POPULATION DYNAMICS MODELING FOR THE EFFECT OF COLLECTIVE BEHAVIOR ON INFORMATION SPREAD

EMMANUEL JESUYON DANSU
B7ID1501

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Graduate School of Information Sciences,
Tohoku University, Sendai, Japan

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candidate
Emmanuel Jesuyon Dansu

candidate id
B7ID1501

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supervisor
Professor Hiromi Seno

affiliation
Department of Computer and Mathematical Sciences
Graduate School of Information Sciences
Tohoku University, Sendai, Japan

dissertation referees
Professor Shigeru Sakaguchi
Professor Kentaro Inui
Associate Professor Naoya Fujiwara

DECLARATION

I, Emmanuel Jesuyon Dansu, declare that this thesis titled, “Population dynamics modeling for the effect of collective behavior on information spread ” and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
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- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Emmanuel Jesuyon Dansu
Sendai, September 2020

Mathematics is Biology's next microscope, only better; Biology is Mathematics' next Physics, only better. In biology, ensemble properties emerge at each level of organization from the interactions of heterogeneous biological units at that level and at lower and higher levels of organization (larger and smaller physical scales, faster and slower temporal scales). New mathematics will be required to cope with these ensemble properties and with the heterogeneity of the biological units that compose ensembles at each level.
— Joel E. Cohen

The propagation of information through social networks bears many similarities to the evolution and transmission of infectious diseases. Analysis of transmission dynamics could therefore provide insight into how misinformation spreads and competes online.
—Adam Kucharski

SUMMARY

Based on the ideas of disease spread dynamics, the thesis examines the diffusion of information subject to certain psychological and sociological situations. We consider two major models, namely *a rejoinder model* in which there are two interacting pieces of information spreading with a time lag between them and *a threshold model* in which a person only begins to spread a piece of information after an acceptable number of people have been spreading it. The analysis of the rejoinder model shows that there is a critical time frame within which an individual, organization or government should release information to properly correct misleading information that has been spreading in a population for some time. In addition, the model shows that a critical portion of a population should be targeted with corrective information for it to be effective. From the threshold model, we discovered that the final proportion of knowers of an information is uniquely determined by the initial proportion of knowers in the population. There are also critical proportions and threshold values which determine how well an information spreads within a population. The models provide theoretical frameworks for the promotion of information literacy in order to combat misinformation and disinformation. Information warfare has become intense due to increasing social activities on the internet.
Keywords: Population dynamics; Rejoinder model; Granovetter's model; Threshold model; Information spread; Collective behavior; Ordinary differential equations

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1

INTRODUCTION

1.1 THE NATURE OF INFORMATION

The whole existence of nature is built on information and its transmission; in fact, things hardly happen in the universe exclusive of the concepts of information and communication. Parker [59] gives one of the most fundamental definitions of information as 'the pattern of organization of matter and energy'. In a sense, Bates [4] sees information as something which exists objectively in our cosmos but is however handled subjectively by individuals in the process of construction, storage and acting upon it. In tandem with Goonatilake [27], the work identifies forms of information as *natural*, *represented* (encoded and embodied), *experienced*, *enacted*, *expressed*, *embedded*, *recorded* and *trace*. These forms are believed to be germane to the information, curatorial and collection sciences. In contrast to Bates concepts of objectivity and subjectivity, Hjørland [37] thinks that information depends on situations and is progressive.

An information philosopher, Luciano Floridi gives the fundamental life cycle of information as episodes of *occurrence*, *transmission*, *processing and management*, and *usage* [22]. He would like us to see information as meaningful data as well as a true factual semantic data which increases knowledge. On the other hand, unintentional untruth leads to misinformation while intentional untruth leads to disinformation. He also shows that information can be mathematical, semantic, physical, biological, economic, etc. Floridi concludes by dealing with the ethics of information with his RPT Model which takes care of ethical issues by viewing information as a resource, a product and a target.

Tracing the history of information, Gleick [26] points out that the world is gradually getting to a point of information overload which might lead to some kind of exhaustion. Satija [65] highlights the ubiquity and necessity of information. Though it has no generally agreed definition, it is seen as ranking after matter and energy. Since information shapes the world, the article emphasizes the importance of information literacy in empowering people globally. Going by Smiraglia [69], information is a dynamic process which alters the behavior of those who interact with or perceive new knowledge in a given context. Due to human interference in the information process, it is quite troublesome that various forms of perversion arise, which may lead to misinformation, namely rumors, gossips, fake news, among others. We look at these information disorders in some depth.

1.1.1 *Rumor*

The following are some definitions and descriptions of rumor.

"*A rumor is a proposition for belief of topical reference, without secure standards of evidence being present. It suffers such serious distortion through the embedding process that it is never under any circumstances a valid guide for belief or conduct*."-Allport and Postman [3]

"*A rumor is unverified information, usually of local or current interest, intended primarily for belief In other words, rumors are propositions or allegations colored by various shades of doubt, because they are not accompanied by corroborative evidence. Thus, rumors scamper about organizations like some mischievous poltergeist, until skillful managers exorcise the allegations or the allegations vanish into thin air*."-DiFonzo, Bordia, and Rosnow [16]

A rumor can be viewed as the result of a succession of individuals transmitting a particular message. The level of accuracy of a rumor is subject to the precision of the initial message as well as that with which it is transmitted down the line. Being probably the first set of people to approach the appraisal of proof scientifically, historians believed the morality, sentiments and source of information of a witness determine his/her reliability [36]. Rumors spread more easily in the face of crises which threaten a 'herd', e.g., wars, earthquakes, and tsunamis. Some are outright lies fabricated to push some hidden agenda. It is important to consider why a rumor tends to have a generic character and where the impulse to spread it comes from.

From the widely acclaimed work of Allport and Postman [3], the importance as well as the ambiguity of a topic determines how far a rumor about it spreads. They suggested that the credibility of sources of information determines whether a rumor subsides or not. The goal-gradient concept, which is a condition under which rumors in certain special circumstances bring about hurried implementation of desired objectives, was pointed out. It was highlighted that rumors spread for the purpose of 'explaining' and 'relieving' people's emotional agitations against others. With reference to the rumors which followed December 1941's Battle of Pearl Harbor, the work showed that tracing of rumor sources, proactive campaigns, rumor clinics and wardens were means by which the United States of America tried to contain the mischievous rumors. Based on their experiment, they identified three inter-connected inclinations in rumors, namely leveling (the easy to grasp version of a rumor), sharpening (the biased summary of an original information) and assimilation (the influence of listener's whims and caprices on rumors). Overall, it was established that the more the people involved with a rumor, the more it is subjected to change until it becomes concise enough to be learned by rote.

DiFonzo, Bordia, and Rosnow [16] studied rumor dynamics in the context of organizations viewing rumors as news without proof. They highlighted the devastating effects of rumors in the workplace and also how experts do not seem to agree on how to handle them. It was shown that rumors undermine firms' tangible assets like sales and mostly intangible ones like credibility. They stated the features of rumors as *lack of proof*, *group interests* and *intention to instill belief*. For organizations to swim against the tide of rumors in the cut throat business environment, functional guidelines based on rumor theory were summed up into *prevention* and *neutralization* of rumors. These interventions cut across the entire rumor process. Alluding to the research by Bordia and Rosnow [8], Bordia [7] established the fact that computer-mediated communication networks (CMC) like the internet have some advantages over face-to-face (FTF) method of sequential transference of information previously used in studying rumors. Such benefits include

- possibility of observing realistic transmission of rumors;
- easy retrieval of data due to automatic retention of discussions;
- availability of nonreactive data which enhances inconspicuous observation.

It was observed that CMC facilitate unrestrained behavior as well continuous, coordinated and relatively more elaborate articulations. It was however instructive to pay serious attention to the issues of privacy and informed consent when handling data from private CMC.

Investigating the impact of rumors on students subjected to a university shutdown, Jones et al. [38] discovered that inconsistent information was more available to those who depended on social media in order to keep abreast of the situation. By extension, the students who believed such unreliable news were more agitated than others. Furthermore, they saw that the longer the gap between updates from University officials, the more rumors and panic spread. In order to have such situations under control, they recommend that administrators should surveil social media networks, like Twitter, in times of crises and provide accurate information to counter the rumors they come across there.

1.1.2 *Gossip*

Grosser et al. [34] admits that there is just a thin line between rumor and gossip while emphasizing that gossips do not necessarily need to have negative effects in the work place. In that light, managers are advised to use gossips to the advantage of their organizations. The paper defines gossip as an "evaluative talk (i.e., concerned with making judgments) between two or more persons about a third party that is absent from the conversation". It was noted that a gossip can serve one or more of the following functions: *getting information*, *gaining influence*, *releasing pent-up emotions*, *providing intellectual stimulation*, *fostering interpersonal intimacy*, and *maintaining and enforcing group values and norms*. To manage gossips in an organization, the following were suggested: communicating information formally; promoting a civil culture; nurturing work place justice; enhancing means of dealing with stress and boredom; and handling excessive gossipers with discretion. In the final analysis, gossips can help leaders to feel the pulse of their firms so that they can act proactively.

In trying to understand the co-evolution of gossip and friendship in organizations, Ellwardt, Steglich, and Wittek [19] found that gossiping increases the likelihood of friendship (the evolutionary view) more than friendship facilitates gossiping (the social capital view). Nevertheless, there is a threshold beyond which gossips limit friendship formation as excessive gossipers are seen as undependable. One very crucial discovery of the study is that discussing other colleagues in their absence can help galvanize informal interactions between co-workers. This in turn helps the organization to grow as there will be greater mutual support and cooperation on a large scale.

Questioning the perception that gossips always have damaging effects and that people tend to be more drawn to malicious gossips about others, Tassiello, Lombardi, and Costabile [71] made an attempt to see if given a valenced gossip (positive or negative; malicious or non-malicious) would more likely spread in a workplace. They established that a gossip is a transmission of unverified information about absent third parties. All the parties concerned in a gossip are known as opposed to what happens in the case of a rumor where the target may not be known. Being a form of informal communication, the research tries to understand whether a positive or negative gossip is likely to be shared within an organization when all the actors concerned have well defined connections. In order to understand how this plays out, the researchers used the Kurland-Pelled model proposed in 2000. The model differentiates types of gossip based on *sign* (positive or negative), *credibility* (truth or falsehood) and *relatedness* (relationship between the gossip receiver and the gossip target).

Using three experimental studies, they discovered that benevolent gossips have better chances of being transmitted if the person whom it is about is in the same social camp with the receiver and when the receiver is likely to fact check the information. It was seen that people are wont to create good impressions of themselves by sharing positive gossips. In fact, the experiments show that gossip can be a tool for building cohesive relationships at work. They concluded by suggesting that some work is required in finding out if some particular group of people derive pleasure from gossips compared to other groups.

1.1.3 *Fake news*

In this information age, fake news is a source of great concern in different fields of human endeavor ranging from information science to politics, health care and economics. Talking about its influence on democratic processes, the following are cases in point: the 'Brexit' referendum of June 23, 2016; the surprise emergence of Donald Trump as POTUS following the November 8, 2016 elections; and the canceled August 8, 2017 Kenyan presidential elections.

In order to get a proper perspective, Wardle and Derakhshan [73] would rather not use the term 'fake news' because of its massive politicization and inadequacy in describing the pollution of information. As a result, they settled for 'information disorder' which manifests as *mis-information*– sharing false information with no harm intended, *dis-information*– sharing false information in order to wreak havoc and *mal-information*– sharing genuine, mostly classified, information for harmful intents. The work was instructive in giving thirty-four suggestions for the consideration of various stakeholders towards managing how wrong information is 'consumed, interpreted and acted upon'.

Lazer et al. [45] define fake news as "fabricated information that mimics news media content in form but not in organizational process or intent". Alongside fake news, they identified "misinformation (false or misleading information)" and "disinformation (false information that is purposely spread to deceive people)" as forms of information irregularities. It was stated that fake news thrives by exploiting the integrity of conventional news agencies. The ubiquity of the internet was highlighted as a key factor in the erosion of virtues like balance and objectivity that characterized journalism starting from the period of the First World War. To curb the spread of fake news, the authors propose (i) *individual empowerment* through proper education and provision of fact checking platforms, (ii) *preemptive detection and prevention* by internet and social media firms like Facebook, Google and Twitter. In all, a far reaching interdisciplinary approach is advocated for handling the menace of fake news.

1.1.4 *Spiral of silence*

Elisabeth Noelle-Nuemann (1916-2010) was a German political scientist and Founder/Director of the Public Opinion Research Center in Allensbach, Germany who popularized the theory of *spiral of silence*. The theory shows how people are coerced into silence about their personal opinions when such opinions are not in consonance with perceived public opinion. This silence tends to project those who hold popular opinion as stronger than they really are while the silent members of society come off as weaker than their population actually suggests. The whole scenario is capable of leading to misinformation about the stands of individuals on controversial issues in society.

Reviewing this interesting concept, Griffin [33] considered the ideas surrounding it. He portrays *public opinion* as having the power to keep people in line. The spiral of silence theory implies that humans have a sixth sense which can be referred to as the *quasi-statistical organ* through which people gauge public opinion. It was revealed that the spiral of silence is propelled by the *fear of isolation* from one's community. The *mass media* is seen as playing a huge role in driving unpopular opinions into oblivion. However, *discriminatory exposure* to the media may lead to *pluralistic ignorance*, a condition in which people erroneously feel that others think like them. Noelle-Nuemann's *plane/train test* helps to show if people would be willing to talk about burning societal situations. As helpful as the test might be, there are popular researchers that question its reliability. As outliers to the theory, *hard-core nonconformists* and *avant-garde* are identified as people who hold on to contrary opinions no matter whose ox is gored. They are radicals who have the capacity of changing the world by standing their grounds against popular opinions.

With an overview of research on the spiral of silence, Yang [76] emphasized the importance of reassessing the theory in view of the reshaping of public opinion due to the advancement in technology in the 21st century. The theory clearly came into existence at at time when the traditional media (newspaper, radio and television) held way as opposed to what is obtainable in modern times when such media have been considerably disrupted by the Internet.

He highlighted the role of the *conformity hypothesis* which postulates that individual opinions tend to give way for what is perceived as public opinion due to the fear of isolation. This establishes Noelle-Nuemann's redefinition of public opinion as that which helps to keep people in line unless they are defiant. The effect of the spiral is seen to be hugely influenced by how emotive and ethical the matter under consideration is. Such emotional issues are generally identified as those related to *political partisanship*, *ideas that threaten a particular group of people* and *unconventional views*. The spiral of silence is also massively affected by the potency of discussion of the issues by popular media. It is shown that further research is required in knowing how the spiral plays out in divisive issues that have roughly equal supporters.

In cyberspace, Yang reveals that various studies show variations in the application of the spiral. There is full applicability in some cases while it is only partial in other scenarios. The theory of the spiral of silence tends to breakdown totally in some cases and this seems to lend credence to the reality of anonymity on the the Internet. In this case, the diffusion theory is better at explaining the opinion climate. Overall, a lot of work is still required to pinpoint the co-evolution dynamics of the social media, individual and public opinion.

Kurambayev and Schwartz-Henderson [43] investigated the culture of self-censorship online and offline in Kyrgyzstan, the Central Asian so called 'island of democracy', as a result of perceived and real consequences. The fuel for this behavior include the fear of isolation like we see in the spiral of silence; power distance which is caused by the uneven distribution of opportunities and wealth thereby rendering the less privileged mute and helpless; and perceived audiences which might be those who would rather fall in line with public opinion or even the unfriendly prying eyes of government agents.

The white paper identified the repercussions of voicing out dissenting opinions on political and social issues as insults/verbal attacks, future professional/career jeopardy, ostracism, the stigma of bad reputation and even physical harm. The powers that be, knowing the impact of the social media, use them to misinform people by spreading propaganda and by trying to suppress ideas that do not agree with their political inclinations. The research finds it quite alarming that tech savvy young people would rather not discuss political and social issues freely on the Internet, an exercise which would have promoted free speech and deepened the democratic process. The authors made it a point of duty to call on international organizations and external bodies to ensure that the culture of silence is halted by providing supports that will make the Kyrgyzstani express themselves without fear thereby making room for an open society.

1.2 THE CHARACTERISTICS OF INFORMATION AGENTS

1.2.1 *Why people spread information*

The attitude of people who have access to information is very vital in its spread. For instance, it has been shown that when people have vested interests in circumstances where there are information gaps, they give in to rumors which go through the process of (i) *generation* which is mainly fueled by uncertainty and anxiety, (ii) *evaluation* based on the knowledge base of the hearers and (iii) *dissemination* inspired by repetition-induced belief and evolution of rumors into more acceptable forms [16]. Jones et al. [38] reported that in the course of active shooting with looming danger and lack of trustworthy information, there is serious vagueness which leads people to seek information from sources, like social media, whose authenticity may not be verifiable. Wardle and Derakhshan [73] investigated the elements of information disorder, namely *agents*, *messages* and *interpreters* while the phases were identified as *creation*, *production* and *distribution*. They highlighted that the spread of polluted information these days is amplified with the help of social bots and cyborgs mimicking humans.

The Customer Insight Group [35] considered what influences people to share information online in order to help marketers get their content shared for increased productivity. They recognized sharing as human nature which has been magnified and made real time as people have transformed from being broadcasters to 'sharecasters'. Sharing is also seen as a means of managing information. The three-phase study included *ethnographies* (in-person interviews), *immersion/deprivation* (a sharing panel) and a *quantitative survey* of 2500 online sharers. It was discovered that relationship building is the underlying factor which make people share online as seen in motivations like *valuable entertainment*, *self definition*, *enhancing relationships*, *personal accomplishment*, *the value of being first to share* as well as *support for causes and brands*. Based on behavior, the research identified six characters of online sharers, namely *altruists* who mostly use emails and are supportive, dependable and considerate; *careerists* commonly found on LinkedIn, they are resourceful and insightful and they have a network of like-minded people; *hipsters* who are not given to emails but they are youthful, well-liked and creative front liners with a strong sense of identity; *boomerangs* who are empowered Twitter and Facebook users driven by likes, comments and controversies; *connectors* that are e-mail and Facebook using planners, they are reflective, inventive and temperate; *selectives* who are imaginative, cautious, instructional e-mail users. It was highlighted that for one's content to get shared, there is need to appeal to consumers' need to connect with other people rather than just the brand; sharers must be able to trust the brand; simplicity will always win; the consumers' sense of humor must be appealed to; a sense of urgency is crucial; the brand must have a reputation of ongoing engagement with consumers; the use e-mails must be prioritized. The following consumer categories were identified: *entertainment*, *finance*, *retail and fashion*, *technology* and *travel*.

1.2.2 *Information agents as epidemic agents*

Motivated by the dearth of literature on the connection between the spread of information and the mode of dissemination among users, Pei et al. [61] carried out in-depth investigation on the diffusion data and social network structure of an online blog community. Amazingly, they discovered that a lot of users show persevering attitude in line with one of *social spreading*, *self-promotion* and *broadcast*. Social spreading is particularly seen as preeminent and resulting to more people receiving the information down the line. The study of the diffusion trees of each kind of attitude reveals that most cases of information transmission are confined to the first few cohorts. In order to advance research in this direction, they suggest exploring the link between user traits, content and the results of the information diffusion on one hand and the dynamics of each behavioral pattern on the other.

The spread of false information through social networks looks so much like the progression of communicable infections, they are both enhanced by social connections. As such, analyzing the dynamics of online propagation and competition of information disorder can be quite insightful. With the growing interest among people to source for news on social media primarily, transmission models together with relevant data might be of great help in understanding this new terrain [42]. From a somewhat psychoanalytic point of view, Linden [50] considered the formation of *echo chambers* of like-minded beliefs due to the increasing use of the internet as news source as well as *filter bubbles* which arise from social media platforms targeting users with information based on their previous online behavior. These were found to reinforce extreme group behavior and polarization of views. Just like in the case of epidemiology, the idea of vaccination was used to mentally inoculate people against fake news and it was found that it could help create herd immunity, thus drastically reducing the impact of falsehood on the society.

Concerned about the readiness of people to believe fake news over mainstream news, Lauzen-Collins [44] viewed the issue from a psychological perspective. It was established that humans tend to believe falsehood because we are naturally *cognitive misers* who like to process information on the surface based on patterns and intuition. Rational, logical thinking requires motivation and effort, as such, we would rather follow the path of least resistance thereby increasing our gullibility. When this tendency to take mental short cuts in problem solving and decision making is coupled with impractical self enhancement, there is a perfect recipe for susceptibility to fake news.

1.3 A MATHEMATICAL BACKGROUND ON INFORMATION SPREAD

Claude Shannon (1916–2001), known as the Father of Information Theory, published a groundbreaking paper [66] where he laid a solid foundation for a mathematical perspective on information. He identifies a communication system as consisting of *information source*, *transmitter*, *channel*, *receiver* and *destination*. Noise, which adds unnecessary data to an original message, is also a prominent concept in his work. It is possible to see noise in terms of electrical signals, cultural biases or emotional disturbances which distort information. He presented the idea of entropy which measures the mean rate of information production by a stochastic data source expressed as

$$H = -k \sum_{i=1}^{n} p_i \log p_i, \tag{1.1}$$

where k is a positive constant; p_i is the probability of occurrence of the ith possible value of the source symbol and H is measured in *bits* per symbol.

Going through literature, we identify three broad approaches in the mathematical models used in representing the spread of information, namely *semi-network population dynamics* as in Solomonoff and Rapoport [70], Rapoport and Rebhun [62]; *network dynamics* as in Lerman [46], Castillo, Chen, and Lakshmanan [11]; and *population dynamics* as in Kermack and McKendrick [40], Chisholm et al. [13]. In this section, we shall review some history of mathematical information epidemics.

