -n + (n-j)v$$

with multiplicity

$$\binom{n}{j} 1^{n-j} (v-1)^{n-j},$$

where $0 \le j \le n$.

8.1. Hamming Graphs

5. Hence

$$\mu_{n,v} = \frac{1}{v^n} \sum_{j=0}^{n} \binom{n}{j} 1^{n-j} (v-1)^{n-j} \delta_{-n+(n-j)v}$$
$$= \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{v}\right)^k \left(1 - \frac{1}{v}\right)^{n-k} \delta_{-n+vk}$$

Namely, $\mu_{n,v}$ is essentially the binomial distribution.

6. By classical theory we know

$$B(n,p) \approx \mathrm{N}(np, np(1-p)), \qquad B(n,p) \approx \mathrm{Po}(np)$$

7. Consider the normalization $\tilde{\mu}_{n,v} \Leftarrow \mathrm{mean}(\mu_{n,v}) = 0$, $\mathrm{var}(\mu_{n,v}) = n(v-1)$

8. Under the proper scaling $n \to \infty$, $v \to \infty$ and $\dfrac{v}{n} \to \tau \geq 0$,

$$\tilde{\mu}_{n,v} \to \begin{cases} \mathrm{N}(0,1), & \tau = 0, \\ \text{affine transform of } \mathrm{Po}(\tau^{-1}), & \tau > 0 \end{cases}$$

▶ Actual proof is based on characteristic functions (Laplace transform).

8.2. Strongly Regular Graphs

Complementary graphs

In general, $\bar{G}$ denotes the *complementary graph* of $G = (V, E)$,i.e., a graph on V with edge set $\bar{E} = \{\{x, y\}\,;\, x, y \in V, x \neq y, \{x, y\} \notin E\}$.

Or equivalently, the adjacency matrix of $\bar{G}$ is defined by

$$\bar{A} = J - I - A. \qquad (J\text{: all-one matrix})$$

Lemma

For a finite graph G with adjacency matrix A we have

$$G \text{ is a regular graph} \quad \Leftrightarrow \quad A\bar{A} = \bar{A}A \quad \Leftrightarrow \quad AJ = JA.$$

Definition

For a finite regular graph G the commutative $*$-algebra generated by $I, A, \bar{A}$, denoted by $\mathcal{A}(G, \bar{G})$, is called the "extended adjacency algebra."

8.2. Strongly Regular Graphs

Definition

$G = (V, E)$ is a *strongly regular graph* with papameter (v, k, λ, μ) if

1. $|V| = v$;
2. G is k-regular;
3. every two adjacent $x, y \in V$ has λ common adjacent vertices;
4. every two non-adjacent $x, y \in V$ has μ common adjacent vertices;
5. (avoiding trivial cases) G is neither complete nor empty, that is, $0 < k < v - 1$.

Note: A strongly regular graph is a distance-regular graph with diameter 2.

8.2. Strongly Regular Graphs

Lemma

If G is a strongly regular graph with papameter (v, k, λ, μ), so is $\bar{G}$ with parameter $(v, \bar{k} = v - k - 1, \bar{\lambda} = v - 2k + \mu - 2, \bar{\mu} = v - 2k + \lambda)$.

Lemma

Let G be a finite regular graph with degree $0 < \kappa < v - 1$. Then the following conditions are equivalent:

1. *G is a strongly regular graph;*
2. *$\mathcal{A}(G, \bar{G})$ is the three-dimensional linear space spanned by $I, A, \bar{A}$.*

For the proof we need only to note that

$$A^2 = kI + \lambda A + \mu \bar{A} = kI + \lambda A + \mu(J - I - A).$$

8.2. Strongly Regular Graphs

Lemma

Let G be a strongly regular graph with (v, k, λ, μ). Then the spectrum of G are given by

$$s < r \le k \qquad \text{with multiplicities } g, f, 1,$$

where

$$s, r = \frac{(\lambda - \mu) \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2},$$

and

$$f = \frac{(v-1)s + k}{s - r}, \qquad g = \frac{(v-1)r + k}{r - s}.$$

The spectrum of $\bar{G}$ are given by

$$\bar{s} = -r - 1 < \bar{r} = -s - 1 \le \bar{k} \qquad \text{with multiplicities } f, g, 1.$$

