A Mathematical Model
on the Efficiency of Regional Lockdown in Epidemic Dynamics

感染症伝染ダイナミクスにおける局所的ロックダウンの効果に関する数理モデル

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1 Introduction

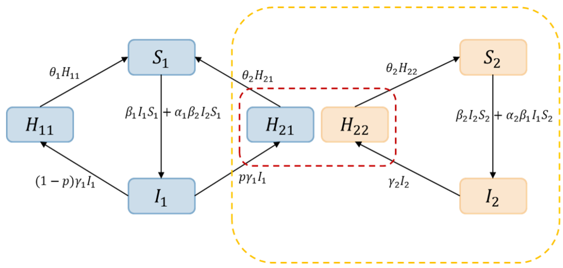
Lockdown is a strategy to prevent the spread of a transmissible disease in a community. Especially in countries and regions with poor medical infrastructure and low emergency response capacity, the lockdown will give the government and decision-makers sufficient time to make plans to control the epidemic (Lytras and Tsiodras, 2021). Although such a strict restriction has played an important role in suppressing the disease transmission among a community, economic development must tend to face with great challenges, as seen in the COVID-19 pandemic (Nicola et al., 2020). Furthermore, Ganesan et al. (2021) mentioned that the prolonged lockdown may cause some problems in the physical and mental health. In this work, we consider a simple mathematical model to theoretically discuss the efficiency of the lockdown, for which we introduce some different types with respect to which social activity is restricted by it. The efficiency is compared according to the endemic size, that is, the number of infective individuals at the endemic equilibrium.

2 Assumptions

- The disease is not fatal;
- The community is composed of the regional area (area 1) and the central area (area 2) with different qualities about the medical treatment for a transmissible disease;
- Susceptible individuals can temporarily visit to the other area;
- Some infective individuals of the regional area (area 1) can get the medical treatment at the central area (area 2), for example, transported by ambulance;
- Recovered individuals become susceptible again;
- The population size is constant in each area according to the epidemic dynamics.

3 Mathematical model

- Susceptibles (S_i): individuals of area i who are healthy and can be infected.
- Infectives (I_i): individuals of area i who have been infected and are able to transmit the disease.
- H_{ij}: infectives of area j who are getting the medical treatment at area i.
- N_i: the population size of area i.



$$\begin{aligned}
\frac{\mathrm{d}S_1}{dt} &= -\beta_1 I_1 S_1 - \alpha_1 \beta_2 I_2 S_1 + \theta_1 H_{11} + \theta_2 H_{21} \\
\frac{\mathrm{d}I_1}{dt} &= \beta_1 I_1 S_1 + \alpha_1 \beta_2 I_2 S_1 - \gamma_1 I_1 \\
\frac{\mathrm{d}H_{11}}{dt} &= (1-p)\gamma_1 I_1 - \theta_1 H_{11} \\
\frac{\mathrm{d}H_{21}}{dt} &= p\gamma_1 I_1 - \theta_2 H_{21} \\
\frac{\mathrm{d}S_2}{dt} &= -\beta_2 I_2 S_2 - \alpha_2 \beta_1 I_1 S_2 + \theta_2 H_{22} \\
\frac{\mathrm{d}I_2}{dt} &= \beta_2 I_2 S_2 + \alpha_2 \beta_1 I_1 S_2 - \gamma_2 I_2 \\
\frac{\mathrm{d}H_{22}}{dt} &= \gamma_2 I_2 - \theta_2 H_{22},
\end{aligned}$$

where β_i is the infection coefficient in area i, which represents the disease transmission from infective to susceptible individuals; $\alpha_i \beta_j$ is the infection coefficient during the temporary visit to area j, which is smaller than β_j ($0 < \alpha_i < 1$); γ_i is the treatment rate of the infective in area i; θ_i is the recovery rate by the medical treatment at area i; p is the proportion of infectives of the regional area, who get the medical treatment in the central area ($0 \le p \le 1$). From the assumption, it holds that $S_1 + I_1 + H_{11} + H_{21} = N_1$, $S_2 + I_2 + H_{22} = N_2$ for any time t with positive constants N_1 and N_2.

With the frequencies $\phi_i = S_i/N_i, \psi_i = I_i/N_i, \zeta_{ij} = H_{ij}/N_j$, the basic reproduction number of the regional area $\mathscr{R}_0^r = \beta_1 N_1/(\gamma_{11} + \gamma_{21})$, and that of the central area $\mathscr{R}_0^c = \beta_2 N_2/\gamma_{22}$, we can transform the original model as follows:

$$
\begin{aligned}
\frac{\mathrm{d}\phi_1}{dt} &= -\mathscr{R}_0^r\gamma_1\psi_1\phi_1 - \mathscr{R}_0^c\gamma_2\alpha_1\psi_2\phi_1 + \theta_1\zeta_{11} + \theta_2\zeta_{21} \\
\frac{\mathrm{d}\psi_1}{dt} &= \mathscr{R}_0^r\gamma_1\psi_1\phi_1 + \mathscr{R}_0^c\gamma_2\alpha_1\psi_2\phi_1 - \gamma_1\psi_1 \\
\frac{\mathrm{d}\zeta_{11}}{dt} &= (1-p)\gamma_1\psi_1 - \theta_1\zeta_{11} \\
\frac{\mathrm{d}\zeta_{21}}{dt} &= p\gamma_1\psi_1 - \theta_2\zeta_{21} \\
\frac{\mathrm{d}\phi_2}{dt} &= -\mathscr{R}_0^c\gamma_2\psi_2\phi_2 - \mathscr{R}_0^r\gamma_1\alpha_2\psi_1\phi_2 + \theta_2\zeta_{22} \\
\frac{\mathrm{d}\psi_2}{dt} &= \mathscr{R}_0^c\gamma_2\psi_2\phi_2 + \mathscr{R}_0^r\gamma_1\alpha_2\psi_1\phi_2 - \gamma_2\psi_2 \\
\frac{\mathrm{d}\zeta_{22}}{dt} &= \gamma_2\psi_2 - \theta_2\zeta_{22}
\end{aligned}
$$

with $\phi_1+\psi_1+\zeta_{11}+\zeta_{21}=1$ and $\phi_2+\psi_2+\zeta_{22}=1$.