1.3.1 *The semi-network population dynamics approach*

One of the earliest works on the mathematical approach to the spread of rumors was by Rapoport and Rebhun [62]. They demonstrated the relationship between the theories of random nets and rumor spread where weak connectivity in a net corresponds with a situation whereby the spread of a rumor is very low. The evolution of rumors over time was emphasized as important and analogous to the average number of points reached every step of the tracing procedure in a random net model. The following groups were identified in connection with rumors: *non-knowers*, *knowers*, *tellers*, *hearers* and *removes*. The number of knowers at the $(t+1)$th remove is derived as

$$x(t+1) = 1 - (1 - x_0)e^{-ax(t)}, \tag{1.2}$$

where x_0 is the initial fraction of knowers; a is the average number of links from each knower; and $x(t)$ is number of knowers at the tth remove.

Building on the works of Rapoport and others, Skvoretz [68] set out to strike a balance between the handiness of the simple approximate models used in random and biased networks and their accuracy. These models are used to try to establish connections between simple local events and complex aggregate patterns. He identified connectivity as a crucial statistical property of networks. This connectivity, often represented as γ, is the limit of the sequence of the fractions of a population reached by a random selection of starters of an epidemics, information or innovation. It reveals the extent to which a particular population is integrated by a network.

For a given population P in a network with a *network density* d and *element density* a, we have $d = a/(P-1)$. Connectivity is measured by the sequence $N_0, N_1, N_2, ...$ where N_i is the proportion of the population newly contacted i steps away from a set of randomly selected starters or the related cumulative sequence $X_0, X_1, X_2, ...$ where $X_i = \sum_{j=0}^{i} N_j$, the proportion reachable from the starter set in i or fewer steps and where $\gamma = X_\infty$ is the limiting fraction ever reached.

Following Skvoretz's argument in the context of information spread, we consider an arbitrary person who is informed about a piece of information but is not in the starter population. The probability they are newly informed at the $i+1$th step is a product of *the probability they have not been contacted on any of the previous steps which is* $(1 - X_i)$ and *the probability that one of the newly informed persons on the* i*th step targets this person*.

The expected number of newly informed people on the ith step is PN_i and each of these has a connections, the targets of which are chosen at random. aPN_iX_i and $aPN_i(1 - X_i)$ of PN_i are expected to go to those that have already been informed and those who have not been informed respectively. The expected number of those who are yet to be informed is $P(1 - X_i)$ and the chance that a particular piece of information goes to the uninformed is $1/[P(1 - X_i)]$ and the chance that it does not get to someone who is uniformed is $1 - 1/[P(1 - X_i)]$.

So, the chance that a piece of information directed to an uniformed person fails to get to someone is $\{1 - 1/[P(1 - X_i)]\}^{P(1-X_i)aN_i}$ and the probability that at least one of the informed persons contacts an uninformed person is one minus the above quantity such that we have

$$N_{i+1} = (1 - X_i)\left[1 - \left(1 - \frac{1}{P(1 - X_i)}\right)^{P(1-X_i)aN_i}\right]. \qquad (1.3)$$

For large values of $P(1 - X_i)$, we have $\{1 - 1/[P(1 - X_i)]\}^{P(1-X_i)} = e^{-1}$ so that

$$N_{i+1} = (1 - X_i)\left(1 - e^{-aN_i}\right). \qquad (1.4)$$

Using (1.4), it can be shown that $\gamma = X_\infty = 1 - (1 - N_0)e^{-a\gamma}$ so that if the starters N_0 are a small fraction of the population $\gamma \cong 1 - e^{-a\gamma}$ and this can be solved by numerical methods.

1.3.2 *The network dynamics approach*

Lerman [46] considers the diffusion of activation on a graph, where each activated node infects neighbors with some probability. In the context of information spread, an activated node can be likened to an individual who is in the know about a piece of information or idea. Lerman points out that network studies based on data are now easy to carry out because it is more convenient to obtain extensive, time-resolved data on social media usage thereby enhancing deeper understanding of social behavior. With such massive data, we can now have better understanding of how information spread on networks, the structure of networks, and how individual users influence the communal attitude of others.

The work was focused on a social news site, Digg where news are voted for such that the mostly voted ones get to the top news page; the micro-blogging site, Twitter where tweets and retweets are analyzed to identify popular trends was also considered. It was made clear that information diffuses on networks through cascade effects. Some typical cascades are identified as branching, chaining and community with the possibilities of collision and degeneracy.

In order to quantify the evolution of cascades, the cascade generating function is quite helpful and it is defined as

$$\varphi(j,\alpha) = \sum_{i \in friend(j)} \alpha\varphi(i,\alpha) \tag{1.5}$$

where j would be an informed person and i are j's contact(s) who introduce him to the information. For instance, given a network defined by $V = 7$ people and $E = 7$ links for information transmission expressed as

$$\begin{aligned} G(V,E) &= G(7,7); \\ V &= \{1,2,3,4,5,6,7\}; \\ E &= \{(1,3),(1,4),(1,6),(1,7),(2,4),(2,5),(3,6)\}, \end{aligned} \tag{1.6}$$

we can evaluate the density generation functions for each person in the network as follows

$$\begin{aligned} \varphi(1,\alpha) &= c_1; \\ \varphi(2,\alpha) &= c_2; \\ \varphi(3,\alpha) &= \alpha\varphi(1,\alpha) = \alpha c_1; \\ \varphi(4,\alpha) &= \alpha\varphi(1,\alpha) + \alpha\varphi(2,\alpha) = \alpha(c_1 + c_2); \\ \varphi(6,\alpha) &= \alpha\varphi(1,\alpha) + \alpha\varphi(3,\alpha) = \alpha(\alpha+1)c_1. \end{aligned} \tag{1.7}$$

It is seen that cascade sizes are constrained by factors like clustering and degree heterogeneity in network structure as well as dynamics such as social contagion mechanism and change in transmissibility. The conservative and non-conservative diffusion classes are identified for networks and their mathematical modeling can help to unify social and epidemic dynamics. Alpha centrality, which measures the number of paths between nodes, each path attenuated by its length with parameter α, is seen as a very important measure of an individual's influence in a network with epidemic or information diffusion. Alpha centrality, as defined in line with Bonacich [6], is given as

$$r^{Alpha}(\alpha) = eA\sum_{k=0}^{\infty}\alpha^k A^k = \frac{eA}{I-\alpha A} \tag{1.8}$$

where I is an identity matrix with the same dimension as A which is the adjacency matrix of the network, e is the eigenvector of A.

Castillo, Chen, and Lakshmanan [11] give very instructive ideas of how information and influence spread on online social networking sites like Facebook, Flickr, Flixster, Last.fm, MySpace, Orkut, Tumblr, Twitter, Wikipedia and YouTube from a graph theoretic and data mining point of view. They established that information is dispersed on social networks as a result of the connection of people who perform actions like creating, posting, sharing, liking, commenting on, rating, linking or retweeting messages.

The basic data model comprises a *graph* G of V users and E links/ties as well as a *log* A of users u_1, actions a_1 and times t_1 respectively represented as

$$\begin{aligned} G = &\quad (V, E); \\ A = &\quad \{u_1, a_1, t_1, ...\} \end{aligned} \tag{1.9}$$

With some staggering data, they listed the tremendous power of social networks in making information go viral and provoking action extremely faster than in the last half of a century. Some example include how our friends' friends' actions have bearings on us, Hotmail's rapid rise to the peak in the 90s, the 2008 Mumbai terror attacks and the rags to riches story of the previously homeless voice over artist Ted Williams in 2011 among others.

They highlighted the application of influence spread in areas like viral marketing, the dispersion of falsehood and rumors, adoption of innovations, human and animal epidemics, social search, etc. The roles of influencers on social networks were identified as very important in making information go viral and it will always be a useful strategy to identify such influencers in order to propagate ideas speedily.

Based on some literature, the tutorial concludes that the idea of influence on social networks can be questionable. For instance, it is possible that people are simply attracted to their likes rather than been influenced. Also, it was pointed out that influence on social networks can be overrated.

1.3.3 *The population dynamics approach*

This approach employs the use of compartmental models in enhancing the understanding of contagions. It was introduced in the work by Kermack and McKendrick [40] and countless studies have been undertaken based on it ever since. These models have been used amply in fields as diverse as epidemiology, ecology, pharmacology, sociology and a host of others in order to demystify relevant population dynamics. Simplicity and elegance are some properties that make compartmental models quite endearing. Since the spread of information follows a contagious process, models that look like the classical Kermack-McKendrick *susceptible-infected-removed (SIR)* model have been used to model it.

Looking at the spread of misinformation as some kind infection from the SIR point of view, we can say that susceptible individuals are those who have the tendency of being misinformed, infected people are those who have been misinformed and are capable of misinforming others, the removed ones are those who have found out the truth. It is important to know if the misinformation dies out after every unexposed person has got the wrong information or not. There is typically a critical mass of unexposed people below whose population wrong information seemingly dies out and above which it spreads.

Considering information spread from the perspective of the SIR model and taking a constant unit time interval t and γ intervals such that the number of newly misinformed individuals in a unit area is $m_{t,\gamma}$ and the total number of those who are misinformed in this interval is $y_t = \sum_0^t m_{t,\gamma}$. we view m_t as the number of those who undergo the process of misinformation during the transition in the interval $(t-1, t)$. Basically, $m_{t,0} = m_t$ except at the point where y_0 individuals have just been misinformed, so that

$$m_{0,0} = m_0 + y_0. \tag{1.10}$$

Suppose that α_γ represents the rate at which misinformed people get to find out the truth, then those who are correctly informed from each γ group at the end of the interval t is $\alpha_\gamma m_{t,\gamma} = m_{t,\gamma} - m_{t+1,\gamma+1}$ and so

$$\begin{aligned} m_{t,\gamma} = & \quad m_{t-1,\gamma-1}(1-\alpha(\gamma-1)) \\ = & \quad m_{t-2,\gamma-2}(1-\alpha(\gamma-1))(1-\alpha(\gamma-2)) \\ = & \quad m_{t-\gamma,0}\beta_\gamma \end{aligned} \tag{1.11}$$

where $\beta_\gamma = (1-\alpha(\gamma-1))(1-\alpha(\gamma-2))...(1-\alpha(0))$.

The number of persons in unit area who got misinformed within the interval t can be given as $m_t = x_t \sum_1^t \lambda_\gamma m_{t,\gamma}$ where x_t denotes the number of individuals who are yet to come in contact with the wrong information and λ_γ is the rate of misinformation at age γ. Unlike in the SIR case, λ_0 is not necessarily zero in the case of information spread as a misinformed person can begin to spread the wrong information as soon as it is received. However, just like for the SIR model, the possibility of a person being misinformation is relative to the number of those already misinformed as well as those who are still susceptible. We can now see that

$$x_t = N - \sum_0^t m_{t,0} = N - \sum_0^t m_t - y_0 \tag{1.12}$$

and $x_t + y_t + z_t = N$ where z_t is the number of those who have found out the truth; N is the initial population density which is considered to be constant over the course of the information dispersion which relatively takes place within a short time.

If we assume the special case where λ and α are constant, we have the system

$$\begin{aligned} \frac{dx}{dt} &= -\lambda xy; \\ \frac{dy}{dt} &= \lambda xy - \alpha y; \\ \frac{dz}{dt} &= \alpha y, \end{aligned} \tag{1.13}$$

with $x + y + z = N$. This system is simple but quite insightful.

An in-depth review of the mathematical modeling of rumor spread was carried out by Ndii, Carnia, and Supriatna [57] so that they could pinpoint vital results and aspects that are necessary to further understand the spread of rumors. It is generally understood that rumors tend to shape public opinion and affect individual behavior. The 620 works examined cover a period of 26 years (1990–2016). The models observed can be generally categorized into *network* and *non-network* models. It is seen that the diffusion of rumors is of interest in various subject areas with the most attention coming from Computer Science, Mathematics, Physics, Engineering and Social Sciences respectively. The assessment shows that most rumor mathematical models are based on compartmental epidemic models where the populations are classified into *ignorants* (X) who are not aware of the rumor; *spreaders* (Y) who who know and spread the rumor; and *stiflers* (Z) who know but do not spread the rumor. One of the most common representations of the model is given by the deterministic version of the Daley-Kendall rumor model proposed by Citrón-Arias and Castillo-Chávez and given by

$$\begin{aligned} \frac{dX}{dt} &= -\lambda \frac{Y}{N} X; \\ \frac{dY}{dt} &= \lambda \frac{Y}{N} X - \alpha \frac{Y+Z}{N} Y; \\ \frac{dZ}{dt} &= \alpha \frac{Y+Z}{N} Y, \end{aligned} \tag{1.14}$$

where $X(t) + Y(t) + Z(t) = N$, λ is the rate at which ignorants become spreaders and α is the rate at which spreaders become stiflers. Most of the papers analyzed show that rumors can be managed through proper education of individuals and public enlightenment by relevant governmental and non-governmental organizations. Based on the literature observed, the role of cliques in the spread of rumors in a network is an area that requires some consideration.

Escalante and Odehnal [20] tried to establish the correspondence in the dynamics of infection-vaccination models and a model of two counteracting rumors by using SIRS type models with short term immunity. The first rumor serves as the infection while the second rumor serves as the vaccination which reduces susceptibility to the first. Owing to the fact that electronic devices are now amply employed to transmit information, the propagation rumor (PR) model, whose standard stochastic form was first introduced by Daley and Kendall [14], was adopted. It is shown that stochastic models depend on probabilities rather than defined rates and they are very good for individual-level modeling though they are not easy to formulate and they require a lot of computational effort so they can give helpful results. On the flip side, deterministic models are easier to develop, they require less data and are user friendly since there are quite a number of software to simulate them. Overall, they give fair representations of large populations.

The PR model which is made up of proportion of the total population who are ignorant about the first rumor (s); the proportion that know and spread the first rumor (i) and the proportion of the population with reduced susceptibility to the first rumor by being exposed to the second (v) such that

$$\begin{aligned}\frac{ds(t)}{dt} &= -\beta s(t)i(t) - \phi s(t) + \gamma(n - i(t) - s(t));\\ \frac{di(t)}{dt} &= \beta s(t)i(t) + \sigma\beta v(t)i(t) - \alpha i(t);\\ \frac{dv(t)}{dt} &= \phi s(t) - \sigma\beta v(t)i(t),\end{aligned} \tag{1.15}$$

with the initial condition $s(0)$, $i(0)$, $v(0)$ and parameters $n = s(t) + i(t) + v(t)$ (constant population size), β (the rate at which the ignorant get to know and spread the first rumor), α (the rate at which spreaders stop spreading the first rumor), ϕ (fraction of the ignorant class to the first rumor which knows the second rumor per unit time), γ (proportionality rate of loss of immunity to the first rumor) and $\sigma : 0 \leq \sigma \leq 1$ (susceptibility reduction factor to the first rumor caused by the second rumor).

An alternative PR model consisting of two subpopulations n_1 and n_2, with variables and parameters corresponding to the first model, was also formulated for the sake of comparison. Here, it is assumed that those who know both rumors have the lessened propensity to spread the first rumor by a factor of η while those who know the second rumor have their susceptibility to the first rumor decreased by a factor of σ. This new model is expressed as

$$\begin{aligned} \frac{ds_1(t)}{dt} &= -\beta_1 s_1(t)[i_1(t) + \eta i_2(t)]; \\ \frac{di_1(t)}{dt} &= \beta_1 s_1(t)[i_1(t) + \eta i_2(t)] - \alpha_1 i_1(t); \\ \frac{ds_2(t)}{dt} &= -\sigma\beta_2 s_2(t)[i_1(t) + \eta i_2(t)]; \\ \frac{di_2(t)}{dt} &= \sigma\beta_2 s_2(t)[i_1(t) + \eta i_2(t)] - \alpha_2 i_2(t), \end{aligned} \tag{1.16}$$

with initial condition $s_1(0)$, $i_1(0)$, $s_2(0)$, $i_2(0)$ such that $s_1(0) + i_1(0) = n_1$, $s_2(0) + i_2(0) = n_2$ and $n_1 + n_2 = n$.

Based on the two models, parameters were estimated and numerical simulations were carried out based on people types, circumstances of rumor spread and quality of rumor. The alternative model is seen to provide a better fit for the estimated data compared to the original infection-vaccination model. The paper lends credence to the fact that rumors can be easily managed by reducing people's susceptibility through the introduction of accurate counter information.

1.4 MATHEMATICAL MODELING OF INFORMATION WARFARE

In a world of rumors, gossips, urban legends, political propaganda and commercial advertisements, there are always loads of information competing for human attention. Many a time, the nature of some information can be quite divisive and get people polarized as they hold on tenaciously to their respective view points. Mathematical modelers have been interested in such warfare in recent times.

Chisholm et al. [13] developed an infectious disease model for the diffusion of two competing opinions of a polarizing view which integrates outside elements and person-to-person connections. The model is derived from both epidemiological and competing species models to understand how members of America's Republican Party supported or were skeptical about the greenhorn candidates in their 2016 primary polls.

The study developed and analyzed two *skeptic, unexposed, proponent (SUP)* models, one basic and one modified. The modified model is applied to a case study using the poll results for candidates Carson, Fiorina, and Trump to fit parameters. The base SUP model is given as

$$\begin{aligned} U' &= -aSU - bPU \\ S' &= aSU \\ P' &= bPU, \end{aligned} \tag{1.17}$$

where $U(t)$, $S(t)$ and $P(t)$ are the *unexposed*, *skeptic*, and *proponent* populations, respectively, at time t and $a, b \geq 0$ are the respective persuasion rates of skeptics and proponents transmitting their viewpoint to the unexposed population.

The modified model known as the *conviction-debate* SUP model was used to understand the population dynamics of competition between support for and opposition to a political candidate in the party primaries. The model is expressed as

$$\begin{aligned} U' &= -aSU - bPU - cU - dU \\ S' &= \quad aSU + \alpha PS + cU - eS + fP \\ P' &= \quad bPU - \alpha PS + dU - fP + eS, \end{aligned} \tag{1.18}$$

where $a, b \geq 0$ are the respective persuasion rates of skeptics and proponents transmitting their viewpoint to the unexposed population; $c, d \geq 0$ are the respective personal conviction rates of the unexposed group to skeptics and proponents; $e, f \geq 0$ are respective mind change conviction rates of skeptics and proponents; α is the debate persuasion rate which may be positive, negative or zero. The authors wish that the model can promote general understanding about the spread of competitive ideas.

The work by Mikhailov, Pronchev, and Proncheva [56] takes a deep dive into various dimensions of an information battle process in which two opposite views about a particular information item spread in the environment. They started by establishing the main difference between the Daley and Kendall [14] and Maki and Thompson [52] rumor models which is that when two spreaders meet in the former, they both become stiflers while in the latter only one of them become a stifler. The interest is to know how many ignorants remain in the long run. Though the models form a basis for an interesting area of research at the interface of mathematics and the social sciences, the stiffling-effect in them has been found not to be realistic so it is ignored in the work. They regard the models by Osei and Thompson [58] and Escalante and Odehnal [20] as good examples of competing rumor models. The impact of mass media and rumor propagation by individuals in the dastardly 1994 Rwanda genocide was highlighted. In an information battle, the winning one is the one with the higher number of spreaders in the long run.

The *basic information attack model* bears resemblance with the single rumor model. Here, an item of information can come from the mass media with the intensity $\alpha > 0$ or by word of mouth with intensity $\beta X (\beta > 0)$ where $X(t)$ is the number of spreaders at time t and N is the total population such that

$$\frac{dX}{dt} = (\alpha + \beta X)(N - X); \quad X(0) = x_0. \tag{1.19}$$

The basic model can be expanded considering a case where the mass media does not fully cover the society. In this case, there are two homogeneous subpopulations N_1 and N_2 where N_1 gets the message from both the mass media and rumors from individuals while N_2 only get it through rumors. The number of spreaders X_1 and X_2 correspond with the two subgroups with no crossing between them. Also, the spreaders are not differentiated in the way they spread information so they are considered together as $X_1 + X_2$ so that we have

$$\begin{aligned}\frac{dX_1}{dt} &= [\alpha + \beta(X_1 + X_2)](N_1 - X_1);\\ \frac{dX_2}{dt} &= \beta(X_1 + X_2)(N_2 - X_2).\end{aligned} \tag{1.20}$$

In some situations, repeated campaigns are needed to make ignorants become spreaders. For instance, if people get to be spreaders in two steps, that is, from ignorants to pre-spreaders $x(t)$ to spreaders $X(t)$ such that

$$\begin{aligned}\frac{dx}{dt} &= (\alpha + \beta X)(N - 2x - X);\\ \frac{dX}{dt} &= x(\alpha + \beta X).\end{aligned} \tag{1.21}$$

If it is possible for spreaders to forget the information thereby behaving like ignorants again, with parameter $\gamma > 0$ as the strength of forgetfulness, we have

$$\frac{dX}{dt} = (\alpha + \beta X)(N - X) - \gamma X. \tag{1.22}$$

A model which takes into consideration the three factors of limited mass media coverage, two-step transition from ignorants to spreaders, and forgetfulness of pre-spreaders and spreaders with intensities δ and γ respectively, we have the model

$$\begin{aligned}\frac{dx_1}{dt} &= [\alpha + \beta(X_1 + X_2)](N_1 - 2x_1 - X_1) - \delta x_1 + \gamma X_1;\\ \frac{dX_1}{dt} &= x_1[\alpha + \beta(X_1 + X_2)] - \gamma X_1;\\ \frac{dx_2}{dt} &= \beta(X_1 + X_2)(N_2 - 2x_2 - X_2) - \delta x_2 + \gamma X_2;\\ \frac{dX_2}{dt} &= \beta x_2(X_1 + X_2) - \gamma X_2.\end{aligned} \tag{1.23}$$

subject to the initial condition $x_1(0) = X_1(0) = x_2(0) = X_2(0) = 0$.