There are many relations among these constants. For example,

$$1 + k + \bar{k} = 1 + f + g = v, \qquad k^2 + fr^2 + gs^2 = kv$$

8.3. Cartesian Product of Strongly Regular Graphs

1. Let G be a strongly regular graph and $\bar{G}$ the complement.
2. Consider the pair $(G^n, \bar{G}^n)$, where
$$G^n = G \times \cdots \times G \quad (n\text{-fold Cartesian power}), \qquad \bar{G}^n = \bar{G} \times \cdots \times \bar{G} \quad (\text{similar}).$$
3. Adjacency matrices:
$$A_{n,G} = \sum_{k=1}^{n} \overbrace{I \otimes \cdots \otimes I}^{k-1} \otimes A \otimes \overbrace{I \otimes \cdots \otimes I}^{n-k}, \quad \bar{A}_{n,G} = (\text{similar}).$$
4. Let $\nu_{n,G}(dxdy)$ be the joint spectral distribution of $(A_{n,G}, \bar{A}_{n,G})$ specified by
$$\frac{1}{v^n}\mathrm{Tr}(A_{n,G}^s \bar{A}_{n,G}^t) = \int_{\mathbb{R}^2} x^s y^t \, \nu_{n,G}(dxdy), \qquad s,t = 0,1,2,\dots.$$

Question (Asymptotic spectral distribution)

$$\nu_{n,G} \to ?? \quad \text{as } n \to \infty \text{ and } |G| \to \infty$$

8.3. Cartesian Product of Strongly Regular Graphs

How we generalized the case of Hamming graphs?

- Outline of our procedure:

1. Consider a strongly regular graph G and its complement $\bar{G}$.
2. Consider a pair of Cartesian powers $(G^n, \bar{G}^n)$
3. and their adjacency matrices $(A_{n,G}, \bar{A}_{n,G})$.
4. The *joint spectral distribution* of $(A_{n,G}, \bar{A}_{n,G})$ is a probability distribution on $\mathbb{R}^2$ specified by

$$\frac{1}{v^n}\mathrm{Tr}(A_{n,G}^s \bar{A}_{n,G}^t) = \int_{\mathbb{R}^2} x^s y^t \, \nu_{n,G}(dxdy), \qquad s,t = 0,1,2,\dots.$$

- Case of Hamming graphs:

Take $G = K_v$. Then $\bar{G}$ is an empty graph,

$$G^n = K_v \times \cdots \times K_v = H(n,v) \qquad \text{(Hamming graph)},$$
$$(A_{n,G}, \bar{A}_{n,G}) = (A_{n,v}, 0).$$

Thus, the spectral distribution is reduced to one-dimension.

8.4. Joint spectral distribution of $(G^n, \bar{G}^n)$

Theorem

The joint spectral distribution of $(G^n, \bar{G}^n)$ *is given by*

$$\nu_{n,G} = \sum_{0\le j+h\le n} \pi(j,h)\delta(\theta_{j,h}, \bar{\theta}_{j,h}), \quad \pi(j,h) = \binom{n}{j,h}\Big(\frac{f}{v}\Big)^j\Big(\frac{g}{v}\Big)^h\Big(\frac{1}{v}\Big)^{n-j-h},$$

$$\theta_{j,h} = (n-j-h)k + jr + hs, \qquad \bar{\theta}_{j,h} = (n-j-h)\bar{k} + j\bar{s} + h\bar{r},$$

$$f = \frac{(v-1)s+k}{s-r}, \qquad g = \frac{(v-1)r+k}{r-s}.$$

Proof: According to $\mathrm{ev}(A_{n,G}) = \{s, r, k\}$ and $\mathrm{ev}(\bar{A}_{n,G}) = \{\bar{r}, \bar{s}, \bar{k}\}$ we have

$$C(G) = \mathbb{C}^v = U_r \oplus U_s \oplus U_k, \quad \dim U_r = f, \quad \dim U_s = g, \quad \dim U_k = 1.$$