4 Different types of lockdown

	α_1	α_2	p
Weak lockdown type 1	0	+	+
Weak lockdown type 2	+	0	+
Strong lockdown	0	0	+
Complete lockdown	0	0	0

5 Disease-free equilibrium

Theorem 5.1. *Disease-free equilibrium $E_0(1,0,0,0,1,0,0)$ is unstable if one of the following is satisfied*

(1) $\mathscr{R}_0^r > 1$,

(2) $\mathscr{R}_0^c > 1$,

(3) $\big(\frac{1}{\mathscr{R}_0^r}-1\big)\big(\frac{1}{\mathscr{R}_0^c}-1\big) < \alpha_1\alpha_2$.

6 Endemic equilibrium

Theorem 6.1.

(1) Under the strong lockdown with $\alpha_1=\alpha_2=0$, the endemic equilibrium E_s^* is globally asymptotically stable if it exists.

(2) Under the complete lockdown with $\alpha_1=\alpha_2=p=0$, the endemic equilibrium E_c^* is globally asymptotically stable if it exists.

7 Endemic size

Endemic size is defined as the total number of infective individuals in the community at the endemic equilibrium. For our model, we define it by $\Psi^* := \psi_1^*+\psi_2^*$ that depends on which type of lockdown the community takes. Let the endemic size under the complete lockdown Ψ_c^*, under the strong lockdown Ψ_s^*, and under the weak lockdown of type 1 Ψ_{w1}^*, of type 2 Ψ_{w2}^*. They can be given as follows:

$$
\begin{aligned}
\Psi_c^* &= \frac{\theta_1}{\theta_1+\gamma_1}\Big(1-\frac{1}{\mathscr{R}_0^r}\Big) + \frac{\theta_2}{\theta_2+\gamma_2}\Big(1-\frac{1}{\mathscr{R}_0^c}\Big); \\
\Psi_s^* &= \frac{\theta_1\theta_2}{\theta_1\theta_2+(1-p)\gamma_1\theta_2+p\gamma_1\theta_1}\Big(1-\frac{1}{\mathscr{R}_0^r}\Big) + \frac{\theta_2}{\theta_2+\gamma_2}\Big(1-\frac{1}{\mathscr{R}_0^c}\Big); \\
\Psi_{w1}^* &= \frac{\theta_1\theta_2}{\theta_1\theta_2+(1-p)\gamma_1\theta_2+p\gamma_1\theta_1}\Big(1-\frac{1}{\mathscr{R}_0^r}\Big) + \frac{\theta_2}{\theta_2+\gamma_2}(1-\phi_2^*); \\
\Psi_{w2}^* &= \frac{\theta_1\theta_2}{\theta_1\theta_2+(1-p)\gamma_1\theta_2+p\gamma_1\theta_1}(1-\phi_1^*) + \frac{\theta_2}{\theta_2+\gamma_2}\Big(1-\frac{1}{\mathscr{R}_0^c}\Big);
\end{aligned}
$$

where ϕ_1^* is the smaller root less than $1/\mathscr{R}_0^r$ of the following equation:

$$
\mathscr{R}_0^r\gamma_1\theta_1\theta_2x^2 - \Big\{(\mathscr{R}_0^r+1)\gamma_1\theta_1\theta_2 + \mathscr{R}_0^c\alpha_1\psi_2^*\gamma_2 \big[\theta_1\theta_2+(1-p)\gamma_1\theta_2+p\gamma_1\theta_1\big]\Big\}x + \gamma_1\theta_1\theta_2 = 0
$$

with $\psi_2^* = \theta_2[1-(1/\mathscr{R}_0^c)]/(\theta_2+\gamma_2)$, and ϕ_2^* is the smaller root less than $1/\mathscr{R}_0^c$ of the following equation:

$$
\mathscr{R}_0^c\gamma_2\theta_2x^2 - \big[(\mathscr{R}_0^c+1)\gamma_2\theta_2 + \mathscr{R}_0^r\alpha_2\psi_1^*\gamma_1(\gamma_2+\theta_2)\big]x + \gamma_2\theta_2 = 0
$$

with $\psi_1^* = \theta_1\theta_2[1-(1/\mathscr{R}_0^c)]/[\theta_1\theta_2+(1-p)\gamma_1\theta_2+p\gamma_1\theta_1]$.

Condition	Result
$\theta_1<\theta_2$	$\Psi_{w\bullet}^* > \Psi_s^* > \Psi_c^*$
$\theta_1=\theta_2$	$\Psi_{w\bullet}^* > \Psi_s^* = \Psi_c^*$
$\theta_1>\theta_2$	$\Psi_{w\bullet}^* > \Psi_s^* < \Psi_c^*$

8 Conclusion

- When the isolation period of the regional area is longer than that of the central area, the complete lockdown is the best choice while the weak lockdown is the worst.
- When the isolation period of the regional area is the same as that of the central area, the weak lockdown is the worst choice.
- When the isolation period of the regional area is shorter than that of the central area, the strong lockdown is the best choice, while the worst choice depends on the proportion of the infectives of the regional area treated in the central area.
- When few people takes the medical treatment in the central area, the complete lockdown is better than the weak lockdown.

References

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