The *information battle model* in which two contrasting messages are sent out with intensities α_1, α_2 and the messages are propagated by individuals with intensities β_1, β_2 so that there are two subgroups of competing spreaders $X(t)$ and $Y(t)$. It is assumed that the spreaders do not cross from one group to another so each group is only increased by recruitment from the group of ignorants given as $N - X(t) - Y(t)$. The model then becomes

$$\begin{aligned}\frac{dX}{dt} &= (\alpha_1 + \beta_1 X)(N - X - Y), \quad X(0) = 0;\\ \frac{dY}{dt} &= (\alpha_2 + \beta_2 X)(N - X - Y), \quad Y(0) = 0.\end{aligned} \tag{1.24}$$

As stated earlier, the condition for the victory of the first group is $\lim_{t\to\infty} X(t) > \lim_{t\to\infty} Y(t)$ so that

$$\frac{\beta_1}{\ln\left(1+\frac{\beta_1 N_0}{2\alpha_1}\right)} > \frac{\beta_2}{\ln\left(1+\frac{\beta_2 N_0}{2\alpha_2}\right)}$$

and vice versa. A generalization of the information battle over m arbitrary groups give

$$\frac{dX_i}{dt} = (\alpha_i + \beta_i X_i)\left(N - \sum_{i=1}^{m} X_i\right), \quad X_i(0) = 0, i = 1, 2, ..., m. \tag{1.25}$$

The model has also been expanded to cater for partial media coverage, two step transition from ignorants to spreaders as well as the factor of forgetfulness.

Brody [9] considers the complexity of the circumstances arising from the spread of disinformation as it relates to elections. By deriving poll statistics as the outputs of his model, a means of forecasting election results in the face of information disorder was provided. The predictions take into account best ways of minimizing false information. With some attention on the spread of rumors on networks that are heterogeneous, Li, Hu, and Jin [48] nonlinear differential equations are derived via probability generating function and pair approximation techniques to demonstrate rumor spread. The favorable results show that the diversity of the network speed up rumor breakout but suppresses the concentration of spreaders. The disparities in rumor and disease dispersion were also investigated.

Kostka, Pignolet, and Wattenhofer [41] show that the ubiquity of social networks impacts a great deal on the spread of information. In addition, it determines the kind of choices people make when they are confronted with multiple options. They reveal that the goal of each information driver is to identify the critical mass of initiators that will drive their agenda as much as possible. In their very interesting trials, Ecker, Lewandowsky, and Tang [18] discovered that people continue to be influenced by wrong information even if they are later corrected. This simply implies that no matter how hard efforts are made to eliminate the effects of false information, it is most likely impossible to achieve that aim completely. As such, the adage that 'prevention is better than cure' also holds true in the case of misinformation. Highlighting the influence of the internet on communication and diffusion of misinformation, Lewandowsky et al. [47] pointed out that individuals, organizations and governments now have a lot more of damage control jobs to do. This is more so because efforts directed towards setting records straight in the public space do not usually yield the desired results. However, there are suggestions on how to deal with undesired information such that an acceptable level of correction can be achieved, at the least.

2

A MATHEMATICAL MODEL FOR THE SPREAD OF TWO INTERACTING PIECES OF INFORMATION

2.1 GENERAL MODEL FORMULATION

We seek to model the spread of two pieces of information who have some kind of relationship with the following variables and parameters.

$U(t)$: population of those who have not been exposed to any of the two pieces of information at time t;

$P_1(t)$: population of those who know and transmit only the first piece of information at time t;

$P_2(t)$: population of those who know and transmit only the second piece of information at time t;

$V_1(t)$: population of those who know both but transmit only the first piece of information at time t;

$V_2(t)$: population of those who know both but transmit only the second piece of information at time t;

$W_0(t)$: population of those who transfer directly from the state U to a state in which they know and transmit both pieces of information t;

$W_1(t)$: population of those who transfer from the states P_1 and P_2 to a state where they know and transmit both pieces of information t;

$W_2(t)$: population of those who transfer from the states V_1 and V_2 to a state where they transmit both pieces of information t;

Λ_1: coefficient of interaction between U and either of P_1 or V_1 leading to U knowing and transmitting only the first piece of information;

Λ_2: coefficient of interaction between U and either of P_2 or V_2 leading to U knowing and transmitting only the second piece of information;

Λ_3: coefficient of interaction between U and any of W_0, W_1 or W_2 leading to U knowing both but transmitting only the first piece of information;

Λ_4: coefficient of interaction between U and any of W_0, W_1 or W_2 leading to U knowing both but transmitting only the second piece of information;

Λ_5: coefficient of interaction between U and any of W_0, W_1 or W_2 leading to U knowing and transmitting both pieces of information;

Γ_1: coefficient of interaction between P_1 and any of P_2, V_2, W_0, W_1 or W_2 leading to P_1 knowing both pieces but transmitting only the first piece of information;

Γ_2: coefficient of interaction between P_1 and any of P_2, V_2, W_0, W_1 or W_2 leading to P_1 knowing both pieces but transmitting only the second piece of information;

Γ_3: coefficient of interaction between P_1 and any of P_2, V_2, W_0, W_1 or W_2 leading to P_1 knowing and transmitting both pieces of information;

Ξ_1: coefficient of interaction between P_2 and any of P_1, V_1, W_0, W_1 or W_2 leading to P_2 knowing both pieces but transmitting only the first piece of information;

Ξ_2: coefficient of interaction between P_2 and any of P_1, V_1, W_0, W_1 or W_2 leading to P_2 knowing both pieces but transmitting only the second piece of information;

Ξ_3: coefficient of interaction between P_2 and any of P_1, V_1, W_0, W_1 or W_2 leading to P_2 knowing and transmitting both pieces of information;

Ψ_1: coefficient of behavioral change in V_1 leading to them transmitting only the second piece of information;

Ψ_2: coefficient of behavioral change in V_2 leading to them transmitting only the first piece of information;

Φ_1: coefficient of behavioral change in V_1 leading to them transmitting both pieces of information;

Φ_2: coefficient of behavioral change in V_2 leading to them transmitting both pieces of information;

where $\Lambda_i, \Gamma_i, \Xi_i, \Psi_i, \Phi_i$ are coefficients related to transition of states and they are functions of relevant subpopulations.

From the foregoing, we formulate the general model as follows

$$\begin{aligned}
\frac{dU}{dt} &= -\Lambda_1 U - \Lambda_2 U - \Lambda_3 U - \Lambda_4 U - \Lambda_5 U; \\
\frac{dP_1}{dt} &= \Lambda_1 U - \Gamma_1 P_1 - \Gamma_2 P_1 - \Gamma_3 P_1; \\
\frac{dP_2}{dt} &= \Lambda_2 U - \Xi_1 P_2 - \Xi_2 P_2 - \Xi_3 P_2; \\
\frac{dV_1}{dt} &= \Lambda_3 U + \Gamma_1 P_1 + \Xi_1 P_2 - \Psi_1 V_1 + \Psi_2 V_2 - \Phi_1 V_1; \\
\frac{dV_2}{dt} &= \Lambda_4 U + \Gamma_2 P_1 + \Xi_2 P_2 + \Psi_1 V_1 - \Psi_2 V_2 - \Phi_2 V_2; \\
\frac{dW_0}{dt} &= \Lambda_5 U; \\
\frac{dW_1}{dt} &= \Gamma_3 P_1 + \Xi_3 P_2; \\
\frac{dW_2}{dt} &= \Phi_1 V_1 + \Phi_2 V_2.
\end{aligned} \tag{2.1}$$

2.2 A REJOINDER MODEL

In our context, a rejoinder is a reply issued to correct an incomplete or misleading piece of information that is already in circulation in a population. According to a study by Akpabio [2] on the direction of rejoinders in two of Nigeria's famous newspapers, it was discovered that the noble attribute of balanced reporting was lacking among some journalists. As a result, there were more of adversarial than mild rejoinders to some news items published about some individuals, organizations and governments.

We hereby propose a rejoinder model in order to understand the nuances of counteracting pieces of information spreading among a population in a typical internet-based social media setting. We introduce the possibility of human skepticism or deliberate negligence towards the corrective information such that some people continue to spread falsehood even after getting the accurate information. This mathematical framework is imperative in this internet-enabled information and post-truth age as multidisciplinary research is required to help contain falsehood in the society.

2.3 ASSUMPTIONS FOR MODELING

TWO PIECES OF INFORMATION The two pieces of information are such that the first one is incomplete and misleading whereas the second one is a *rejoinder* which is complete and corrective to the first.

TRANSMISSION AND SPREAD OF INFORMATION The accurate piece of information is only shared alongside the wrong one such that a non-knower either gets to know and transmit only the first piece or both pieces of information at any given time. We assume that interactions on the Internet tend to happen very fast and do not significantly rely on the detailed structure of networks. So, complete mixing is

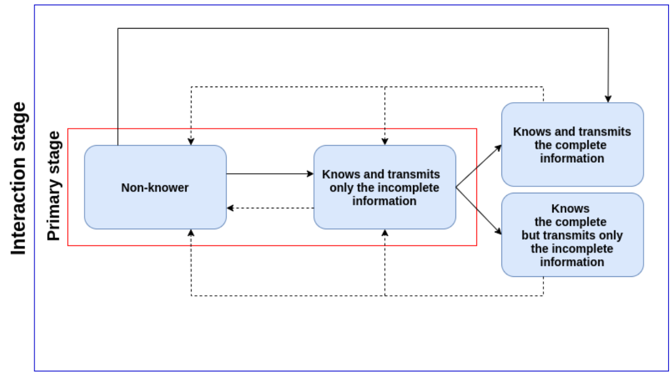


Figure 2.1: Stage transition of individuals and their relation according to information transmission in the rejoinder model.

assumed among those who are unexposed and transmitters who are either misinformed or correctly informed.

TWO STAGES OF THE INFORMATION SPREAD We consider the spread of information in two steps as shown in Figure 2.1: first is the *primary stage* when the misinformation is introduced and begins to spread till a given time $t = t_s$; afterwards, we have the *interaction stage* after the complete piece of information is introduced at $t = t_s$. In the interaction stage, the complete piece of information spreads together with the wrong piece of information from time t_s which is the moment of rejoinder introduction.

DIFFERENT ATTITUDES TOWARDS THE REJOINDER Of those who get to know the complete information after being misled initially, some get reinforced in transmitting the misleading information thereby going into the subpopulation V while the rest go directly into the subpopulation W_+ where they transmit the complete information (see Figures 2.1 and 2.2). We assume that the wrongly informed people are expected to keep transmitting the incomplete and misleading information with probability b even after knowing the complete one (the transition to V). Further, we assume a strengthened motivation for V to be dogmatic and not want to stop transmitting the misleading information after getting to know the second piece. As such, the wrongly informed person comes to believe and transmit the complete information with a probability $1 - b$ after getting it (the transition to W_+).

2.4 MODEL

Primary stage

When the first piece of information, which is misleading, is the only one in circulation, we have the following dynamics of information spread for $t \in [0, t_s)$:

$$\frac{dU}{dt} = -\beta \frac{P}{N} U; \quad \frac{dP}{dt} = \beta \frac{P}{N} U \tag{2.2}$$

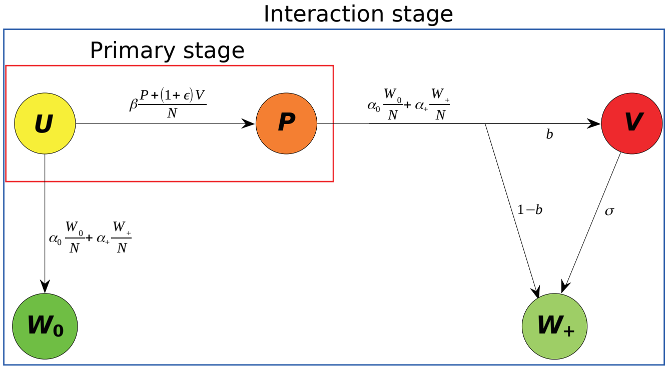


Figure 2.2: State transition in the population dynamics for the spread of two interacting pieces of information. In the primary stage, we have the system (2.2) consisting of U and P, and at the interaction stage, we have (2.5).

with initial condition $(U(0), P(0)) = (U_0, P_0)$. It is assumed that the total population size N is constant independently of time, so that the system (2.2) satisfies $U(t) + P(t) = N$ for any $t \in [0, t_s)$. $U = U(t)$ is the population size of those who have not been exposed to the considered piece of information at time t, while $P = P(t)$ is the population size of those who come to know and transmit the misleading information at time t with transmission coefficient $\beta > 0$. We can derive the following closed one-dimensional differential equation in terms of P from (2.2):

$$\frac{dP}{dt} = \beta\frac{P}{N}(N-P) \tag{2.3}$$

with initial condition $P_0 = N - U_0 > 0$. The solution is easily obtained as

$$P(t) = \frac{N}{1+\frac{U_0}{P_0}e^{-\beta t}}, \tag{2.4}$$

so that $U(t) = N - P(t)$ for $t \in [0, t_s]$. This is monotonically increasing towards N in time.

Interaction stage

After the rejoinder is introduced at time $t = t_s$, we have the following dynamics of information spread for $t > t_s$ (see Figure 2.2):

$$\begin{aligned}
\frac{dU}{dt} &= -\beta\frac{P}{N}U - (1+\epsilon)\beta\frac{V}{N}U - \alpha_0\frac{W_0}{N}U - \alpha_+\frac{W_+}{N}U; \\
\frac{dP}{dt} &= \beta\frac{P}{N}U + (1+\epsilon)\beta\frac{V}{N}U - \alpha_0\frac{W_0}{N}P - \alpha_+\frac{W_+}{N}P; \\
\frac{dV}{dt} &= b\alpha_0\frac{W_0}{N}P + b\alpha_+\frac{W_+}{N}P - \sigma V; \\
\frac{dW_0}{dt} &= \alpha_0\frac{W_0}{N}U + \alpha_+\frac{W_+}{N}U; \\
\frac{dW_+}{dt} &= (1-b)\alpha_0\frac{W_0}{N}P + (1-b)\alpha_+\frac{W_+}{N}P + \sigma V,
\end{aligned} \tag{2.5}$$

with $U(t) + P(t) + V(t) + W_0(t) + W_+(t) = N$ for any $t > t_s$. Every parameter is a positive constant whose meaning is explained in the following part.

As the continuity between the primary and the interaction stages, we define

$$U(t_s - 0) = \lim_{t \to t_s - 0} U(t); \quad U(t_s + 0) = \lim_{t \to t_s + 0} U(t) \tag{2.6}$$

as well as the other variables. It marks the end of the primary stage (represented by $t_s - 0$) on the interval $[0, t_s)$ and the beginning of the interaction stage (given by $t_s + 0$). With θ $(0 < \theta < 1)$ representing the portion of people that get to learn about the complete information at time t_s, we define the initial condition at $t = t_s + 0$ as

$$\begin{aligned}&(U(t_s + 0), P(t_s + 0), V(t_s + 0), W_0(t_s + 0), W_+(t_s + 0))\\ &= ((1-\theta)U(t_s - 0), (1-\theta)P(t_s - 0), b\theta P(t_s - 0), \theta U(t_s - 0), (1-b)\theta P(t_s - 0)),\end{aligned} \tag{2.7}$$

where

$$\theta N = \theta[U(t_s - 0) + P(t_s - 0)] = V(t_s + 0) + W_0(t_s + 0) + W_+(t_s + 0). \tag{2.8}$$

The values of $P(t_s - 0)$ and $U(t_s - 0)$ are given by (2.4):

$$P(t_s - 0) = \frac{N}{1 + \frac{U_0}{P_0} e^{-\beta t_s}}; \quad U(t_s - 0) = N - P(t_s - 0). \tag{2.9}$$

At this stage, as shown in Figure 2.2, the non-knowers of U may get either the misleading information only or the complete one. After an individual of U gets the complete information, such a person is assumed to always come to transmit it; this is defined as the transition from the state U to W_0. $W_0(t)$ is the population size of those who have not been misled but transit from the state U to the state in which they know and transmit the complete (correct) information at time t.

$P(t)$ is the population size of those who know ONLY the misleading information and transmit it at time t. $W_+(t)$ is the population size of those who get misled before knowing and transmitting the complete information at time t. $V(t)$ is the population size of those who know the second piece of information but transmit ONLY the first piece of information at time t. It is now assumed that even such an individual, after getting the complete information, may get hardened in spreading the misleading information *with a probability* b, which is now defined as the transition from the state P to V.

The coefficient α_0 is for the transmission of complete information to U by those of W_0; α_+ is for the transmission of complete information to U by those of W_+; σ is the transition rate from V to W_+. It represents the change of thoughts to transmit the complete information after insisting on spreading the misleading first piece of information despite knowing the complete one before. ϵ is the increment of the transmission coefficient for the individuals of V, because of their tendency for information transmission psychologically stimulated or excited by receiving the second piece of information as mentioned in Section 2.

With a set of non-dimensionally transformed variables and parameters $u := U/N$, $p := P/N$, $v := V/N$, $w_0 := W_0/N$, $w_+ := W_+/N$, $\tau := \beta t$, $\tau_s := \beta t_s$, $a_0 := \alpha_0/\beta$, $a_+ := \alpha_+/\beta$, and $c := \sigma/\beta$, the system (2.5) becomes the following non-dimensionalized system for $\tau > \tau_s$:

$$\begin{aligned}
\frac{du}{d\tau} &= -pu - (1+\epsilon)vu - a_0 w_0 u - a_+ w_+ u; \\
\frac{dp}{d\tau} &= pu + (1+\epsilon)vu - a_0 w_0 p - a_+ w_+ p; \\
\frac{dv}{d\tau} &= b a_0 w_0 p + b a_+ w_+ p - cv; \\
\frac{dw_0}{d\tau} &= a_0 w_0 u + a_+ w_+ u; \\
\frac{dw_+}{d\tau} &= (1-b) a_0 w_0 p + (1-b) a_+ w_+ p + cv,
\end{aligned} \tag{2.10}$$

with $u(\tau) + p(\tau) + v(\tau) + w_0(\tau) + w_+(\tau) = 1$ for any $\tau > \tau_s$, and from (2.7) and (2.9), the initial condition

$$\begin{aligned}
&(u(\tau_s+0), p(\tau_s+0), v(\tau_s+0), w_0(\tau_s+0), w_+(\tau_s+0)) \\
&= ((1-\theta)u(\tau_s-0), (1-\theta)p(\tau_s-0), b\theta p(\tau_s-0), \theta u(\tau_s-0), (1-b)\theta p(\tau_s-0)),
\end{aligned} \tag{2.11}$$

where

$$p(\tau_s - 0) = \frac{1}{1 + \frac{U_0}{P_0} e^{-\tau_s}}; \quad u(\tau_s - 0) = 1 - p(\tau_s - 0). \tag{2.12}$$

2.5 TERMINAL STATE

The system (2.10) has three equilibrium states: $(u^*, p^*, v^*, w_0^*, w_+^*) = (1,0,0,0,0)$, $(0,1,0,0,0)$, and $(0,0,0,w_0^*, 1-w_0^*)$. By the standard local stability analysis, it can be easily shown that the first two equilibrium states are always unstable. In Appendix A.1, we show that the system necessarily converges to the third equilibrium state as indicated by the numerical calculations given in Figure 2.3. Thus, the terminal state of the information spread in our model is characterized by the terminal population size w_0^* of non-misinformed people, or alternatively the size w_+^* of misinformed people.

The introduction of rejoinder is to correct the misinformation. However, it is far more important to ensure that as many people as possible escape from being misinformed in the first place. This makes prevention our ultimate goal so that the value of w_0^* at the equilibrium state is a critical estimator of the efficiency of rejoinder introduction. The smaller w_0^* is, the more unfavorable it is owing to the aim for rejoinder introduction.

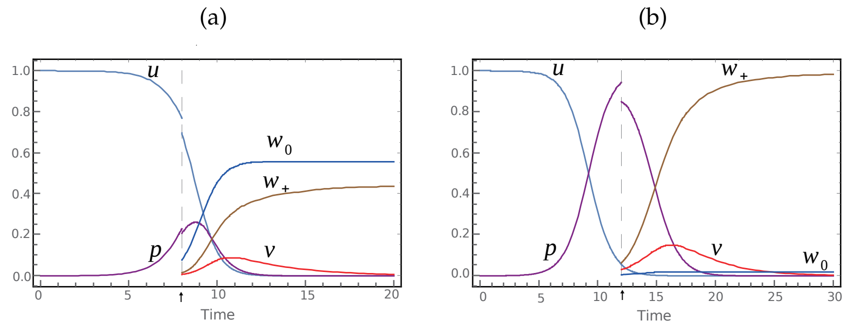


Figure 2.3: Temporal variation showing the primary stage for the spread of the misleading information as represented by (2.12) and the interaction stage after the rejoinder is introduced at $\tau = \tau_s$ represented by the system (2.10) with (2.11). (a) $\tau_s = 8.0$; (b) $\tau_s = 12.0$. Commonly, $p(0) = 0.0001$, $u(0) = 1 - p(0)$, $\theta = 0.1$, $\epsilon = 0.05$, $a_0 = 2.00$, $a_+ = 1.10$, $b = 0.33$, $c = 0.30$.

2.6 INSTANTANEOUS RESPONSE TO THE REJOINDER INTRODUCTION

In this section, we consider the response of the misled population just after the rejoinder introduction at $\tau = \tau_s$. In many cases involving the news media, the efficiency of an operation taken against a wrong information is likely to be estimated/criticized by a relatively short-term response following the introduction of such an action. However, we will show later that such short-term response could not be an appropriate index to estimate the efficiency of rejoinder introduction in our model.