Then look at

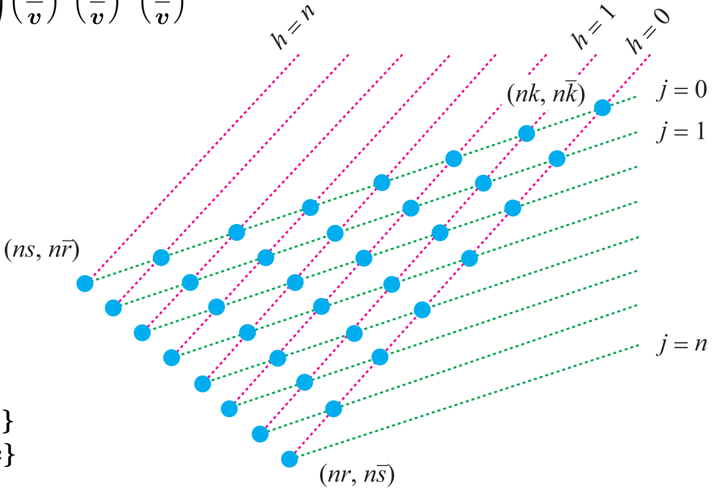
$$A_{n,G} = \sum I \otimes \cdots \otimes A \otimes \cdots \otimes I,$$

$$C(G^n) = (U_r \oplus U_s \oplus U_k) \otimes \cdots \otimes (U_r \oplus U_s \oplus U_k).$$

8.4. Joint spectral distribution of $(G^n, \bar{G}^n)$

$$\nu_{n,G} = \sum_{0 \le j+h \le n} \pi(j,h)\delta(\theta_{j,h}, \bar{\theta}_{j,h})$$

$$\pi(j,h) = \binom{n}{j,h}\left(\frac{f}{v}\right)^j \left(\frac{g}{v}\right)^h \left(\frac{1}{v}\right)^{n-j-h}$$



$$\mathrm{ev}(G) = \{s, r, k\}$$
$$\mathrm{ev}(\bar{G}) = \{\bar{r}, \bar{s}, \bar{k}\}$$

8.5. Asymptotic Joint Spectral Distributions

$n \to \infty$, $v \to \infty$ and some balance conditions

- Hamming graphs: $H(n,v) = K_v \times \cdots \times K_v$ (n-fold Cartesian power)

$$\frac{v}{n} \to \tau \qquad \text{and automatically} \quad \frac{-1}{n} \to 0, \quad \frac{v-1}{n} \to \tau.$$

these are conditions for eigenvalues!

- Growing pair of strongly regular graphs: $(G^n, \bar{G}^n)$

Recall: $\mathrm{ev}(G) = \{s, r, k\}$, $\mathrm{ev}(\bar{G}) = \{\bar{r}, \bar{s}, \bar{k}\}$

$$1 + k + \bar{k} = v, \qquad \bar{s} = -r - 1, \qquad \bar{r} = -s - 1.$$

The proper scaling is given by

$$\frac{k}{n} \to \kappa, \quad \frac{\bar{k}}{n} \to \bar{\kappa}, \quad \frac{r}{n} \to \rho, \quad \frac{s}{n} \to \sigma, \quad \frac{v}{n} \to \kappa + \bar{\kappa} \equiv \omega.$$

- Note: $\rho = 0$ or $\sigma = 0$ follows.

8.5. Asymptotic Joint Spectral Distributions

Theorem (Morales-Obata-Tanaka (2019+))

Let ν be the limit of the joint spectral distribution of $\left(\frac{A_{n,G}}{\sqrt{nk}}, \frac{\bar{A}_{n,G}}{\sqrt{n\bar{k}}}\right)$. Then,

1. *If $\kappa > 0$, $\bar{\kappa} = -\sigma > 0$, $\rho = 0$, then ν is an affine transformation of the bivariate Poisson distribution:*

$$\nu\left(\left(\frac{\kappa j - \bar{\kappa} h}{\sqrt{\kappa}}, \frac{\bar{\kappa} j + \bar{\kappa} h - 1}{\sqrt{\bar{\kappa}}}\right)\right) = e^{-1/\bar{\kappa}}\left(\frac{1}{\omega}\right)^j \left(\frac{\kappa}{\omega\bar{\kappa}}\right)^h \frac{1}{j!h!}$$

2. *If $\kappa = \rho > 0$, $\bar{\kappa} > 0$, $\sigma = 0$, then similar as above.*
3. *If $\kappa > 0$ or $\bar{\kappa} > 0$, and if $\rho = \sigma = 0$, then ν is an affine transformation of the product of Gaussian and Poisson distributions:*

$$\int_{\mathbb{R}^2} f(x)\nu(dx) = \sqrt{\frac{\omega}{2\pi}}\, e^{-1/\omega} \sum_{h=0}^{\infty} \left(\frac{1}{\omega}\right)^h \frac{1}{h!} \int_{-\infty}^{+\infty} f(x_{h,t}) e^{-\omega t^2/2} dt$$