For $dp(\tau)/d\tau < 0$ at $\tau \to \tau_s + 0$ such that $p(\tau)$ declines immediately after the rejoinder is introduced, we can get the following necessary and sufficient condition from (2.10):

$$p(\tau_s+0)[u(\tau_s+0) - a_0 w_0(\tau_s+0) - a_+ w_+(\tau_s+0)] + (1+\epsilon)v(\tau_s+0)u(\tau_s+0) < 0.$$

From the initial condition (2.11) with (2.12), this condition can be rewritten as follows:

$$\frac{1}{\theta} - \frac{1}{\theta_c} < a_+(1-b)\frac{P_0}{U_0}\left(e^{\tau_s} - 1\right), \tag{2.13}$$

where

$$\theta_c := \frac{1}{a_+(1-b)\frac{P_0}{U_0} + a_0 + (1-b) - \epsilon b}. \tag{2.14}$$

If $0 < \theta_c \leq \theta$, the inequality (2.13) holds independently of τ_s, so that the introduction of the rejoinder is highly efficient to immediately reduce the size of the misinformed subpopulation. When $0 < \theta < \theta_c$, the misled population size increases or decreases depending on τ_s (see Figure 2.4(a-2)). If $\theta_c < 0$, that is, if

$$a_+(1-b)\frac{P_0}{U_0} + a_0 < \epsilon b - (1-b),$$

such that ϵ or b is sufficiently large, there are sufficiently many misled people to actively spread the misleading information after knowing the complete one. In such a situation, the population size of misled individuals still increases after the introduction of the rejoinder independently of θ, if the rejoinder is introduced before the following critical moment τ_c, that is, when $\tau_s < \tau_c$ (Figures 2.3(a) and 2.4(a-1)):

$$\tau_c := \ln \left| 1 + \frac{1}{a_+(1-b)} \frac{\theta_c - 1}{\theta_c} \frac{U_0}{P_0} \right| . \tag{2.15}$$

It is interesting that for sufficiently small τ_s in such a case, the misled population size increases, no matter how small. That is, no matter how early the rejoinder is introduced and no matter how large the portion of people who get the correct information at that moment, the misled population size continues to increase even after rejoinder introduction.

Figure 2.4(b-1,2) shows the dependence of the instantaneous response on parameters ϵ and b, where $\epsilon_c := a_0 - 1/\theta$ and

$$b_c := 1 - \frac{\frac{1}{\theta} - a_0}{1 + a_+ \frac{P_0}{U_0} e^{\tau_s}} . \tag{2.16}$$

These results clearly imply that human psychological and sociological tendencies like skepticism or deliberate negligence towards the corrective information would contribute significantly to the instantaneous social response.

2.7 EFFICIENCY OF REJOINDER INTRODUCTION ON THE TERMINAL STATE

The numerical calculation as shown in Figure 2.5 indicates the existence of a specific range of τ_s for which a pronounced switch over of the value of w_0^* can be observed. Such a prominent range could not be identified for the dependence of the value of w_0^* on any other parameter, though the value of w_0^* depends on the other parameters in a much more moderate manner. As we can intuitively expect from the meanings of the parameters, the value of w_0^* monotonically increases in terms of θ, while it monotonically decreases in terms of b and ϵ (see numerical results given in Figure 2.6).

This result implies that the aim of keeping people away from being misinformed is significantly achieved when the rejoinder is introduced earlier than a certain critical period. Otherwise, when the rejoinder is launched later, it has little or no impact in suppressing the population of misinformed people. The critical period for the moment of rejoinder introduction depends on the other parameters as numerically shown in Figures 2.6(a–c), though the dependence appears rather weak. This implies that the moment of rejoinder introduction itself is the most relevant factor which affects the extent to which misinformation spreads.

Our result conjectures that introducing the rejoinder after a critical period leads to little effect on the terminal population size of non-misinformed people. In contrast, introducing it earlier than the

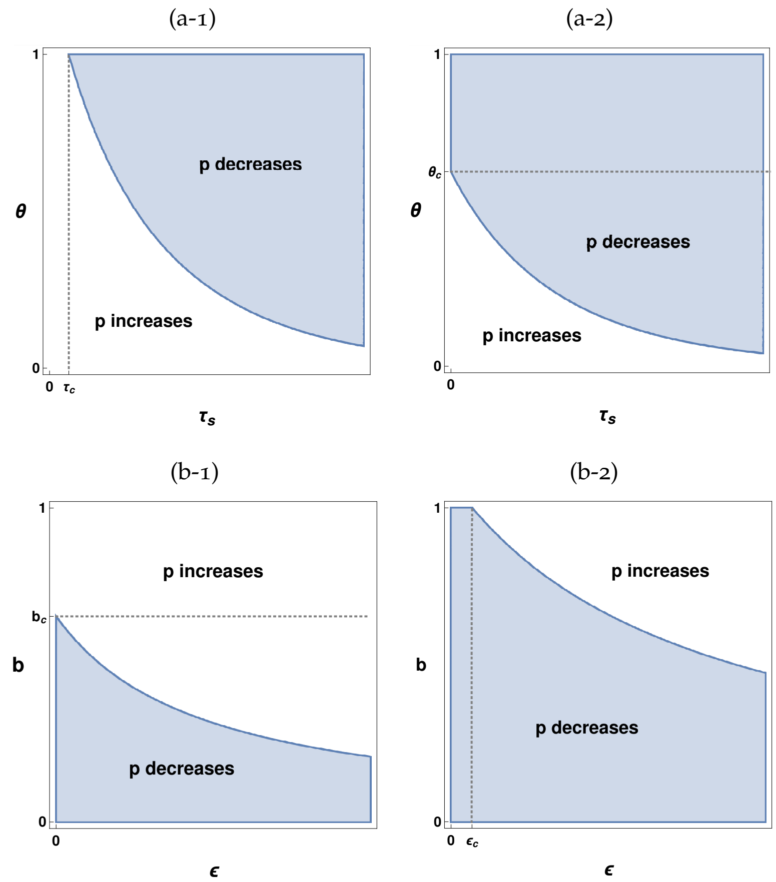


Figure 2.4: Parameter dependence of the instantaneous response to the introduction of rejoinder, based on the inequality (2.13): (a-1) $\theta_c < 0$ or $\theta_c \geq 1$; (a-2) $0 < \theta_c < 1$; (b-1) $a_0\theta < 1$; (b-2) $a_0\theta \geq 1$. The shaded parts are regions of decrease while the unshaded parts are regions of instantaneous increase of the misled population size just after the introduction of rejoinder.

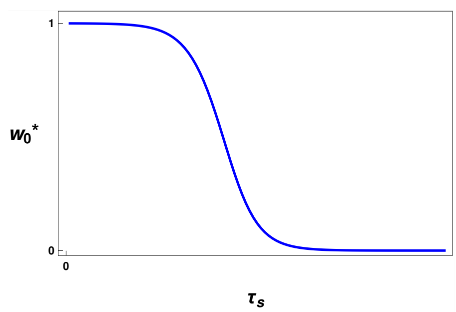


Figure 2.5: Dependence of the equilibrium value w_0^* on the moment of rejoinder introduction τ_s. A numerical result with $\theta = 0.10$, $b = 0.33$, $\epsilon = 0.05$, $p(0) = 0.0001$, $u(0) = 1 - p(0)$, $a_0 = 2.00$, $a_+ = 1.10$, and $c = 0.30$. w_+^* is given by $1 - w_0^*$.

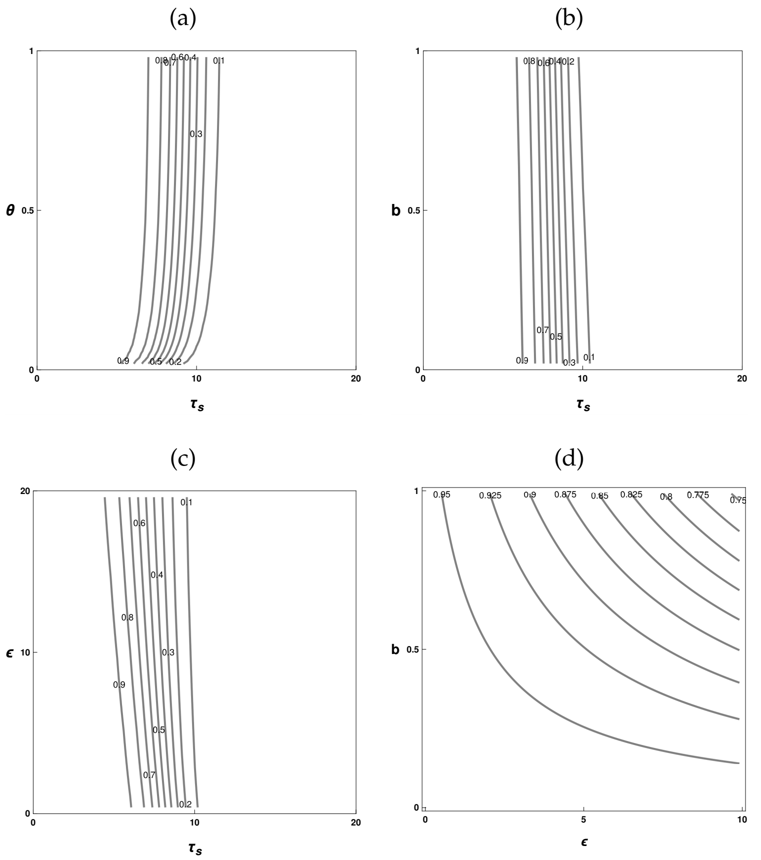


Figure 2.6: Contour plot of the dependence of w_0^* on (a) (τ_s, θ) with $b = 0.33$ and $\epsilon = 0.05$; (b) (τ_s, b) with $\theta = 0.10$ and $\epsilon = 0.05$; (c) (τ_s, ϵ) with $\theta = 0.10$ and $b = 0.33$; (d) (ϵ, b) with $\theta = 0.10$ and $\tau_s = 5$. Numerically drawn commonly with $p(0) = 0.0001$, $u(0) = 1 - p(0)$, $a_0 = 2.00$, $a_+ = 1.10$, and $c = 0.30$.

critical period may result in a rather large terminal population size of non-misinformed people. Releasing the rejoinder within the critical period, that is, the specific critical range of τ_s, the terminal population size of non-misinformed people is sensitively determined by the actual moment, so that the earlier introduction of rejoinder can result in significantly larger terminal population size of non-misinformed people.

2.8 DISCUSSION

The model proposed by Liu, Zeng, and Luo [51] shows that when there is increased spread of truth, the propagation of rumor can be obliterated; it was also seen that the rumor and the truth can coexist for long given certain circumstances. To some extent, these findings tend to agree with the results from our model since the introduction of a rejoinder and the possible cynicism of potential spreaders can accelerate or delay the elimination of misleading information. Feria et al. [21] established the importance of the early introduction of truth by relevant spreaders to make it endemic in a population. This seems to correspond with the early rejoinder introduction in our model, with similar effect. The instantaneous response of the population dynamics just after the rejoinder introduction does not match up to the consequence of interaction between pieces of information as shown by Figures 2.4 and 2.6. Although the earlier introduction of rejoinder can result in the more preferable consequence of saving people from being misinformed in the long run, it tends to cause such an instantaneous response that the misled people still increase after the rejoinder is introduced. This implies that the short-term response to rejoinder introduction would not be an appropriate index about its efficiency eventually.

A special case of the rejoinder model is that of cyber criminals who send phishing emails to customers of established organizations in order to harvest vital data. This case is special because $b = 0$ as every misinformed person will almost surely accept the correct information from the organization about the activities of fraudsters. It should be noted that b measures the effect of personal belief in the correctness of a rejoinder.

The rejoinder model has some similarities with the classical Kermack-McKendrick SIR model about the population dynamics of transmissible diseases. For instance, the population sizes of non-knowers (U), knowers and transmitters of the misleading piece of information (P, V), knowers and transmitters of the complete information (W_0, W_+) in the rejoinder model correspond respectively with the population sizes of susceptibles (S), infectives (I), removed (R) in the SIR model. The population size W_0 can be considered, for example, as those who are shielded from infection through vaccination while W_+ are like those who develop natural immunity having been previously infected. Though the two models are similar in structure, they are different in dynamics. This is demonstrated by the fact that the population size

R in the SIR model has no effect on the population sizes S and I. However, in the rejoinder model, the population sizes W_0, W_+ have direct impact on the population sizes U, P and V.

Our model shows the effect of mass action that better estimates interactions on the Internet in comparison with earlier rumor models which are more biased towards network theory. This is because people are not necessarily connected following the traditional theory of networks [12]. On social media platforms like Facebook, Instagram, Twitter, etc., there are lots of misleading information about governmental and non-governmental organizations. Sometimes, there are also misleading information from such institutions in form of propaganda [43]. So, it has become imperative to be able to tell apart correct and wrong information. It has been widely agreed that the problems of misinformation and disinformation can be mitigated by promoting information literacy through multidisciplinary collaborations (see [44, 45, 73]).

The extension of our information spread model to accommodate human heterogeneity in the handling of information is shown in the threshold model in Chapter 4. The concept is based on Granovetter's threshold model discussed in Chapter 3.

3

GRANOVETTER'S THRESHOLD MODEL

Granovetter [28, 30] highlight the strength of weak ties by reviewing many interesting studies in that line. The overall idea is that, in certain scenarios, strong ties which exist in closely-knitted relationships (e.g. family, friends) may not be as important as weak ties which exist in loose relationships (e.g. acquaintances, friends of friends). For instance, close friends have a tendency to always access the same kind of ideas and as such they are apparently closed up. However, weak ties may be important in injecting new ideas into such circles, thereby making them better aware. The importance of weak ties can be seen in the spread of information about innovation, jobs, culture, goods, services as well as integration of diverse groups.

Granovetter [29] developed the models of collective behavior for circumstances where people have to make one of two clearly different choices such that the merit or demerit of each choice depends on the number of individuals who decide for or against it. The whole idea is that of a threshold proportion of people taking a course of action before a particular person does the same. When this threshold is reached, the advantages of taking the decision begin to outweigh the disadvantages for the given individual. Threshold here is analogous to credulity and vulnerability in the spread of rumors and diseases respectively. The models find application in areas like diffusion of innovation, public protests, migration, voting, market trends and information spread. For instance, a radical who is capable of single-handedly starting a riot can be said to have a threshold of 0% as they are able to riot even if nobody else toes that line. On the other hand, a conservative might have a threshold close to 100% depending on their level of reluctance to join a riot. As a sociological concept, the Granovetter model has some similarities with the idea of behavioral contagion in psychology and the cultural phenomenon of bandwagon effect.

As an extension of the 1978 work, Granovetter and Soong [31] emphasized the reality of complex heterogeneity in collective behavior as opposed to the earlier simplifying assumptions of homogeneous individuality and mixing in the adoption and spread of ideas. They showed the importance of threshold models as lying in the not-so-simple connection between individual choices and overall steady results. The work also refers to the importance of bandwagon effect in which people adopt a new concept because a given number of people are into it and a snob effect in which some people drop the idea once a certain number of people sign up. In this case, there are two threshold values: one minimum inspiring the bandwagon and one maximum leading to snobbish behavior.

The relevance of threshold models in economics is seen in the occurrence of interpersonal influence on the demand for goods and services as highlighted by Granovetter and Soong [32]. They showed that someone may buy or not buy depending on the demand of people around them for the same commodities. A model was formulated by assuming disparities in individuals in the way they react to the behavior of others. The impact of price changes on these threshold behaviors was also considered. It was discovered that markets could still be unpredictable in spite of perfect competition. [75] presents a difference equation model that mathematically represents Granovetter's idea as

$$r(k+1) = F(r(k)), \tag{3.1}$$

where r is the percentage of members of a group who can take a decision, $F(\xi)$ is the cumulative distribution of members who have threshold value less than ξ.

From the foregoing, the overarching assumption of the Granovetter model is that the strength $Q = Q(P)$ of social effect on decision making is proportional to the ratio P of those who have already taken the decision within the population, that is $Q(P) = \alpha P$ where α is a positive constant and $0 \leq \alpha P \leq \alpha$ since $0 \leq P \leq 1$. We assuming that $Q(P)$ is always increasing though that is not always the case. More so, there is a certain threshold value ξ for the strength $Q(P)$ such that

$$\begin{cases} \xi \leq Q(P) \implies \text{decision may be made;} \\ \xi > Q(P) \implies \text{decision is not made.} \end{cases} \tag{3.2}$$

The cumulative distribution function (CDF) for individual threshold value ξ in the population can be expressed as

$$F(z) = \text{Prob}(\xi \leq z) = \int_{-\infty}^{z} f(\xi) d\xi. \tag{3.3}$$

$F(z)$ is the proportion of individuals with threshold value less than z and $f(\xi)$ is the frequency distribution function (FDF) of the threshold value.

Let us assume a total population size N; proportion P_t of those who have made the decision at time t; $Q(P_t) = \alpha P_t$ which is the strength of the social effect at time t and $F(\alpha P_t)$, the proportion of individuals with threshold value less than αP_t. Also, let

$$P_0 = \int_{-\infty}^{\infty} \theta(\xi) f(\xi) d\xi \tag{3.4}$$

which is the proportion of decision takers at time $t = 0$ and $0 \leq \theta(\xi) \leq 1$ is the ratio/proportion of initial decision takers with threshold value ξ.

Suppose that γ is the probability of decision making by an individual under the condition that the threshold is less than the strength of social effect of the decision and $B(P_t)$ is the probability or likelihood of an individual to have a chance of making the decision, the increase in the number of decision takers in the interval $[t, t+1)$ can be expressed as

$$\begin{aligned}
P_{t+1}N - P_tN &= \gamma B(P_t)\left[NF(\alpha P_t) - N\int_{-\infty}^{\alpha P_t}\theta(\xi)f(\xi)d\xi - N(P_t - P_0)\right] \\
P_{t+1} - P_t &= \gamma B(P_t)\left[F(\alpha P_t) - \int_{-\infty}^{\alpha P_t}\theta(\xi)f(\xi)d\xi - (P_t - P_0)\right] \\
&= \gamma B(P_t)\left[F(\alpha P_t) - \int_{-\infty}^{\alpha P_t}\theta(\xi)f(\xi)d\xi - P_t + \int_{-\infty}^{\infty}\theta(\xi)f(\xi)d\xi\right] \\
&= \gamma B(P_t)\left[F(\alpha P_t) + \int_{\alpha P_t}^{\infty}\theta(\xi)f(\xi)d\xi - P_t\right]. \qquad (3.5)
\end{aligned}$$

If we assume uniform distribution such that the FDF is given as

$$f(\xi) = \begin{cases} 0, & \xi \leq 0; \\ \frac{1}{\alpha}, & 0 < \xi < \alpha; \\ 0, & \xi \geq \alpha \end{cases} \qquad (3.6)$$

with the corresponding CDF

$$F(z) = \begin{cases} 0, & z \leq 0; \\ \frac{z}{\alpha}, & 0 < z < \alpha; \\ 1, & z \geq \alpha. \end{cases} \qquad (3.7)$$

Going by (3.7), we have $F(\alpha P_t) = \alpha P_t/\alpha = P_t$. In addition, if $\theta(\xi)$ takes a uniform value θ_0 meaning that initial decision takers are set up for each threshold value independently of the threshold value, then (3.5) becomes

$$\begin{aligned}
P_{t+1} &= P_t + \gamma B(P_t)\left[P_t + \theta_0\int_{\alpha P_t}^{\infty}f(\xi)d\xi - P_t\right] \\
&= P_t + \gamma B(P_t)\theta_0\left[\int_{\alpha P_t}^{\infty}f(\xi)d\xi\right] \\
&= P_t + \gamma B(P_t)\theta_0\left[\int_{\alpha P_t}^{\alpha}\frac{1}{\alpha}d\xi\right] \\
&= P_t + \gamma B(P_t)\theta_0(1 - P_t). \qquad (3.8)
\end{aligned}$$

Assuming that $B(P_t)$ is proportional to P_t meaning that the higher the proportion of decision takers at time t, the higher the likelihood of others to have a chance to make the decision, that is, $B(P_t) = kP_t$ $(0 < k \leq 1)$ so that we have

$$P_{t+1} = P_t + k\gamma\theta_0 P_t(1 - P_t). \qquad (3.9)$$

Furthermore,

$$P_{t+1} = k\gamma\theta_0 \left(1 + \frac{1}{k\gamma\theta_0}\right) P_t \left(1 - \frac{P_t}{1 + \frac{1}{k\gamma\theta_0}}\right)$$

so that if $X_t = P_t / \left(1 + \frac{1}{k\gamma\theta_0}\right)$, we now have the logistic map

$$X_{t+1} = AX_t(1 - X_t) \tag{3.10}$$

where $A = 1 + k\gamma\theta_0$ and $0 < A < 2$. We see that as $t \to \infty$, $X_t \to X^* = 1 - \frac{1}{1+k\gamma\theta_0}$ and $P_t \to P^* = 1$. Logistic maps like (3.10) find applications in the biological and social sciences and they have an array of interesting dynamical behaviors like stable points, stable cycles and chaos [54, 55].

Kaempfer and Lowenberg [39] pointed out the importance of individuals contributing their quota in collective action towards public good. It was also highlighted that free riders are liabilities when it comes to the collective action. The crux of the work was to find out how international persuasion is capable of inspiring collective action in target countries. The authors tried to see how foreign economic policies can raise critical masses for desired change in countries of interest. They observed a myriad of ways in which influences from abroad catalyze local groups to overcome barriers in organizing collective action for political change. Taking a cue from the Ryan-Gross adopter categories, Valente [72] formulated a social network threshold model of innovation spread. The model used social networks for categorizing early adopters, early majority, late majority and laggards. It was established that these four adopter categories can be based on the entire social system or an individual's personal network. Overall, it was shown that network threshold models can explain the success or failure of collective action and the diffusion of innovations. Delre, Jager, and Janssen [15] emphasized the possibility of formalizing the spread of novel products and technologies through social networks as the diffusion of infectious diseases. They presented a model of spread of decision making influenced by social connections and word of mouth. They revealed that the speed of transmission changes with the the degree of randomness in social networks. It was also shown that population heterogeneity enhances diffusion. These discoveries are seen to be quite helpful to marketing experts who want to introduce new products and services in unpredictable and stylish markets.

With a discrete dynamical system, Bischi and Merlone [5] modeled two-way choice games with external influences. They introduced an explicit adjustment technique for the analysis of some oscillatory time sequence and problems regarding selection of equilibrium. They were able to carry out analyses that explained extreme situations in social systems and the attendant cyclical behaviors. [10] highlights the relevance of statistical physics to other areas of learning apart from physics. The application of concepts in the field to the study of collective behavior in social systems was seen to be fast emerging. The work delved into many interesting areas of social dynamics like how hierarchies come about, how people spread out and how languages evolve. Their model results were compared with empirical data from social systems. Assuming a network that is random and non-finite with weak connections, Whitney [74] tried to understand diffusion (of information or innovation) on the network using generating functions. The theory proposed is based on a threshold rule which ensures that a node only changes state after a fraction of nearby nodes, surpassing a particular limit, have previously flipped over. Some of the results obtained from the Markov model show behaviors that are novel compared to previous simulations.

Working on a generalized linear threshold model, which happens to be a quickly blending Markov process, Pathak, Banerjee, and Srivastava [60] examined multifarious cascades in a grid which allows switching of nodes. The equilibrium states are used to gauge the most possible situations of the diffusion of cascades in the network. The results are seen to perform very well in actual circumstances. In order to make sense of the concept of social influence, Dodds and Watts [17] reasoned that it can be viewed as a result of making decisions based on a series of binary possibilities. They showed that binary choices receive a lot attention by many social scientists compared to other choice types. This has led to popular sociological and economic models like diffusion, segregation, coordination, social learning, Welch, threshold and generalized contagion models. Most of these models, however, were seen to not provide an encompassing framework for theoretical studies. As such, it was necessary to clarify model assumptions and how models are inter-related. They concluded that Granovetter's model was more generalizable compared to others since they allow for the heterogeneity of each individual and their equilibrium states can easily be found.

Akhmetzhanov, Worden, and Dushoff [1] extended the Granovetter model to consider a network of individuals in a square lattice with each one having a state and a specified threshold for change in behavior. Asynchronous system simulation was done by picking an individual at random and updating its state or switching it with another individual selected at random, thereby giving rise to mixing. The evolution of the system is described analytically in the fast-mixing limit via mean-field approximation and it is checked numerically under finite mixing. The dynamics was seen to approach a state space manifold determining possible equilibrium states. It was revealed that the impact of the network can be grouped under finite-neighborhood effects and finite-mixing-rate effects which are equally prone to making the system move to the ground state from a less desirable equilibrium. Changes in attitude and the impact of information diffusion on the persistence of antisocial practices can be better understood by their results. A utility-psychological threshold model based on the Granovetter's threshold model was introduced by Li and Tang [49]. They studied the critical shift in phase of group behavior by taking into account rational utility and psychological thresholds under the influence of space and intensity of social network. They discovered that the model shows more stability in phase transition compared to classic models. It was also seen that space and social network have negligible influence on the steady state of group actions.

With a simple social behavior model, Shrestha and Moore [67] considered the manner in which fads and viewpoints advance in networks in which an individual only embraces the trend after being informed by a threshold number of adopter neighbors. A reliable method which provides complete time evolution of each person's chance of adopting the trend in a random network was developed using a message-passing procedure. The technique is robust enough to accommodate different types of scenarios. Robertson [63] points out that personal and group inclinations determine a person's involvement with social trends. It is emphasized that dispositions towards the likes of the Arab Spring uprising, Kickstarter promotions and online memes can be modeled according to Granovetter's model of collective behavior in which each individual require a critical level of engagement by others before they can join. The spread of thresholds as determined by the mean and standard deviation of thresholds are seen to determine the variety of results obtainable. The work establishes that for a social movement to thrive or wane, instigators are needed to drive the process. The shape of the threshold distribution also has a huge role to play. A methodical appraisal of seven threshold models of transitions in technology was carried out by Zeppini, Frenken, and Kupers [77]. They considered economic factors like price, performance and profit vis-à-vis social factors like expectations, verbal referrals and influence that explain technological transitions. The models (which may serve as bases for specific models of interest) for such evolution in technology are identified to include hyperselection, increasing returns, informational cascades, coordination game, co-evolution, percolation and social influence. Specific to each model is a transition threshold which is determined by a critical mass, a fitness value or a critical price.

Garulli, Giannitrapani, and Valentini [24] considers the asymptotic behaviors of threshold models of collective action in social systems in which each agent is confronted with a binary decision. The chosen option depends on the decision made by neighboring agents as well as a constantly updated threshold. A measure of self-confidence is introduced in the model which has impact on threshold changes and decision making by individuals. The limiting characteristics of the network and their causative conditions are observed. As highlighted by Gao et al. [23], the behavior of a cascade is critically determined by the given cascade model or societal impact as well as the topology of the social network. They examined the general threshold model which captures many models that have been studied in the past, namely independent cascade model, the linear threshold model and the k-complex model, among others. They gave analytical and empirical results for cascades from the model to diffuse in a growing network with preferential attachment and generalized cases, showing that if initial seeds are taken as early arriving nodes, contagion can spread to a critical proportion of the network determined by the stationary points of a function obtained exclusively from the given distribution.

It was established that the stochastic attachment graph model was a better estimator of contagion behavior on real data sets compared to configuration models.

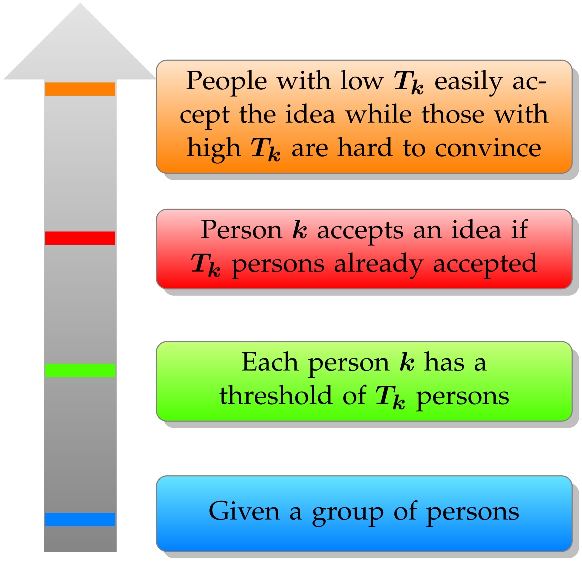
Gavrilets and Richerson [25] designed a model in which individuals who decide to take part in a collective action try to maximize a situation which depends on the possibility of imbibing the prevailing norms. It is found that cooperation seemingly becomes second nature when participation is encouraged and non-compliant members of the group are punished. While average levels of norm acceptance are more common, there are extreme cases of under-socialization and over-socialization, not minding the cost of such behaviors. For wide reaching collaboration among humans, it is seen that norm internalization ability is imperative. Rossi, Como, and Fagnani [64] pointed out that the diffusion of new ideas in socio-economic systems are usually directed by cascading effects thereby following the principle of contagion. With attention to the threshold model of cascades based on Granovetter's work, they analyzed threshold model dynamics on large-scale networks with nonhomegenous agents. By using a local mean-field approach, a one-dimensional, non-linear recursive equation was obtained to estimate the changing states of the system dynamics of networks of given sizes, degrees of distribution and thresholds. They obtained numerical results that quite agree with their theory that on most networks which become trivial with increase in size, the proportion of adopters of an action is randomly near the output of the observed recursion. Owing to the fact that the idea of collective behavior is complicated due to the non-linear interactions that occur between individuals, Marshall, Reina, and Bose [53] designed a framework which would not only be accessible to sophisticated experts. Their modeling tool known as MuMoT (Multiscale Modeling Tool) is designed to facilitate experts' rigorous analysis and to make such near-elusive modeling accessible to people with basic knowledge.

4

THRESHOLD DISTRIBUTION MODEL FOR THE DYNAMICS OF INFORMATION SPREAD

Based on the idea of Granovetter's model, we proceed to formulate a population dynamics threshold model for information spread.

4.1 SET-UP OF THE MODEL



4.1.1 *Assumptions*

H1 There is a piece of information spreading within a population with a given strength of social recognition effect Q;

H2 The strength of social recognition effect Q increases with the proportion/frequency P of knowers of the information;

H3 Each individual has a threshold value ξ which determines whether the information is accepted or ignored by that individual;

H4 The threshold value ξ, which characterizes each individual, is constant independently of time and social situation;

H5 Every knower keeps transmitting the information at any time t;

H6 Knowers never return to being non-knowers.

4.1.2 *Model formulation*

$Q = Q(P)$: the strength of the social recognition effect, which is a function of the frequency P of knowers in the population. It is assumed to be non-decreasing in terms of P; $Q(0) = 0$, $Q \geq 0$.

ξ: the threshold value for Q, specifying the individual independently of time.

$$\begin{cases} \xi \leq Q \longrightarrow \text{The individual may accept the information to transmit to others;} \\ \xi > Q \longrightarrow \text{The individual ignores the information.} \end{cases}$$

The value of ξ is generally defined on $(-\infty, \infty)$. Persons with negative threshold values always satisfy the first rule thereby being prone to the possibility of accepting the information.

$F(x)$: the cumulative distribution function (CDF) of the threshold value ξ in the population.

$$F(x) = \int_{-\infty}^{x} f(\xi) d\xi$$

where $f(\xi)$ is the frequency distribution function (FDF) of the threshold value ξ in the population, such that $f(\xi)\Delta\xi$ with sufficiently small $\Delta\xi > 0$ means the frequency of individuals with the threshold in the range $[\xi, \xi + \Delta\xi]$ in the population. The value $F(x)$ means the frequency of individuals with the threshold value ξ not beyond x in the population.

If we assume that $f(\xi) = 0$ for any $\xi > \xi_m$ with a finite value ξ_m, then ξ_m is the upper bound for the threshold value ξ.

The functions F and f satisfy the conditions:

- F and f are independent of time t;
- $f(\xi)$ is non-negative and integrable for any $\xi \in \mathbb{R}$;
- $F(x)$ is non-negative, non-decreasing, and continuous for any $x \in \mathbb{R}$;
- $\lim_{x \to \infty} F(x) = \int_{-\infty}^{\infty} f(\xi) d\xi = 1$ and $\lim_{x \to -\infty} F(x) = 0$;
- $\lim_{\xi \to \infty} f(\xi) = 0$ and $\lim_{\xi \to -\infty} f(\xi) = 0$.

$P(t)$: the frequency of knowers in the population at time t.

$$P(t) = \int_{-\infty}^{\infty} p(\xi, t) d\xi,$$

where $p(\xi, t)$ is the FDF of knower's threshold value ξ in the population, such that $p(\xi, t)\Delta\xi$ with sufficiently small $\Delta\xi > 0$ is the frequency of knowers with the threshold in the range $[\xi, \xi + \Delta\xi]$ at time t in the population.

$U(t)$: the frequency of non-knowers in the population at time t.

$$U(t) = \int_{-\infty}^{\infty} u(\xi, t) d\xi,$$

$$P(t) + U(t) = 1$$

independently at time t, and further

$$p(\xi, t) + u(\xi, t) = f(\xi)$$

for any $\xi \in \mathbb{R}$ and any t.

$\Xi(P)$: the set of threshold values satisfying $\xi \leq Q(P)$, defined as follows:

$$\Xi(P) := \{\xi \mid \xi \leq Q(P)\},$$

and the complementary set of $\Xi(P)$, $\overline{\Xi}(P)$, is defined by

$$\overline{\Xi}(P) := \{\xi \mid \xi > Q(P)\}.$$

$\mathscr{B}(\xi, P)\Delta t$: the transition probability that the non-knower with the threshold value ξ gets the information and transits to the knower population in $[t, t + \Delta t]$ with sufficiently small Δt. $\mathscr{B}(\xi, P)$ is the coefficient of information transmission under the situation with the knower frequency P given by

$$\mathscr{B}(\xi, P) = \begin{cases} B(P), & \xi \in \Xi(P); \\ 0, & \xi \in \overline{\Xi}(P). \end{cases}$$

Now, $B(P)$ is the coefficient of information transmission for the non-knower with the threshold value of $\Xi(P)$ with $B(0) = 0$, $B(P) > 0$ for $P \in [0, 1]$.

4.1.3 *Temporal change of the non-knower frequency*

From the above setup, we can immediately get the following equation

$$u(\xi, t + \Delta t)\Delta\xi - u(\xi, t)\Delta\xi = -\mathscr{B}(\xi, P(t))\Delta t \cdot u(\xi, t)\Delta\xi,$$

where the left side means the change of the frequency of non-knowers during $[t, t + \Delta t]$ with sufficiently small Δt, with the threshold value in the range $[\xi, \xi + \Delta\xi]$ with sufficiently small $\Delta\xi$. It corresponds to the number of non-knowers becoming knowers by getting the information, so that it should be equal to the right hand side given the expected reduction of the non-knower frequency by the transition probability defined above.

From the above equation, we can get the following equations as $\Delta t \to 0$,

$$\frac{\partial u(\xi, t)}{\partial t} = -\mathscr{B}(\xi, P(t))u(\xi, t).$$

Therefore, integrating both sides in terms of ξ over $\mathbb{R}$, we have

$$\begin{aligned} \frac{d}{dt}\int_{-\infty}^{\infty} u(\xi, t)d\xi &= \frac{dU(t)}{dt} \\ &= -\int_{-\infty}^{\infty} \mathscr{B}(\xi, P(t))u(\xi, t)d\xi = -\int_{\Xi(P(t))} B(P(t))u(\xi, t)d\xi. \end{aligned}$$

4.1.4 *Temporal change of the knower frequency*

$$\frac{dP(t)}{dt} = -\frac{dU(t)}{dt} = B(P(t)) \int_{\Xi(P(t))} u(\xi, t) d\xi$$

We do not assume the knower stops transmitting information at any time or under any condition. That is, the knower remains at the knower state for any time t, and the frequency of knowers does not decrease in time. Hence, correspondingly the frequency of non-knowers does not increase with time.

4.1.5 *Initial condition*

As the initial condition at $t = 0$, we assume a portion of knowers in the population, who play the role of initial transmitters of information. We now assume that the initial knowers are given independently of their threshold values, aside how they become knowers.

We give the initial distribution of the knower frequency by

$$p(\xi, 0)\Delta\xi = \theta(\xi) f(\xi)\Delta\xi,$$

where $\theta(\xi)$ determines the ratio of initial knowers in the subpopulation with the threshold value ξ, such that $0 \leq \theta(\xi) \leq 1$. Since $p(\xi, 0)\Delta\xi + u(\xi, 0)\Delta\xi = f(\xi)\Delta\xi$, we have the following about the intial distribution of the non-knower frequency at the same time:

$$u(\xi, 0)\Delta\xi = \{1 - \theta(\xi)\} f(\xi)\Delta\xi.$$

Consequently, we have

$$P(0) = \int_{-\infty}^{\infty} \theta(\xi) f(\xi) d\xi; \quad U(0) = \int_{-\infty}^{\infty} \{1 - \theta(\xi)\} f(\xi) d\xi = 1 - P(0). \tag{4.1}$$

4.1.6 *Non-knower frequency of* $\overline{\Xi}(P)$

When there are such non-knowers that have the threshold value over the value of Q at time t, the frequency is given by

$$U(t) - \int_{\Xi(P(t))} u(\xi, t) d\xi = \int_{\overline{\Xi}(P(t))} u(\xi, t) d\xi,$$

with a non-empty set $\overline{\Xi}(P(t))$. From the assumption H6, $P(t)$ is non-decreasing in time. Since $P(t)$ is non-decreasing in time and $Q(P)$ is non-decreasing in terms of P, we note that the non-empty set $\overline{\Xi}(P(t))$ identifies all non-knowers who have not experienced any moment at which the value of $Q(P(t))$ is more than the threshold value until time t. Thus, the above integral gives the frequency of such non-knowers at time t when $\overline{\Xi}(P(t))$ is non-empty.

Therefore, when $\overline{\Xi}(P(t))$ is non-empty, the non-knowers belonging to the above integral is only those who remain at the non-knower state from the initial time to time t. That is,

$$u(\xi, t) = u(\xi, 0) \quad \text{for} \quad \xi \in \overline{\Xi}(P(t)),$$

so that

$$\int_{\overline{\Xi}(P(t))} u(\xi,t)d\xi = \int_{\overline{\Xi}(P(t))} u(\xi,0)d\xi = \int_{\overline{\Xi}(P(t))} \{1-\theta(\xi)\}f(\xi)d\xi.$$

4.1.7 *Closed equation for the knower frequency*

From the preceding modeling arguments, we have

$$\begin{aligned}\frac{dP(t)}{dt} &= B(P(t)) \int_{\Xi(P(t))} u(\xi,t)d\xi \\ &= B(P(t)) \left\{ U(t) - \int_{\overline{\Xi}(P(t))} u(\xi,t)d\xi \right\} \\ &= B(P(t)) \left\{ U(t) - \int_{\overline{\Xi}(P(t))} \{1-\theta(\xi)\}f(\xi)d\xi \right\} \\ &= B(P(t)) \left[1 - P(t) - \int_{\overline{\Xi}(P(t))} \{1-\theta(\xi)\}f(\xi)d\xi \right]. \end{aligned} \tag{4.2}$$

The equation is closed in terms of $P(t)$, which can be regarded as an autonomous ordinary differential equation to describe the temporal change of knower frequency in the population.

Now we note that the more concrete formula of the above equation depends on the limit $\lim_{P(t)\to 1} Q(P(t))$ which is now formally equal to $\sup_{P(t)} Q(P(t))$ because the function of $Q(P(t))$ is assumed to be non-decreasing in terms $P(t)$. Here we take account of the case when $\lim_{P(t)\to 1} Q(P(t)) = \infty$.

If $\lim_{P(t)\to 1} Q(P(t)) < \infty$, that is, if $Q(1)$ is a finite value, the set $\overline{\Xi}(P(t))$ becomes empty for a certain value of $P(t) = P_c \leq 1$, when $f(\xi) = 0$ for any $\xi > \xi_m$ with a finite value ξ_m, and $Q(P_c) \geq \xi_m$. In such a case, the integral in the last formula of the above equation necessarily becomes zero because $f(\xi) = 0$ for any $\xi \geq Q(P_c)$. Therefore, the above equation holds even when the set $\overline{\Xi}(P(t))$ is empty. Now, we can express (4.2) as

$$\frac{dP(t)}{dt} = \begin{cases} B(P(t)) \left[1 - P(t) - \displaystyle\int_{Q(P(t))}^{\xi_m} \{1-\theta(\xi)\}f(\xi)d\xi \right], & Q(P(t)) < \xi_m; \\ B(P(t)) \Big[1 - P(t) \Big], & Q(P(t)) \geq \xi_m. \end{cases} \tag{4.3}$$

4.2 INVARIANCE OF $P(t)$

To guarantee the limits of $P(t)$ for the reasonableness of (4.2), we have the following theorem (Appendix B.1).

Theorem 4.2.1. *For any $P(0)$ such that $0 \leq P(0) \leq 1$, $P(t)$ satisfies the condition that $0 \leq P(t) \leq 1$ for all $t > 0$.*

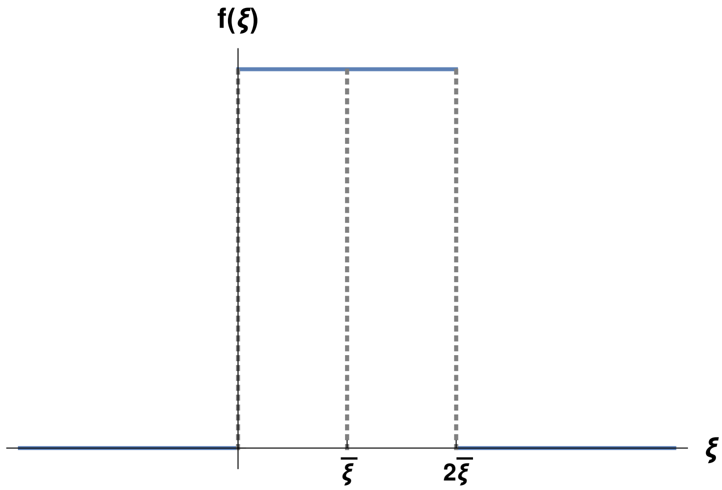


Figure 4.1: Graph of the frequency distribution function $f(\xi)$ against the threshold value ξ of the social recognition effect given by (4.4).

4.3 THE CASE OF COMPACT SUPPORT UNIFORM DISTRIBUTION

In this case, the distribution of ξ is uniform with $f(\xi)$ given as

$$f(\xi) = \begin{cases} 0, & \xi < 0; \\ \frac{1}{2\bar{\xi}}, & 0 \leq \xi \leq 2\bar{\xi}; \\ 0, & \xi > 2\bar{\xi}, \end{cases} \tag{4.4}$$

with mean $\bar{\xi}$. The graph of this frequency distribution is seen in Figure 4.1.