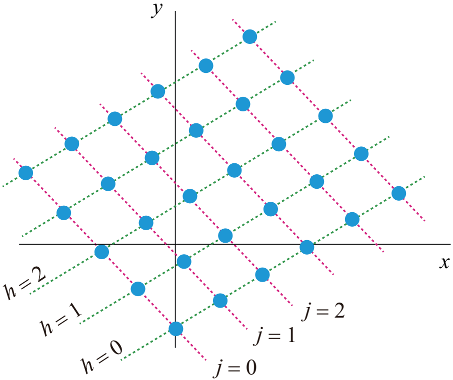
$$x_{h,t} = \left(\sqrt{\kappa}\, h + \sqrt{\bar{\kappa}}\, t - \frac{\sqrt{\kappa}}{\omega}, \sqrt{\bar{\kappa}}\, h - \sqrt{\kappa}\, t - \frac{\sqrt{\bar{\kappa}}}{\omega}\right)$$

4. *If $\kappa = \bar{\kappa} = \rho = \sigma = 0$, ν is the bivariate Gaussian distribution.*

8.5. Asymptotic Joint Spectral Distributions

Bivariate Poisson distribution

$$\nu\left(\left(\frac{\kappa j - \bar{\kappa} h}{\sqrt{\kappa}}, \frac{\bar{\kappa} j + \bar{\kappa} h - 1}{\sqrt{\bar{\kappa}}}\right)\right) = e^{-1/\bar{\kappa}} \left(\frac{1}{\omega}\right)^{j} \left(\frac{\kappa}{\omega \bar{\kappa}}\right)^{h} \frac{1}{j!h!}$$

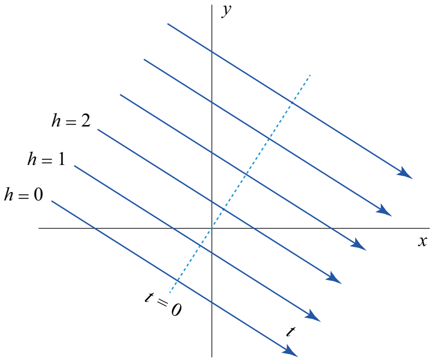


8.5. Asymptotic Joint Spectral Distributions

Gauss × Poisson distribution

$$\int_{\mathbb{R}^2} f(x)\nu(dx) = \sqrt{\frac{\omega}{2\pi}}\, e^{-1/\omega} \sum_{h=0}^{\infty} \left(\frac{1}{\omega}\right)^h \frac{1}{h!} \int_{-\infty}^{+\infty} f(x_{h,t}) e^{-\omega t^2/2} dt$$

$$x_{h,t} = \left(\sqrt{\kappa}\, h + \sqrt{\bar{\kappa}}\, t - \frac{\sqrt{\kappa}}{\omega}, \sqrt{\bar{\kappa}}\, h - \sqrt{\kappa}\, t - \frac{\sqrt{\bar{\kappa}}}{\omega}\right)$$



8.6. Bivariate Orthogonal Polynomials

Extended Adjacency Algebra $\mathcal{A}(G^n, \bar{G}^n)$

For $0 \le \alpha + \beta \le n$ we put

$$A_{\alpha,\beta} = \sum I \otimes \cdots \otimes A \otimes \cdots \otimes \bar{A} \otimes \cdots \otimes I,$$

A appears α times and $\bar{A}$ appears β times

In particular, the adjacency matrices of $(G^n, \bar{G}^n)$ are

$$A[G^n] = A_{n,G} = A_{1,0}, \qquad A[\bar{G}^n] = \bar{A}_{n,G} = A_{0,1}.$$

$\mathcal{A}(G^n, \bar{G}^n)$: unital $*$-algebra generated by $A_{n,G}$ and $\bar{A}_{n,G}$.

Lemma

$\mathcal{A}(G^n, \bar{G}^n)$ *is a linear span of* $\{A_{\alpha,\beta}\,;\, 0 \le \alpha + \beta \le n\}$.