Based on this $f(\xi)$, (4.3) can be expressed as

$$\frac{dP(t)}{dt} = \begin{cases} B(P(t)) \left[1 - P(t) - (1-\theta_0) \left(\int_{Q(P(t))}^{2\bar{\xi}} \frac{1}{2\bar{\xi}} d\xi \right) \right], & Q(P(t)) \leq 2\bar{\xi}; \\ B(P(t))[1 - P(t)], & Q(P(t)) > 2\bar{\xi}, \end{cases}$$

so that

$$\frac{dP(t)}{dt} = \begin{cases} B(P(t)) \left[\theta_0 - P(t) + \frac{1}{2\bar{\xi}}(1-\theta_0) Q(P(t)) \right], & Q(P(t)) \leq 2\bar{\xi}; \\ B(P(t))[1 - P(t)], & Q(P(t)) > 2\bar{\xi}. \end{cases}$$

If we define $Q(P(t)) := \alpha P(t)$, we have

$$\frac{dP(t)}{dt} = \begin{cases} B(P(t)) \left[\theta_0 - \left(1 - \frac{\alpha}{2\bar{\xi}}(1-\theta_0) \right) P(t) \right], & \alpha P(t) \leq 2\bar{\xi}; \\ B(P(t))[1 - P(t)], & \alpha P(t) > 2\bar{\xi}. \end{cases} \tag{4.5}$$

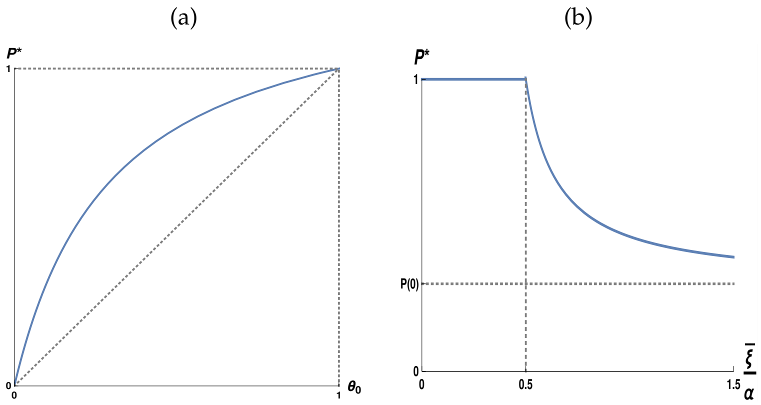


Figure 4.2: Bifurcation diagram for P^* with parameter (a) θ_0, $\bar{\xi}/\alpha > 1/2$ and (b) $\bar{\xi}/\alpha$ based on (4.6) and (4.8). $P(0) = \theta_0$.

For $P(t) \leq \frac{2\bar{\xi}}{\alpha}$, the equilibrium state in the interval $\left(0, \frac{2\bar{\xi}}{\alpha}\right]$ is determined by

$$\mathcal{H}_1(P) = \theta_0 - \left(1 - \frac{\alpha}{2\bar{\xi}}(1-\theta_0)\right) P = 0 \tag{4.6}$$

such that

$$P = P^* = \frac{\theta_0}{1 - \frac{\alpha}{2\bar{\xi}}(1-\theta_0)}. \tag{4.7}$$

It is necessary that $\bar{\xi}/\alpha > (1-\theta_0)/2$ for this equilibrium state to be positive as required.

For $P(t) > \frac{2\bar{\xi}}{\alpha}$, the equilibrium state in the interval $\left(\frac{2\bar{\xi}}{\alpha}, 1\right]$ is determined by

$$\mathcal{H}_2(P) = 1 - P = 0 \tag{4.8}$$

and it is $P = P^* = 1$.

Lemma 4.3.1. *When $\bar{\xi}/\alpha > 1/2$, the equilibrium state (4.7) exists in $(0,1)$ and it is necessarily locally asymptotically stable.*

Lemma 4.3.2. *When $\bar{\xi}/\alpha \leq 1/2$, there is no equilibrium state in $(0, 2\bar{\xi}/\alpha) \subseteq (0,1)$. On the other hand, there is the equilibrium state $P = P^* = 1$. P increases monotonically with time so that $P = P^* = 1$ is globally stable.*

Theorem 4.3.3. *The system (4.5) has a unique equilibrium state which is globally asymptotically stable.*

Proof. When $\bar{\xi}/\alpha > 1/2$, (4.7) is always positive. Since $\mathcal{H}_1(0) = \theta_0 > 0$, then P^* is locally asymptotically stable. It is clear that there is no equilibrium state in $[2\bar{\xi}/\alpha, 1]$ since $\bar{\xi}/\alpha > 1/2$.

When $\bar{\xi}/\alpha \leq 1/2$, the equilibrium state (4.7) is non existent since it becomes negative. As such the equilibrium state is in $[2\bar{\xi}/\alpha, 1]$. For all $P < 1$, $\mathcal{H}_2(P)$ is positive and so P is monotonically increasing with time.

Given that $\bar{\xi}/\alpha \leq 1/2$ and that P is monotonically increasing with an upper bound $P = 1$, then we have the equilibrium state $P = P^* = 1$ which is asymptotically stable.

Overall, we have a unique global equilibrium state for the system (4.5) depending on $2\bar{\xi}/\alpha$. For $\bar{\xi}/\alpha > 1/2$, $P \to P^*$ given by (4.7). For $\bar{\xi}/\alpha \leq 1/2$, $P \to P^* = 1$. □

$\bar{\xi}/\alpha > 1/2$ represents a high threshold value for knowing and accepting the information. People with such high threshold values are so difficult to convince that they never accept the information. This may be due to a high level of education or the perceived lack of trustworthiness of the source of information. On the other hand, with $\bar{\xi}/\alpha \leq 1/2$, everyone gets to know the information in the long run due to their low threshold values of acceptance. This may be a result of gullibility on the part of the population or the reliability of the source. This tends to correspond to Granovetter's conceptual model and the dynamics is similar to the logistic map.

4.4 THE CASE OF EVERYWHERE POSITIVE DISTRIBUTION

For mathematical simplicity, we assume that initial knowers are distributed independently of the threshold value ξ. Further, $\theta(\xi)$ is given as a constant value θ_0. This means that initial knowers are randomly given with probability θ_0. We also assume that $Q(P)$ is continuous and differentiable in terms of P.

4.4.1 *Existence of equilibrium states*

In this case, we assume that $f(\xi) > 0$ for any ξ such that (4.2) becomes

$$\frac{dP(t)}{dt} = B(P(t))G(P(t)), \tag{4.9}$$

where

$$G(P) := 1 - P - (1 - \theta_0) \int_{Q(P)}^{\infty} f(\xi) d\xi \tag{4.10}$$

with $G(P)$ continuous in terms of P. At the equilibrium state $P = P^*$ where $dP(t)/dt = 0$, we have $G(P^*) = 0$ since $B(P) > 0$.

Theorem 4.4.1. *There is at least one equilibrium state $P = P^*$ for (4.9) such that $0 < P^* \leq 1$.*

Proof. Since $G(P)$ is continuous, we see that

$$G(0) = 1 - (1 - \theta_0) \int_{Q(1)}^{\infty} f(\xi) d\xi > 0$$

for $Q(1) \geq 0$. On the other hand, if $Q(1) < \infty$, then

$$G(1) = -(1 - \theta_0) \int_{Q(1)}^{\infty} f(\xi) d\xi < 0 \tag{4.11}$$

since $\int_{Q(1)}^{\infty} f(\xi)d\xi > 0$. So, there is at least one value of $P = P^*$ such that $0 < P^* < 1$.

If $Q(1) = \infty$, then $\int_{Q(1)}^{\infty} f(\xi)d\xi = 0$ so that

$$G(1) = -(1-\theta_0)\int_{Q(1)}^{\infty} f(\xi)d\xi = 0.$$

This is sufficient condition that there is one equilibrium value $P = P^* = 1$ because $\left.\frac{dP}{dt}\right|_{P=1} = 0$. Conversely, we have the necessary condition that if $G(1) = 0$, $\int_{Q(1)}^{\infty} f(\xi)d\xi = 0$ due to the fact that $f(\xi)$ is everywhere positive for $\xi < \infty$, then $Q(1) = \infty$. Overall, there is at least one equilibrium state $P = P^*$ for (4.9) such that $0 < P^* \leq 1$.

Furthermore, given that $Q(1) < \infty$, from (4.11) it is clear that $P = 1$ cannot become an equilibrium state. □

We have the following corollaries from the theorem.

Corollary 4.4.2. *$P = 1$ can become an equilibrium state if and only if $Q(1) = \infty$.*

Corollary 4.4.3. *If $Q(1) < \infty$, $P = 1$ cannot become an equilibrium state and there is at least one equilibrium state $P = P^*$ such that $0 < P^* < 1$.*

From the idea of standard local stability, we have the following result for an equilibrium state $P = P^* \in (0, 1)$.

Theorem 4.4.4. *Suppose that there is an equilibrium state $P = P^*$ for (4.9) such that $0 < P^* < 1$, then it is locally asymptotically stable if $(1-\theta_0)\frac{dQ(P^*)}{dP} f(Q(P^*)) < 1$.*

Proof. Given that such a $P^* < 1$ exists, then from (4.9), we have

$$L(P) = \frac{dP(t)}{dt} = B(P)G(P) \tag{4.12}$$

such that P^* is asymptotically stable if $dL(P^*)/dP < 0$. Now,

$$\frac{dL(P)}{dP} = \frac{dB(P)}{dP}G(P) + B(P)\frac{dG(P)}{dP}$$

$$\frac{dL(P^*)}{dP} = \frac{dB(P^*)}{dP}G(P^*) + B(P^*)\frac{dG(P^*)}{dP}$$

$$= B(P^*)\frac{dG(P^*)}{dP}.$$

$dL(P^*)/dP < 0$ implies that $dG(P^*)/dP < 0$ since $B(P) > 0$. So, we have

$$\begin{aligned}\frac{dG(P)}{dP} &= -1 - (1-\theta_0)\frac{d}{dP}\int_{Q(P)}^{\infty} f(\xi)d\xi \\ &= -1 - (1-\theta_0)\frac{dQ(P)}{dP}\frac{d}{dQ}\int_{Q}^{\infty} f(\xi)d\xi \\ &= -1 - (1-\theta_0)\frac{dQ(P)}{dP}[-f(Q(P))] \\ &= -1 + (1-\theta_0)\frac{dQ(P)}{dP}f(Q(P))\end{aligned}$$

such that $dG(P^*)/dP < 0$ results to

$$(1-\theta_0)\frac{dQ(P^*)}{dP}f(Q(P^*)) < 1.$$

This completes the proof for the establishment of sufficient condition for local stability. □

More so, we have the following result for the stability of the equilibrium state $P = P^* = 1$.

Theorem 4.4.5. *If $P = 1$ is an equilibrium state for (4.9), then it is necessarily locally asymptotically stable.*

Proof. Since $0 \leq P(t) \leq 1$ from Theorem 4.2.1, it is satisfactory to consider the pertubation such that $P(t) = 1 - \varepsilon(t)$ with $0 < \varepsilon \ll 1$. Then around $P = 1$, we have

$$\frac{dP(t)}{dt} = \frac{d(1-\varepsilon(t))}{dt} = -\frac{d\varepsilon}{dt} = B(1-\varepsilon(t))G(1-\varepsilon(t)). \tag{4.13}$$

Again, the sign of $G(1-\varepsilon(t))$ determines the sign of $\frac{d\varepsilon(t)}{dt}$ since $B(1-\varepsilon(t)) > 0$ by definition. Going by (4.10), we have

$$G(1-\varepsilon(t)) = \varepsilon(t) - (1-\theta_0)\int_{Q(1-\varepsilon(t))}^{\infty} f(\xi)d\xi.$$

Taking the limit of the integral as $s \to 1+0$, then

$$G(1-\varepsilon(t)) = \varepsilon(t) - (1-\theta_0)\lim_{s\to 1+0}\int_{Q(s-\varepsilon(t))}^{\infty} f(\xi)d\xi.$$

The Taylor expansion of the integral gives

$$G(1-\varepsilon(t)) = \varepsilon(t) - (1-\theta_0)\lim_{s\to 1-0}\left[\int_{Q(s)}^{\infty} f(\xi)d\xi - f(Q(s))\varepsilon(t) + o(\varepsilon(t))\right].$$

The first two terms of the expansion vanish because $\lim_{s\to 1-0}\int_{Q(s)}^{\infty} f(\xi)d\xi = 0$ since $Q(s) \to \infty$ as $s \to 1-0$ from Corollary 4.4.2 and $\lim_{s\to 1-0} f(Q(s)) = 0$ from the nature of function f. Consequently,

$$G(1-\varepsilon(t)) = \varepsilon(t) - (1-\theta_0)o(\varepsilon(t)) > 0.$$

Since $G(1-\varepsilon(t)) > 0$, then $d\varepsilon/dt < 0$ in (4.13). As such, $P = 1$ is a locally asymptotically stable equilibrium state. □

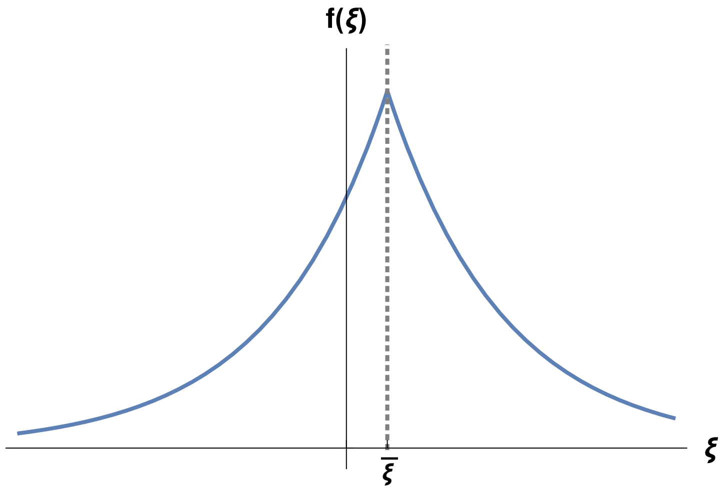


Figure 4.3: Graph of the frequency distribution function $f(\xi)$ against the threshold value ξ of the social recognition effect given by (4.14).

4.4.2 *A specific model*

We consider the distribution defined as

$$f(\xi) = \frac{1}{\sigma\sqrt{2}} e^{-\sqrt{2}\frac{|\xi-\bar{\xi}|}{\sigma}} = \begin{cases} \frac{1}{\sigma\sqrt{2}} e^{\sqrt{2}\frac{(\xi-\bar{\xi})}{\sigma}}, & \xi < \bar{\xi}; \\ \frac{1}{\sigma\sqrt{2}} e^{-\sqrt{2}\frac{(\xi-\bar{\xi})}{\sigma}}, & \xi \geq \bar{\xi}, \end{cases} \tag{4.14}$$

with mean $\bar{\xi}$ and variance σ^2. The graph of this function is found in Figure 4.3.

In addition, we define $Q(P) = \alpha P$ $(\alpha > 0)$, $Q(1) = \alpha < \infty$. Following Corollary 4.4.3, there is at least one equilibrium state $P = P^*$ such that $0 < P^* < 1$ since $Q(1) < \infty$. In terms of (4.9) and (4.10), we now have

$$G(P) = \begin{cases} \mathcal{G}_1(P) := \theta_0 - P + \frac{1}{2}(1-\theta_0)e^{\sqrt{2}\frac{\alpha}{\sigma}\left(P-\frac{\bar{\xi}}{\alpha}\right)}, & P < \bar{\xi}/\alpha; \\ \mathcal{G}_2(P) := 1 - P - \frac{1}{2}(1-\theta_0)e^{-\sqrt{2}\frac{\alpha}{\sigma}\left(P-\frac{\bar{\xi}}{\alpha}\right)}, & P \geq \bar{\xi}/\alpha. \end{cases} \tag{4.15}$$

This function is continuous since

$$\lim_{P\to\frac{\bar{\xi}}{\alpha}} G(P) = \lim_{P\to\frac{\bar{\xi}}{\alpha}-0} \mathcal{G}_1(P) = \lim_{P\to\frac{\bar{\xi}}{\alpha}+0} \mathcal{G}_2(P) = \frac{1}{2}(1+\theta_0) - \frac{\bar{\xi}}{\alpha}. \tag{4.16}$$

The continuity of $G(P)$ is further demonstrated in Figure 4.4. A detailed analysis of (4.15) for the existence and number of equilibrium states is found in Appendix B.2 and the summary of the result is shown in Figure 4.5.

The symbol $\langle i, j\rangle$ in Figures 4.4 and 4.5 represents the respective numbers of equilibrium states in each of the intervals $(0, \bar{\xi}/\alpha)$ and $[\bar{\xi}/\alpha, 1]$. This means that the total number of equilibrium states in the complete interval $(0, 1]$ is given by $i + j$. We have the following necessary and sufficient conditions for $\langle i, j\rangle$ based on the conditions given in Appendix B.2.

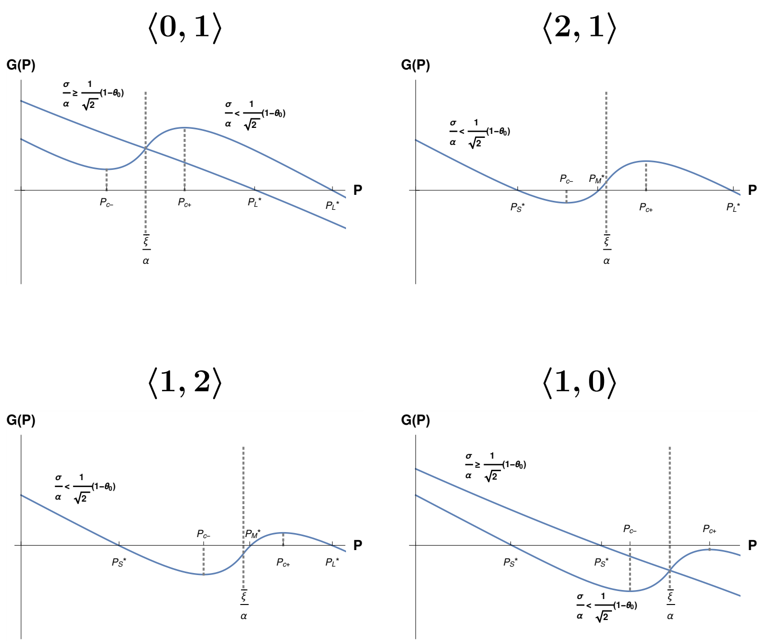


Figure 4.4: Graph of $G(P)$ against P. In each figure, $\langle i, j\rangle$ represents the pair of numbers of equilibrium states i and j in the intervals $(0, \bar{\xi}/\alpha)$ and $[\bar{\xi}/\alpha, 1]$ respectively. P_S^*, P_M^*, P_L^* are equilibrium states formally obtained in each case.

- For $\langle 0, 1\rangle$, we have either of

$$\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0) \quad \text{and} \quad \frac{\bar{\xi}}{\alpha} < \theta_0 + \frac{\sigma}{\alpha\sqrt{2}}\left[1 + \ln\frac{1-\theta_0}{\sqrt{2}} - \ln\frac{\sigma}{\alpha}\right], \tag{4.17}$$

$$\frac{\sigma}{\alpha} \geq \frac{1}{\sqrt{2}}(1-\theta_0) \quad \text{and} \quad \frac{\bar{\xi}}{\alpha} < \frac{1}{2}(1+\theta_0). \tag{4.18}$$

- For $\langle 2, 1\rangle$, we have

$$\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0) \quad \text{and} \quad \theta_0 + \frac{\sigma}{\alpha\sqrt{2}}\left[1 + \ln\frac{1-\theta_0}{\sqrt{2}} - \ln\frac{\sigma}{\alpha}\right] < \frac{\bar{\xi}}{\alpha} < \frac{1}{2}(1+\theta_0). \tag{4.19}$$

- For $\langle 1, 2\rangle$, we have

$$\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0) \quad \text{and} \quad \frac{1}{2}(1+\theta_0) < \frac{\bar{\xi}}{\alpha} < 1 - \frac{\sigma}{\alpha\sqrt{2}}\left[1 + \ln\frac{1-\theta_0}{\sqrt{2}} - \ln\frac{\sigma}{\alpha}\right]. \tag{4.20}$$

- For $\langle 1, 0\rangle$, we have either of

$$\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0) \quad \text{and} \quad \frac{\bar{\xi}}{\alpha} > 1 - \frac{\sigma}{\alpha\sqrt{2}}\left[1 + \ln\frac{1-\theta_0}{\sqrt{2}} - \ln\frac{\sigma}{\alpha}\right], \tag{4.21}$$

$$\frac{\sigma}{\alpha} \geq \frac{1}{\sqrt{2}}(1-\theta_0) \quad \text{and} \quad \frac{\bar{\xi}}{\alpha} > \frac{1}{2}(1+\theta_0). \tag{4.22}$$

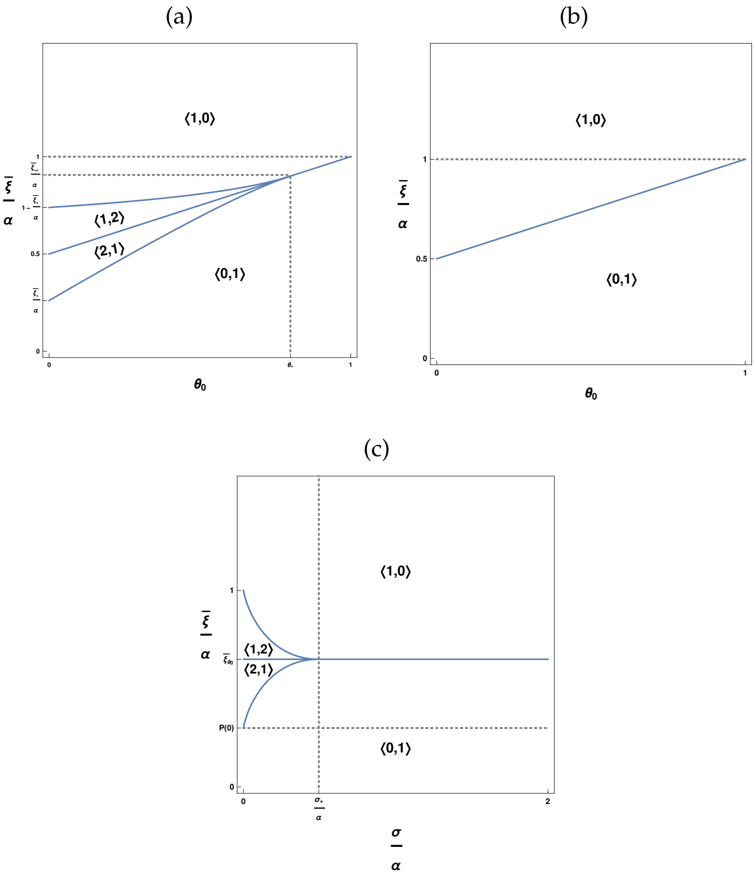


Figure 4.5: Parameter dependence for the formal existence of equilibrium states. (a) $\left(\theta_0, \frac{\bar{\xi}}{\alpha}\right)$ with $\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}$, (b) $\left(\theta_0, \frac{\bar{\xi}}{\alpha}\right)$ with $\frac{\sigma}{\alpha} \geq \frac{1}{\sqrt{2}}$, (c) $\left(\frac{\sigma}{\alpha}, \frac{\bar{\xi}}{\alpha}\right)$. $\theta_* = 1 - \frac{\sigma\sqrt{2}}{\alpha}$, $\frac{\bar{\xi}_*}{\alpha} = \frac{\sigma}{\alpha\sqrt{2}}\left(1 + \ln\frac{\alpha}{\sigma\sqrt{2}}\right)$, $\frac{\bar{\xi}_{**}}{\alpha} = 1 - \frac{\sigma}{\alpha\sqrt{2}}$, $\frac{\sigma_*}{\alpha} = \frac{1}{\sqrt{2}}(1-\theta_0)$ and $\bar{\xi}_{\theta_0} = \frac{1}{2}(1+\theta_0)$. The respective numbers of equilibrium states i and j in the intervals $\left(0, \frac{\bar{\xi}}{\alpha}\right)$ and $\left[\frac{\bar{\xi}}{\alpha}, 1\right]$ are given by $\langle i, j\rangle$.

Stability of equilibrium states

Based on Theorem 4.4.4, we can obtain the sufficient condition for local asymptotic stability for the specific model given by (4.14) and (4.15). Since $Q(P) = \alpha P$, we have

$$f(Q(P)) = \frac{1}{\sigma\sqrt{2}} e^{-\sqrt{2}\frac{\alpha}{\sigma}\left|P-\frac{\bar{\xi}}{\alpha}\right|} = \begin{cases} \frac{1}{\sigma\sqrt{2}} e^{\sqrt{2}\frac{\alpha}{\sigma}\left(P-\frac{\bar{\xi}}{\alpha}\right)}, & P < \bar{\xi}/\alpha; \\ \frac{1}{\sigma\sqrt{2}} e^{-\sqrt{2}\frac{\alpha}{\sigma}\left(P-\frac{\bar{\xi}}{\alpha}\right)}, & P \geq \bar{\xi}/\alpha. \end{cases}$$

As such, for $P^* < \bar{\xi}/\alpha$, $P = P^*$ is locally asymptotically stable if

$$P^* < P_{c-} = \frac{\bar{\xi}}{\alpha} - \frac{\sigma}{\alpha\sqrt{2}} \ln \frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0)$$

defined in (B.3). Otherwise, for $P^* \geq \bar{\xi}/\alpha$, $P = P^*$ is locally asymptotically stable if

$$P^* > P_{c+} = \frac{\bar{\xi}}{\alpha} + \frac{\sigma}{\alpha\sqrt{2}} \ln \frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0)$$

defined in (B.4).

Furthermore, we consider the global stability analysis based on (4.15). From Figure 4.4, we have equilibrium states $\langle 0,1\rangle$, $\langle 2,1\rangle$, $\langle 1,2\rangle$ and $\langle 1,0\rangle$ in the intervals $(0,\bar{\xi}/\alpha)$ and $[\bar{\xi}/\alpha,1]$ respectively. We see that when there is only one equilibrium state in the whole interval $(0,1]$, it is always globally asymptotically stable. On the other hand, when there are three equilibrium states, the smallest and the largest are necessarily stable while the middle one is unstable. The results obtained for local asymptotic stability correspond with the global stability of (4.15).

Critical parameter values

The bifurcation branches with parameter θ_0 is obtained by first solving $\mathcal{G}_1(P^*) = 0$ and $\mathcal{G}_2(P^*) = 0$ for θ_0. This gives

$$\theta_0 = \mathcal{F}_1(P^*) = \begin{cases} \mathcal{F}_{11}(P^*) := \frac{P^* - \frac{1}{2}e^{\sqrt{2}\frac{\alpha}{\sigma}\left(P^*-\frac{\bar{\xi}}{\alpha}\right)}}{1-\frac{1}{2}e^{\sqrt{2}\frac{\alpha}{\sigma}\left(P^*-\frac{\bar{\xi}}{\alpha}\right)}}, & P^* < \bar{\xi}/\alpha; \\ \mathcal{F}_{12}(P^*) := \frac{P^*-1+\frac{1}{2}e^{-\sqrt{2}\frac{\alpha}{\sigma}\left(P^*-\frac{\bar{\xi}}{\alpha}\right)}}{\frac{1}{2}e^{-\sqrt{2}\frac{\alpha}{\sigma}\left(P^*-\frac{\bar{\xi}}{\alpha}\right)}}, & P^* \geq \bar{\xi}/\alpha. \end{cases} \tag{4.23}$$

In order to obtain P_c^* for (4.23), we solve $d\mathcal{F}_1(P^*)/dP^* = 0$ which results to

$$\begin{cases} \sqrt{2}(1-P_c^*) + \frac{\sigma}{\alpha}\left(1-2e^{-\sqrt{2}\frac{\alpha}{\sigma}\left(P_c^*-\frac{\bar{\xi}}{\alpha}\right)}\right) = 0, & P^* < \bar{\xi}/\alpha; \\ P_c^* := 1 - \frac{\sigma}{\alpha\sqrt{2}}, & P^* \geq \bar{\xi}/\alpha. \end{cases} \tag{4.24}$$

From this, we obtain critical values of θ_0 as

$$\theta_c = \begin{cases} \mathcal{F}_{11}(P_c^*), & P^* < \bar{\xi}/\alpha; \\ 1 - \dfrac{\sigma\sqrt{2}}{\alpha} e^{\sqrt{2}\frac{\alpha}{\sigma}\left(1 - \frac{\sigma}{\alpha\sqrt{2}} - \frac{\bar{\xi}}{\alpha}\right)}, & P^* \geq \bar{\xi}/\alpha. \end{cases} \tag{4.25}$$

With respect to $\bar{\xi}/\alpha$ as bifurcation parameter, we have

$$\frac{\bar{\xi}}{\alpha} = \mathcal{F}_2(P^*) = \begin{cases} \mathcal{F}_{21}(P^*) := P^* - \dfrac{\sigma}{\alpha\sqrt{2}} \ln\left[\frac{2(P^*-\theta_0)}{1-\theta_0}\right], & P^* < \bar{\xi}/\alpha; \\ \mathcal{F}_{22}(P^*) := P^* - \dfrac{\sigma}{\alpha\sqrt{2}} \ln\left[\frac{1-\theta_0}{2(1-P^*)}\right], & P^* \geq \bar{\xi}/\alpha \end{cases} \tag{4.26}$$

with

$$P_c^* = \begin{cases} \theta_0 + \dfrac{\sigma}{\alpha\sqrt{2}}, & P^* < \bar{\xi}/\alpha; \\ 1 - \dfrac{\sigma}{\alpha\sqrt{2}}, & P^* \geq \bar{\xi}/\alpha \end{cases} \tag{4.27}$$

so that we obtain the critical values of $\bar{\xi}/\alpha$ to be

$$\frac{\bar{\xi}_c}{\alpha} = \begin{cases} \theta_0 + \dfrac{\sigma}{\alpha\sqrt{2}}\left[1 + \ln\frac{\alpha(1-\theta_0)}{\sigma\sqrt{2}}\right], & P^* < \bar{\xi}/\alpha; \\ 1 - \dfrac{\sigma}{\alpha\sqrt{2}}\left[1 + \ln\frac{\sigma\sqrt{2}}{\alpha(1-\theta_0)}\right], & P^* \geq \bar{\xi}/\alpha. \end{cases} \tag{4.28}$$

With respect to σ/α as bifurcation parameter, we have

$$\frac{\sigma}{\alpha} = \mathcal{F}_3(P^*) = \begin{cases} \mathcal{F}_{31}(P^*) := \dfrac{\sqrt{2}\left(P^* - \frac{\bar{\xi}}{\alpha}\right)}{\ln\left[\frac{2(P^*-\theta_0)}{1-\theta_0}\right]}, & P^* < \bar{\xi}/\alpha; \\ \mathcal{F}_{32}(P^*) := \dfrac{\sqrt{2}\left(P^* - \frac{\bar{\xi}}{\alpha}\right)}{\ln\left[\frac{1-\theta_0}{2(1-P^*)}\right]}, & P^* \geq \bar{\xi}/\alpha \end{cases} \tag{4.29}$$

with the implicit functions of P_c^* given by

$$\begin{cases} \ln\frac{2(P_c^*-\theta_0)}{1-\theta_0} = \frac{\left(P_c^* - \frac{\bar{\xi}}{\alpha}\right)}{P_c^*-\theta_0}, & P^* < \bar{\xi}/\alpha; \\ \ln\frac{1-\theta_0}{2(1-P_c^*)} = \frac{\left(P_c^* - \frac{\bar{\xi}}{\alpha}\right)}{1-P_c^*}, & P^* \geq \bar{\xi}/\alpha \end{cases} \tag{4.30}$$

so that we obtain the critical values of σ/α to be

$$\frac{\sigma_c}{\alpha} = \begin{cases} \mathcal{F}_{31}(P_c^*), & P^* < \bar{\xi}/\alpha; \\ \mathcal{F}_{32}(P_c^*), & P^* \geq \bar{\xi}/\alpha. \end{cases} \tag{4.31}$$

The bifurcation of the system with respect to parameters θ_0, $\bar{\xi}/\alpha$ and σ/α are shown in Figures 4.6, 4.7, 4.8.

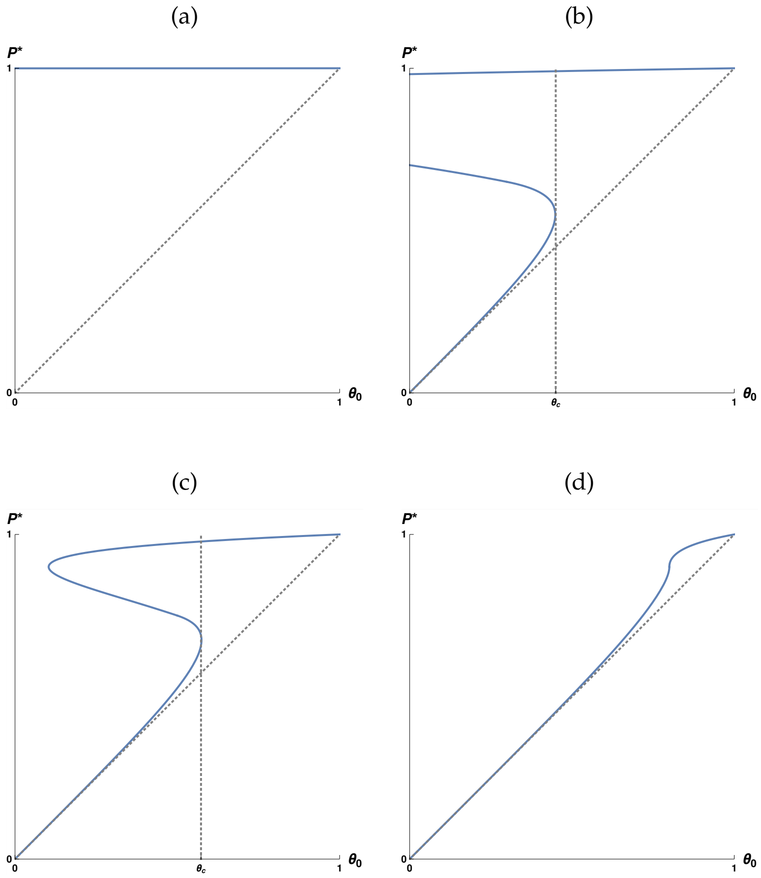


Figure 4.6: Bifurcation diagram for P^* with parameter θ_0 given by (4.23). (a) $0 < \bar{\xi}/\alpha < \bar{\xi}_*/\alpha$, (b) $0.5 < \bar{\xi}/\alpha < 1 - \bar{\xi}_*/\alpha$, (c) $\bar{\xi}/\alpha = 1 - \bar{\xi}_*/\alpha$, (d) $1 - \bar{\xi}_*/\alpha < \bar{\xi}/\alpha < 1.0$. Commonly, $\frac{\bar{\xi}_*}{\alpha} = \frac{\sigma}{\alpha\sqrt{2}}\left(1 + \ln\frac{\alpha}{\sigma\sqrt{2}}\right)$, $P(0) = \theta_0$ and $\sigma/\alpha = \sqrt{2}/10$. θ_c is defined by (4.24) and (4.25).

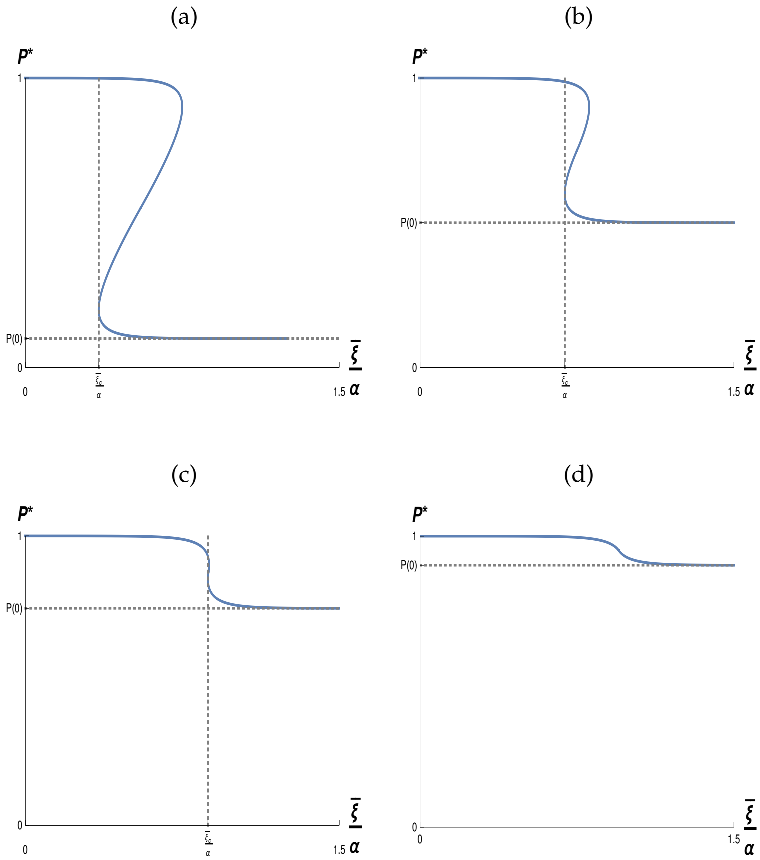


Figure 4.7: Bifurcation diagram for P^* with parameter $\overline{\xi}/\alpha$ given by (4.26). (a) $P(0) = 0.10$, (b) $P(0) = 0.50$, (c) $P(0) = 0.75$, (d) $P(0) = 0.90$. Commonly, $\sigma/\alpha = \sqrt{2}/10$. $\overline{\xi}_c/\alpha$ is defined by (4.27) and (4.28).

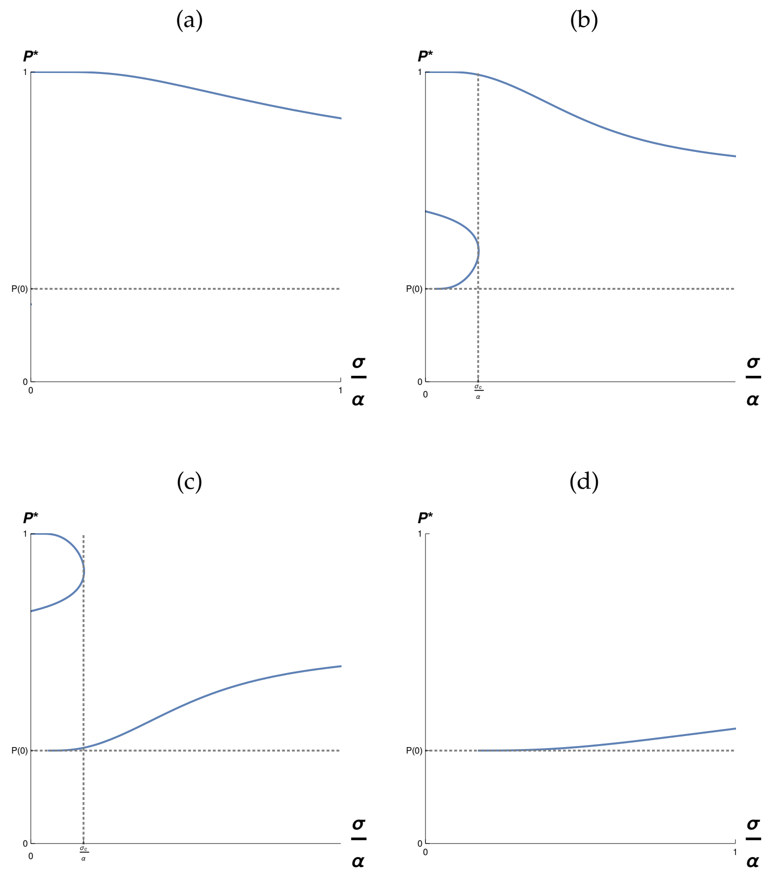


Figure 4.8: Bifurcation diagram for P^* with parameter σ/α given by (4.29). (a) $\overline{\xi}/\alpha = 0.25$, (b) $\overline{\xi}/\alpha = 0.55$, (c) $\overline{\xi}/\alpha = 0.75$, (d) $\overline{\xi}/\alpha = 1.50$. Commonly, $P(0) = \theta_0 = 0.30$. σ_c/α is defined by (4.30) and (4.31).

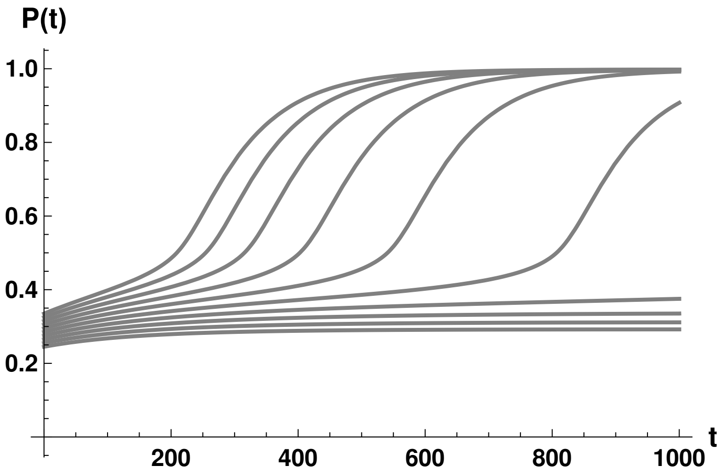


Figure 4.9: Temporal variation of $P(t)$ based on (4.15) with varying initial values $P(0) = \theta_0$. Commonly, we have $B(P) = 0.011P$, $\sigma/\alpha = \sqrt{2}/10$ and $\bar{\xi}/\alpha = 0.50$.

Equilibrium value of ***P***

Going by (4.1) and our assumption that $\theta(\xi) = \theta_0$, we see that $P(0) = \theta_0$ and it determines the equilibrium state to which the system converges in the case of seeming bistability. The equilibrium state is uniquely determined by the initial condition. As stated in subsection 4.1.4, the frequency of knowers does not decrease in time. The temporal variation, showing the convergence of the system, for various initial conditions is shown in Figure 4.9.

Theorem 4.4.6. *The system converges to the equilibrium state at which the equilibrium value of P is greater than $\bar{\xi}/\alpha$ only in the case of $\langle 0, 1\rangle$. In any other case, the equilibrium value of P is necessarily smaller than $\bar{\xi}/\alpha$.*

Proof. For the case of $\langle 1, 2\rangle$, the second condition for its existence is given as expressed in the second part of (4.20). Since $\theta_0 < (1 + \theta_0)/2$ for any $\theta_0 \in (0, 1)$, this condition requires that $\theta_0 < \bar{\xi}/\alpha$. This means that the case of $\langle 1, 2\rangle$ is only valid for the initial value satisfying that $\theta_0 < \bar{\xi}/\alpha$.