Lemma (Orthogonal relation)

$$\frac{1}{v^n}\mathrm{Tr}(A_{\alpha,\beta}A_{\alpha',\beta'}) = k_{\alpha,\beta}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'}, \quad k_{\alpha,\beta} = \binom{n}{\alpha, \beta} k^{\alpha}\bar{k}^{\beta}.$$

8.6. Bivariate Orthogonal Polynomials

Lemma (Mizukawa–Tanaka (PAMS 2004))

The eigenvalues of $A_{\alpha,\beta}$ are given in the form:

$$k_{\alpha,\beta} P_{\alpha,\beta}(j,h) \quad \text{with multiplicity} \quad \binom{n}{j,h} f^j g^h,$$

Bivariate Krawtchouk Polynomials

$$P_{\alpha,\beta}(j,h) = \sum_{0\le\nu_1+\cdots+\nu_4\le n} \frac{(-\alpha)_{\nu_1+\nu_3}(-\beta)_{\nu_2+\nu_4}(-j)_{\nu_1+\nu_2}(-h)_{\nu_3+\nu_4}}{(-n)_{\nu_1+\nu_2+\nu_3+\nu_4}} \frac{t_1^{\nu_1} t_2^{\nu_2} t_3^{\nu_3} t_4^{\nu_4}}{\nu_1!\nu_2!\nu_3!\nu_4!},$$

where

$$t_1 = 1-\frac{r}{k}, \quad t_2 = 1-\frac{\bar{s}}{\bar{k}}, \quad t_2 = 1-\frac{s}{k}, \quad t_4 = 1-\frac{\bar{r}}{\bar{k}}.$$

- This is a particular case of *Aomoto-Gelfand hypergeometric function* of $(3,6)$-type.
- Pochhammer symbol: $(a)_n = a(a+1)(a+2)\cdots(a+n-1)$

8.6. Bivariate Orthogonal Polynomials

Then the orthogonal relation becomes

$$\sum_{0\le j+h\le n} \sqrt{k_{\alpha,\beta}}\, P_{\alpha,\beta}(j,h)\sqrt{k_{\alpha',\beta'}} P_{\alpha',\beta'}(j,h)\pi(j,h) = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}\,.$$

Using integral form and applying variable change:

$$\nu_{n,G} = \sum_{0\le j+h\le n} \binom{n}{j,h} \pi(j,h)\delta(\theta_{j,h},\bar{\theta}_{j,h}),$$

$$\theta_{j,h} = (n-j-h)k + jr + hs, \qquad \bar{\theta}_{j,h} = (n-j-h)\bar{k} + j\bar{s} + h\bar{r},$$

$$x = \frac{\theta_{j,h}}{\sqrt{nk}}\,, \qquad y = \frac{\bar{\theta}_{j,h}}{\sqrt{n\bar{k}}}\,,$$

we obtain polynomials $\{\tilde{P}_{\alpha,\beta}(x,y)\}$ such that

$$\int_{\mathbb{R}^2} \tilde{P}_{\alpha,\beta}(x,y)\tilde{P}_{\alpha',\beta'}(x,y)\tilde{\nu}_{G,n}(dxdy) = \delta_{\alpha,\alpha'}\delta_{\beta,\beta'}$$

8.7. Bivariate Orthogonal Polynomials in the Limit

- We consider the Gauss × Poisson case

Let

$$R_{\alpha,\beta}(x,y) = \lim \tilde{P}_{\alpha,\beta}(x,y)$$

under the scaling

$$\frac{k}{n} \to \kappa > 0 \quad \text{or} \quad \frac{\bar{k}}{n} \to \bar{\kappa} > 0, \quad \frac{r}{n} \to \rho = 0, \quad \frac{s}{n} \to \sigma = 0,$$

Then we have

$$\int_{\mathbb{R}^2} R_{\alpha,\beta}(x,y) R_{\alpha',\beta'}(x,y) \nu(dxdy) = \delta_{\alpha,\alpha'} \delta_{\beta,\beta'}$$

Theorem (Morales-Obata-Tanaka (2019+))

$\{R_{\alpha,\beta}(x,y)\}$ *are the orthogonal polynomials with respect to the Gauss × Poisson distribution* ν.