Next, for the case of $\langle 2, 1\rangle$, the second condition for its existence is expressed in the second part of (4.19). From the first condition for its existence, we have

$$\ln \frac{1 - \theta_0}{\sqrt{2}} - \ln \frac{\sigma}{\alpha} > 0,$$

so that the second condition for its existence requires that $\theta_0 < \bar{\xi}/\alpha$ similar to the previous case. Therefore, the case of $\langle 2, 1\rangle$ is also only valid for the initial value satisfying that $\theta_0 < \bar{\xi}/\alpha$. Further, the first inequality of the second condition in (4.19) can be rewritten to be

$$\theta_0 + \frac{\sigma}{\alpha\sqrt{2}} < P_{c-}.$$

Hence we find the other necessary condition for the case of $\langle 2, 1\rangle$ that $\theta_0 < P_{c-}$.

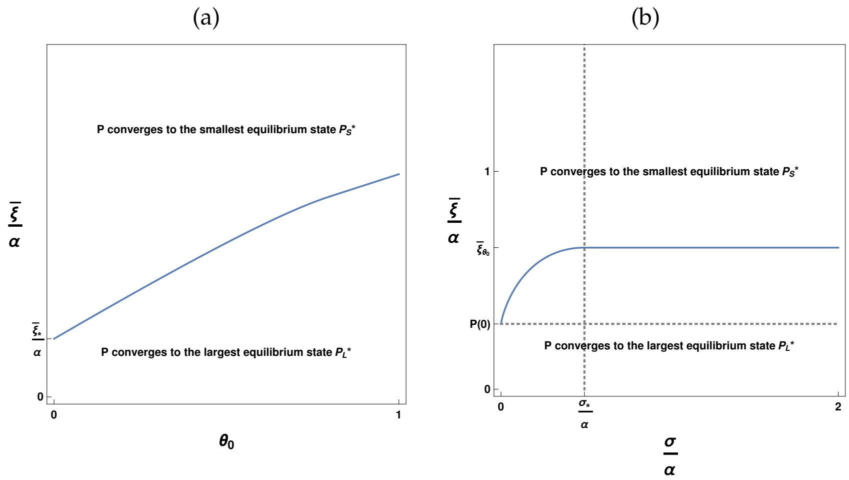


Figure 4.10: Parameter dependence for the convergence of P. (a) $\left(\theta_0, \frac{\bar{\xi}}{\alpha}\right)$ with $\frac{\bar{\xi}_*}{\alpha} = \frac{\sigma}{\alpha\sqrt{2}}\left(1 + \ln\frac{\alpha}{\sigma\sqrt{2}}\right)$. (b) $\left(\frac{\sigma}{\alpha}, \frac{\bar{\xi}}{\alpha}\right)$ with $\frac{\sigma_*}{\alpha} = \frac{1}{\sqrt{2}}(1-\theta_0)$, $P(0) = \theta_0$ and $\bar{\xi}_{\theta_0} = \frac{1}{2}(1+\theta_0)$.

Now, since $G(\theta_0) = G(P(0)) > 0$, we can find that the value of P necessarily converges to the smallest equilibrium state P_s^* given by the smaller root of $\mathcal{G}_1(P) = 0$ in both cases of $\langle 1, 2\rangle$ and $\langle 2, 1\rangle$.

From Figure 4.4 in the case of $\langle 1, 2\rangle$, it is clear that P must converge to it since $P(0) = \theta_0 < \bar{\xi}/\alpha$ as shown in the above argument. Similarly, for the case of $\langle 2, 1\rangle$ in Figure 4.4, since $P(0) = \theta_0 < P_{c-}$ as shown in the above argument, it is clear that P must converge to the smallest equilibrium state. □

In Figure 4.10, we represent the convergence profile of P based on the dependence of the key parameters of the model. Figure 4.11 shows the numerically obtained convergence for P with parameters θ_0, $\bar{\xi}/\alpha$ and σ/α. Figure 4.12 is the three-dimensional representation of the dependence of P^* on θ_0 and $\bar{\xi}/\alpha$.

From our analyses, we found a mathematically bistable situation for P, however, P always converges to a unique equilibrium state depending on the initial value θ_0. Figure 4.10 shows that P converges to the smallest equilibrium state P_S^* when the mean threshold value is sufficiently large. For small mean threshold values, P converges to the largest equilibrium state P_L^*. 4.10(a) shows that a sufficiently large initial value θ_0 is required for a higher information spread. From 4.10(b), we see that there is a higher chance of an information spreading among just a small proportion of the population if the variance is below a critical value.

4.4.3 *Discussion*

The general model with everywhere positive distribution shows that the proportion of the population that ends up knowing an information largely depends on the strength of social recognition effect. A very large value of this effect on the population leads to the circulation of

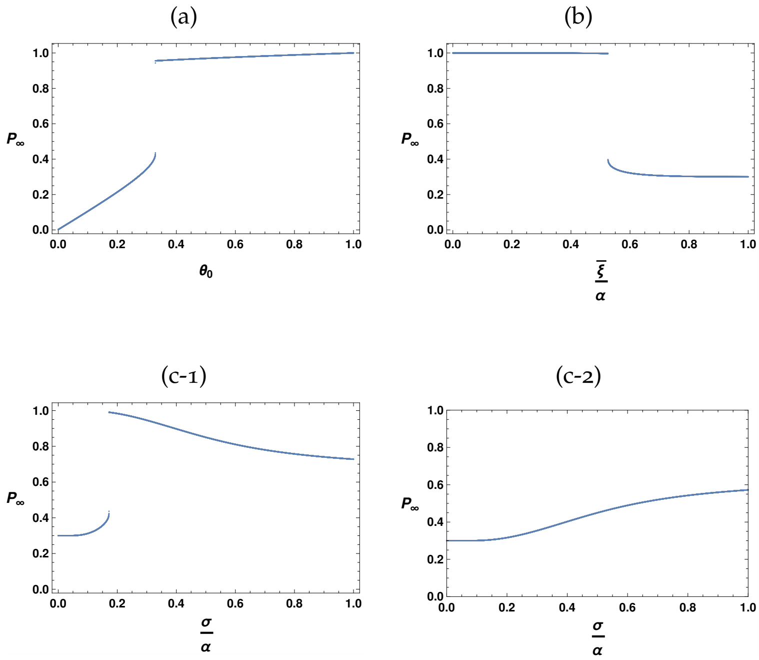


Figure 4.11: Numerically obtained convergence for P with parameter (a) θ_0 with $\bar{\xi}/\alpha = 0.55$, $\sigma/\alpha = \sqrt{2}/10$, (b) $\bar{\xi}/\alpha$ with $\theta_0 = 0.3$, $\sigma/\alpha = \sqrt{2}/10$, (c-1) σ/α with $\theta_0 = 0.3$, $\bar{\xi}/\alpha = 0.55$, (c-2) σ/α with $\theta_0 = 0.3$, $\bar{\xi}/\alpha = 0.75$. Here, we assume the function $B(P) = P$.

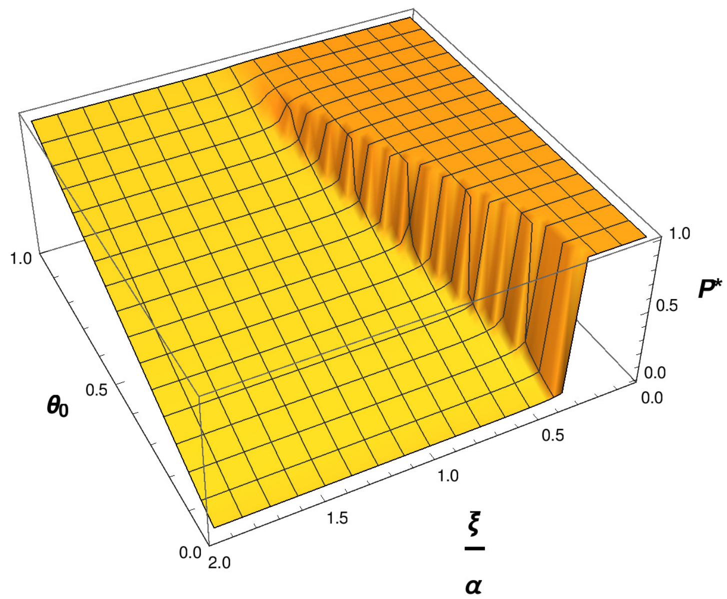


Figure 4.12: Dependence of P^* on $\left(\theta_0, \frac{\bar{\xi}}{\alpha}\right)$. $B(P) = P$ and $\sigma/\alpha = \sqrt{2}/10$.

the information among a lower proportion of the population in the long run. Conversely, a large proportion of the population will end up knowing the information if the strength of the social recognition effect on the population is relatively small.

From the model with specific distribution, the most important parameters are θ_0, $\bar{\xi}$ and σ. θ_0, the proportion of initial knowers, represents those who have the task to carry out an initial operation to circulate a certain information within a population. From our model dynamics, it is worthy of note that the final level of spread of a piece of information depends largely on the specific value of the initial proportion of knowers even when there is apparent bistability of equilibrium states. Coincidentally, this proportion corresponds with the initial condition for our model. As such, it is both a parameter and an initial value from which the proportion of knowers continue to increase since we do not consider forgetfulness on the part of knowers. The initial proportion of knowers depends on the nature and situation of information. For instance, a single person can begin to spread a rumor; a syndicate may be the initiator of a fake news; a new idea can begin with a pilot group within a population (e.g., the use of masks in preventing epidemics); top security secrets are known by very few people; and information from the mass media can be known initially by a large proportion of a population.

The mean threshold value of a population's social recognition effect, given as $\bar{\xi}$, characterizes how a community reacts to a specific kind of information. It is a kind of peak/mode behavior that is representative of the community. A large mean threshold indicates that a society is closed or conservative towards a particular kind of information, e.g., old people's attitude towards a hip hop concert. On the other hand, a small threshold mean shows that a society freely transmits a given information. The standard deviation, σ, is a measure of the degree of scattering or variance of threshold values within the population. A small variance shows that the threshold values are steeply spread while a large one implies that the threshold values are wide apart. In our analyses, $\bar{\xi}$ and σ describe the heterogeneity of the threshold values of individuals in the community. These two parameters are re-scaled by α which characterizes the strength of social response to a piece of information.

For each of the parameters, there are critical values which determine whether a substantial proportion of the population gets to know the information or whether it is confined to an insignificant proportion.

The system of ordinary differential equations describing the threshold model is special because it explicitly depends on the initial value $P(0)$. This means that the temporal variation of proportion of knowers P is affected by the initial condition. As such, the equilibrium state of the system is determined by the given initial value so that the equilibrium state varies for different initial values as seen in Figure 4.9.

The results show that each of the model parameters (initial value, mean threshold value for social response and standard deviation/-variance) have critical values which determine the equilibrium state to which the system converges. However, the effect of variance is not as significant as that of the other two. From Figure 4.11, it is seen that the proportion of knowers is increases in terms of initial knowers and decreases in terms of mean threshold value of the social recognition effect. On the other hand, the effect of variance depends on the mean threshold value. For a relatively small mean threshold value, the proportion of knowers drastically increase once a critical variance is exceeded. A decrease of the proportion of knowers is then seen for large values of variance. For relatively large mean threshold value, the proportion of knowers rises continuously and peaks moderately.

Figure 4.6 reveals that, in case of seeming bistability (b,c), the system always converges to the lower equilibrium state for initial values below the critical value while it goes to the upper equilibrium state for initial values which exceed the critical value. This means that a very large proportion of the population gets to know the information when there is a large enough proportion of initial knowers.

Figure 4.7 shows there is a critical mean threshold value ξ_c for the population. So, a mean threshold value below the critical one, makes the system converge to the upper equilibrium state while any one above it makes the system converge to the lower equilibrium state. This is understandable since the mean threshold value measures the acceptability of the information to the population. The proportion of knowers becomes drastically small when the community has a high mean threshold value thereby making the information highly unacceptable. A low mean threshold value indicates that the information is readily welcome in the society.

When bistability seems to appear in Figure 4.8 as seen in (b), there is a critical variance. So, at a variance below the critical variance, the system converges to the lower equilibrium state. A variance above the critical variance drives the system to the upper equilibrium state with some decrease afterwards.

From the results discussed, it may be possible to control the initial proportion of knowers in order to achieve a purpose which involves the spread of information. On the other hand, individual threshold values and their distribution within a population can hardly be controlled. Such a control may only be possible under special conditions.

The analyses show that people can be stubborn in accepting a piece of information until a critical threshold value is reached. When the mean threshold value falls below the critical mean threshold value, there is a drastic increase in the frequency of knowers of the information due to an increasing level of sociability/acceptability. This scenario is commonly seen in the way people respond to most innovative ideas. Individuals always tend to resist potential changes to their ways of life but over time, with persistent awareness, they embrace change and the new idea becomes well circulated within the population.

5 CONCLUDING REMARKS

This thesis shows that the ways in which individuals think (Psychology) and the attitude of the societies to which they belong (Sociology) play important roles in the way they interact with information in terms of acceptance and spreading.

The nature of the information under consideration and the world-view of the society in which it is introduced are vital in determining how it spreads. News about sports and hip hop music will do well among youths compared to aged people, for instance. Furthermore, it is easier for misinformation to spread in a society with low literacy rate compared with one with high literacy rate.

From the foregoing, we can infer that demographic groupings (e.g. age) and education have impact on an individual's thought process. An educated person is prone to thinking better thoughts and when there are many people having that high quality of thinking in a society, their collective behavior is well informed.

More so, there are situations where an otherwise stubborn society becomes favorably disposed towards accepting a piece of information if they had had encounter with a similar piece of information. For example, it is easy to get people to take quick measures against the spread of a virus when they have been confronted with a similar situation.

In terms of future research, possible extensions to the threshold model can be done by reformulating the model to introduce

- a second piece of information; or
- a change of individual behavior by modifying the strength of social recognition effect $Q(P)$; or
- another threshold such that we have lower and upper thresholds as seen in [75].

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A
APPENDIX FOR THE REJOINDER MODEL

A.1 CONVERGENCE TO THE EQUILIBRIUM STATE FOR REJOINDER MODEL

Since $u(\tau_s + 0) > 0$ and $w_0(\tau_s + 0) > 0$ from the initial condition assumed in our model, and since $dw_0/d\tau > 0$ for any $\tau \in (\tau_s, \infty)$, we have $w_0(\tau) > 0$ for any $\tau > \tau_s$. This argument can also be applied for w_+, so that $w_+(\tau) > 0$ for any $\tau > \tau_s$. As long as u is positive for (2.10) and any of p, w_0 and w_+ is also positive, $du/d\tau$ is always negative. This is contrary if $u \to u^* > 0$, as such, $u \to 0$. So, as $u \to 0$, the right hand side of $dp/d\tau$ must become negative since the first term becomes significantly small compared to the other terms, for sufficiently large $\tau > \tau_s$. It is now clear that p decreases such that $p \to 0$ and $v \to 0$. From this analysis, we have $(u, p, v, w_0, w_+) \to (0, 0, 0, w_0^*, w_+^*)$ as the convergence state.

Since $u + p + v + w_0 + w_+ = 1$ for any $\tau > \tau_s$, the convergence as $\tau \to \infty$ means that w_0 and w_+ converge to some positive values w_0^* and w_+^* such that $w_0^* + w_+^* = 1$. The convergent value w_0^* or w_+^* depends on the initial condition at $\tau = \tau_s + 0$, which we could not determine analytically.

As an extremal mathematical supposition, if $u(\tau_s - 0) = 1$ and $p(\tau_s - 0) = 0$, then $u(\tau_s + 0) \leq 1$ and $p(\tau_s + 0) = 0$ so $dp/d\tau = 0$ and $p(\tau) = 0$ for any $\tau > \tau_s$ at the interaction stage. This means that $dw_+/d\tau = 0$ for any $\tau > \tau_s$ at this stage since $w_+(\tau) = w_+(\tau_s + 0) = 0$ for any $\tau > \tau_s$, so $w_+^* = 0$ and $w_0^* = 1$. On the other hand, if we assume the extremum situation where $u(\tau_s - 0) = 0$ and $p(\tau_s - 0) = 1$, then $u(\tau_s + 0) = 0$ and $p(\tau_s + 0) \leq 1$ since $du/d\tau = 0$ and $u(\tau) = 0$ for any $\tau > \tau_s$ at the interaction stage. This means that $dw_0/d\tau = 0$ for any $\tau > \tau_s$ at this stage since $w_0(\tau) = w_0(\tau_s + 0) = 0$ for any $\tau > \tau_s$ so $w_0^* = 0$. Overall, $w_0^* \in (0, 1]$.

B

APPENDIX FOR THE THRESHOLD MODEL

B.1 PROOF FOR THEOREM 4.2.1

If $P = 0$, $B(0) = 0$ and $\left|\int_{\Xi(0)} \{1-\theta(\xi)\}f(\xi)d\xi\right| < \infty$, so from (4.2), $\left.\frac{dP(t)}{dt}\right|_{P=0} = 0$. As such, $P \equiv 0$ is a solution for (4.2). From the uniqueness of solution for (4.2), if $P(0) = 0$, then $P(t) = 0$ for all $t > 0$.

If $P(0) > 0$, then $P(t) > 0$ for all $t > 0$ since $P(t)$ is non decreasing in time.

More so,

$$\left.\frac{dP(t)}{dt}\right|_{P=1} = -B(1)\int_{\Xi(1)} \{1-\theta(\xi)\}f(\xi)d\xi \leq 0. \tag{B.1}$$

In addition,

$$\left.\frac{dP(t)}{dt}\right|_{P>1} = B(P)\left[1 - P - \int_{\Xi(P)} \{1-\theta(\xi)\}f(\xi)d\xi\right] < 0 \tag{B.2}$$

since $1 - P(t) < 0$.

This establishes the invariance of $P(t)$ such that $0 \leq P(t) \leq 1$ for all $t > 0$.

B.2 ANALYSIS FOR THE EXISTENCE AND NUMBER OF EQUILIBRIUM STATES FOR $G(P)$

Suppose there is an equilibrium state $P^* < \bar{\xi}/\alpha$, it must satisfy $\mathcal{G}_1(P) = 0$. We see that $\mathcal{G}_1(0) = \theta_0 + \frac{1}{2}(1-\theta_0)e^{-\sqrt{2}\frac{\bar{\xi}}{\sigma}} > 0$. When $\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0)$, the function $\mathcal{G}_1(P)$ is concave in terms of P with minimum point

$$P = P_{c-} = \frac{\bar{\xi}}{\alpha} - \frac{\sigma}{\alpha\sqrt{2}}\ln\frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0) \tag{B.3}$$

so that $\mathcal{G}_1(P_{c-}) = \theta_0 + \frac{\sigma}{\alpha\sqrt{2}}\left[1 + \ln\frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0)\right] - \frac{\bar{\xi}}{\alpha}$. On the other hand, $\mathcal{G}_1(P)$ is monotonically decreasing when $\frac{\sigma}{\alpha} \geq \frac{1}{\sqrt{2}}(1-\theta_0)$. $\mathcal{G}_1(P_{c-}) = 0$ results to $\frac{\bar{\xi}_c}{\alpha} = \theta_0 + \frac{\sigma}{\alpha\sqrt{2}}\left[1 + \ln\frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0)\right]$ which is the same in (4.28).

The equilibrium state $P^* \geq \bar{\xi}/\alpha$, if any exists, has to satisfy $\mathcal{G}_2(P) = 0$. We see that $\mathcal{G}_2(1) = -\frac{1}{2}(1-\theta_0)e^{-\sqrt{2}\frac{\alpha}{\sigma}\left(1-\frac{\bar{\xi}}{\alpha}\right)} < 0$. When $\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0)$, the function $\mathcal{G}_2(P)$ is convex in terms of P with maximum point at

$$P = P_{c+} = \frac{\bar{\xi}}{\alpha} + \frac{\sigma}{\alpha\sqrt{2}}\ln\frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0) \tag{B.4}$$

and $\mathcal{G}_2(P_{c+}) = 1 - \frac{\sigma}{\alpha\sqrt{2}}\left[1 + \ln\frac{\alpha}{\sigma\sqrt{2}}(1-\theta_0)\right] - \frac{\bar{\xi}}{\alpha}$. On the other hand, $\mathcal{G}_2(P)$ is monotonically decreasing when $\frac{\sigma}{\alpha} \geq \frac{1}{\sqrt{2}}(1-\theta_0)$.

From Figure 4.4, we have the following conditions for the existence of $\langle i, j\rangle$, where i and j are the numbers of equilibrium states in the intervals $(0, \bar{\xi}/\alpha)$ and $[\bar{\xi}/\alpha, 1]$ respectively.

When $\frac{\sigma}{\alpha} < \frac{1}{\sqrt{2}}(1-\theta_0)$,

- $\langle 0, 1\rangle$: $\mathcal{G}_1(P_{c-}) > 0$.
- $\langle 2, 1\rangle$: $\mathcal{G}_1(P_{c-}) < 0$ and $G\left(\frac{\bar{\xi}}{\alpha}\right) > 0$.
- $\langle 1, 2\rangle$: $G\left(\frac{\bar{\xi}}{\alpha}\right) < 0$ and $\mathcal{G}_2(P_{c+}) > 0$.
- $\langle 1, 0\rangle$: $\mathcal{G}_2(P_{c+}) < 0$.

When $\frac{\sigma}{\alpha} \geq \frac{1}{\sqrt{2}}(1-\theta_0)$,

- $\langle 0, 1\rangle$: $G\left(\frac{\bar{\xi}}{\alpha}\right) > 0$.
- $\langle 1, 0\rangle$: $G\left(\frac{\bar{\xi}}{\alpha}\right) < 0$.

These equilibrium states agree with theorem 4.4.1 and corollaries 4.4.2 and 4.4.3.