8.7. Bivariate Orthogonal Polynomials in the Limit

Explicit form

1. We start with the generating function:

$$\sum_{0\le\alpha+\beta\le n} k_{\alpha,\beta}P_{\alpha,\beta}(j,h)\xi_1^{\alpha}\xi_2^{\beta} = (1+k\xi_1+\bar{k}\xi_2)^{n-j-h}(1+r\xi_1+\bar{s}\xi_2)^{j}(1+s\xi_1+\bar{r}\xi_2)^{h}$$

2. Changing variables and taking the limit, we have

$$\sum_{\alpha,\beta=0}^{\infty} \frac{R_{\alpha,\beta}(x,y)}{\sqrt{\alpha!\beta!}}\,\xi_1^{\alpha}\xi_2^{\beta} = (1+\sqrt{\kappa}\,\xi_1+\sqrt{\bar{\kappa}}\,\xi_2)^{(\sqrt{\kappa}\,x+\sqrt{\bar{\kappa}}\,y+1)/\omega} \times \exp\left\{-\frac{\sqrt{\kappa}\,\xi_1+\sqrt{\bar{\kappa}}\,\xi_2}{\omega}-\frac{(\sqrt{\bar{\kappa}}\,\xi_1-\sqrt{\kappa}\,\xi_2)^2}{2\omega} + \frac{(\sqrt{\bar{\kappa}}\,x-\sqrt{\kappa}\,y)(\sqrt{\bar{\kappa}}\,\xi_1-\sqrt{\kappa}\,\xi_2)}{\omega}\right\}$$

8.7. Bivariate Orthogonal Polynomials in the Limit

Five-term recurrence relation

1. We start with

$$
\begin{aligned}
AA_{\alpha,\beta} &= (\alpha+1)A_{\alpha+1,\beta} + (\alpha+1)(\bar{k}-\bar{\mu})A_{\alpha+1,\beta-1} \\
&\qquad + (\alpha\lambda + \beta(k-\mu))A_{\alpha,\beta} + (\beta+1)\mu A_{\alpha-1,\beta+1} \\
&\qquad\qquad + (n-\alpha-\beta+1)kA_{\alpha-1,\beta},
\end{aligned}
$$

$$
\begin{aligned}
\bar{A}A_{\alpha,\beta} &= (\beta+1)A_{\alpha,\beta+1} + (\alpha+1)\bar{\mu}A_{\alpha+1,\beta-1} \\
&\qquad + (\alpha(\bar{k}-\bar{\mu}) + \beta\bar{\lambda})A_{\alpha,\beta} + (\beta+1)(k-\lambda)A_{\alpha-1,\beta+1} \\
&\qquad\qquad + (n-\alpha-\beta+1)\bar{k}A_{\alpha,\beta-1}.
\end{aligned}
$$

2. Use the correspondence:

$$
\frac{A_{\alpha,\beta}}{\sqrt{k_{\alpha,\beta}}} \quad \leftrightarrow \sqrt{k_{\alpha,\beta}}\, P_{\alpha,\beta}(j,h)
$$

we obtain the five-term recurrence relation for $\{P_{\alpha,\beta}(j,h)\}$.

3. Changing variables and taking the limit, we have

8.7. Bivariate Orthogonal Polynomials in the Limit

Theorem (Five-term recurrence relation)

$$\begin{aligned} xR_{\alpha,\beta} &= \sqrt{\alpha+1}\,R_{\alpha+1,\beta} + \sqrt{(\alpha+1)\beta}\,\frac{\kappa\sqrt{\bar{\kappa}}}{\omega}\,R_{\alpha+1,\beta-1} \\ &\quad + (\alpha\kappa+\beta\bar{\kappa})\frac{\sqrt{\kappa}}{\omega}\,R_{\alpha,\beta} + \sqrt{\alpha(\beta+1)}\,\frac{\kappa\sqrt{\bar{\kappa}}}{\omega}\,R_{\alpha-1,\beta+1} + \sqrt{\alpha}\,R_{\alpha-1,\beta}, \end{aligned}$$

$$\begin{aligned} yR_{\alpha,\beta} &= \sqrt{\beta+1}\,R_{\alpha,\beta+1} + \sqrt{(\alpha+1)\beta}\,\frac{\kappa\sqrt{\bar{\kappa}}}{\omega}\,R_{\alpha+1,\beta-1} \\ &\quad + (\alpha\kappa+\beta\bar{\kappa})\frac{\sqrt{\bar{\kappa}}}{\omega}\,R_{\alpha,\beta} + \sqrt{\alpha(\beta+1)}\,\frac{\bar{\kappa}\sqrt{\kappa}}{\omega}\,R_{\alpha-1,\beta+1} + \sqrt{\beta}\,R_{\alpha,\beta-1}. \end{aligned}$$

- This would be a good example for a bivariate spectral analysis of growing graphs.
- The next step is to derive a bivariate extension of quantum decomposition.

Life is short, but there is always time enough for mathematics!

THANK YOU VERY MUCH!

谢谢你，再见